Stochastic Differential Equations with Reflecting Boundary Conditions

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Introduction

In this paper we solve stochastic differential equations with reflecting boundary conditions by a direct approach based on the Skorokhod problem. Before explaining more precisely our method and our results, let us recall a few results and methods concerning such equations: first the case of reflecting diffusion processes in a half-space (or a half-line) has been investigated by many authors (see A. V. Skorokhod [18], H. P. McKean [15], [16], N. Ikeda and S. Watanabe [10], S. Watanabe [22], N. El Karoui [6], N. El Karoui and M. Chaleyat-Maurel [7], N. El Karoui, M. Chaleyat-Maurel and B. Marchal [8]). In the case of a general domain $\mathcal{O}$ only the following particular cases were treated: (i) the existence of weak solutions in a smooth domain $\mathcal{O}$ is proved in D. W. Stroock and S. R. S. Varadhan [19]; (ii) in a smooth domain, the equations are solved by a heavy localization procedure in A. Bensoussan and J. L. Lions [2]; (iii) the case of a convex domain is treated by a direct approach based on the Skorokhod problem in H. Tanaka [21], using the convexity of the domain and a variational inequality formulation in an essential way (see also A. Bensoussan and J. L. Lions [2], P. L. Lions, J. L. Menaldi and A. S. Sznitman [14]).

Our goal here will be two-fold: (i) we give a direct approach to the solution of stochastic differential equations with reflecting boundary conditions (without localization arguments) and (ii) reductions to the case including nonsmooth ones (provided convenient restrictions are assumed).

Let us describe now more precisely the results and methods we present below: first, in the case of a normal reflection inside, say, a smooth bounded open set in $\mathbb{R}^d$, we show that for each $w_0 \in C([0, \infty[; \mathbb{R}^d)$ with $w(0) \in \mathcal{O}$ there exists a unique solution $(x_t, k_t)$ of the Skorokhod problem:
\[ x_t \in C([0, \infty[, \mathcal{F}), \quad k_t \in C([0, \infty[; \mathbb{R}^d), \quad k_t \in BV(0, T) \quad \text{for all} \quad T < \infty, \]

\begin{equation}
(S) \quad x_t + k_t = w_t \quad \text{for all} \quad t \geq 0,
\end{equation}

\[ k_t = \int_0^t n(x_s) \, d[k_s], \quad |k_t| = \int_0^t 1_{\{x_s \in \partial \mathcal{O}\}} \, d[k_s] \quad \text{for} \quad t \geq 0,
\]

where \( k_t \) stands for the total variation of \( k \) on \([0, t]\) and \( n(x) \) is the unit outward normal to \( \partial \mathcal{O} \) at \( x \).

This kind of deterministic problem was first studied in [7], [8] in the case of a half-line and in [21] in the case of a convex domain: let us point out that in the case of the half-line \( \mathcal{O} = \{x > 0\} \), \( S \) is solved by the explicit formulas

\[ k_t = - \sup_{s \in [0, t]} w_s - , \quad x_t = w_t - k_t. \]

On the other hand, the case of a convex domain was handled in [21] using in an essential way the convexity of the domain (and, for example, the projection onto \( \mathcal{O} \)).

Here we are able to treat the general case by using two ingredients:

(i) We remark that there exists a constant \( C_0 \geq 0 \) such that

\begin{equation}
(VI) \quad \forall x \in \partial \mathcal{O}, \quad \forall x' \in \mathcal{O}, \quad (n(x), x - x') + C_0 |x - x'|^2 \geq 0;
\end{equation}

and this inequality immediately yields the uniqueness of the \((x_t, k_t)\) solution of \( S \), we also use it in order to obtain \textit{a priori} estimates on \((x_t, k_t)\) (see Section 1).

(ii) For smooth \( w \), we solve \( S \) by a penalty method (see Section 2).

Having solved the deterministic problem \( S \), we are now able to solve \textit{stochastic differential equations with reflection along the normal} as for example: on some given probability space \((\Omega, F, F_t, P, B_t)\) satisfying the usual assumptions, \( B_t \) being an \( F_t \)-Brownian motion, find a continuous adapted semimartingale \( X_t \) such that

\[ X_t = x + \int_0^t \sigma(X_s) \, dB_s + \int_0^t b(X_s) \, ds - k_t, \]

\begin{equation}
(SDE) \quad X_t \in \mathcal{O} \quad \text{for all} \quad t \geq 0, \ a.s., \ k_t \quad \text{is a bounded variation process,}
\end{equation}

\[ |k_t| = \int_0^t 1_{\{x_s \in \partial \mathcal{O}\}} \, d[k_s], \quad k_t = \int_0^t n(X_s) \, d[k_s] \quad \text{for all} \quad t \geq 0, \ a.s.
\]

We prove that \textit{there exists a unique solution} \((x_t, k_t)\) of \( SDE \) provided, for example, \( \sigma, b \) are Lipschitz on \( \mathcal{O} \) and \( \mathcal{O} \) is smooth. To do so, we need only remark that an \( X_t \) solution of \( SDE \) is a fixed point of the mapping sending \( Y_t \) onto the process \( X_t \) obtained by taking for almost all paths the solution of \( S \) corresponding to

\[ w_t = x + \int_0^t \sigma(Y_s) \, dB_s + \int_0^t b(Y_s) \, ds. \]
And we prove by a usual iteration method the existence (and uniqueness) of such a fixed point using the above inequality (VI) in an essential way (see Section 3). Furthermore, the method of proof for solving (S) and (SDE) is good enough to treat the case of more general domains essentially those satisfying (VI) (with a convenient definition of \( n(x) \)); see Sections 1–3 for more details.

Next (cf. Section 4) we consider the case of oblique reflections, that is, the case when \( n \) in the above problems is to be replaced by \( \gamma \), where \( \gamma \) is a smooth (say \( C^2 \)) vector field on \( \mathbb{R}^d \) such that

\[
\exists \nu > 0, \quad \forall x \in \partial \Omega, \quad (\gamma(x), n(x)) \geq \nu > 0.
\]

Then we prove

(i) the existence of an \((x_1, k_1)\) solution of (S);
(ii) the uniqueness of an \((x_1, k_1)\) solution of (S) if \( w \) has bounded variation or, more generally, if \( w \) is a semimartingale;
(iii) the existence and uniqueness of an \( X_t \) solution of (SDE).

It can be seen that the only remaining question we were unable to treat is the uniqueness of the \((x_1, k_1)\) solution of (S) for general continuous \( w \). The main new idea in this case is the use of a symmetric matrix-valued function \((a_{ij}(x))_{1 \leq i, j \leq d}\) satisfying:

\[
\forall x \in \partial \Omega, \quad a_{ij}(x) \gamma_j(x) = n_i(x) \quad \text{for} \quad 1 \leq i \leq d, \quad a_{ij}(x) \in C^2_b(\mathbb{R}^d),
\]

\[
\exists \alpha > 0, \quad \forall x \in \mathbb{R}^d, \quad (a_{ij}(x)) \geq \alpha I_d.
\]

Then conveniently plugging this matrix into the estimates obtained in Sections 1–3, we are able to treat problems (S) and (SDE). We also indicate a few extensions to nonsmooth domains (or nonsmooth vector fields) but these extensions are necessarily limited in view of the counterexamples discussed briefly at the end of the paper.

We shall often use properties of the function “distance to the boundary” and of the field of normals in a smooth domain and we refer the interested reader to J. Serrin [17], D. Gilbarg and N. S. Trudinger [9].

Finally, concerning the study of various properties of solutions of stochastic differential equations with reflecting boundary conditions (determination of the support, small perturbations of deterministic motion, large deviations, optimal control), we refer to R. F. Anderson and S. Orey [1], H. Doss and P. Priouret [4], [5], P. L. Lions [12], [13].

1. Main Estimates on the Skorokhod Problem

Here and everywhere below \( \Omega \) will be an open set in \( \mathbb{R}^d \) that we assume to be bounded (only to simplify the presentation). Let \( n \) be a vector field (not necessarily single-valued) on \( \partial \Omega \) such that \( \forall \xi \in n(x), \forall x \in \partial \Omega, |\xi| = 1 \). We shall use the following assumption which, in the case of a smooth domain \( \partial \Omega \), merely
expresses the fact that $n(x)$ is the unit outward normal to $\partial \mathcal{O}$ at the point $x$:

(i) \[ \exists C_0 \geq 0, \quad \forall x \in \partial \mathcal{O}, \quad \forall x' \in \mathcal{O}, \quad \forall k \in n(x), \]
\[ (x-x', k) + C_0 |x-x'|^2 \geq 0, \]

(1) \[ \forall x \in \partial \mathcal{O}, \text{if } \exists C \geq 0, \exists k \in \mathbb{R}^d: \]
\[ \forall x' \in \mathcal{O}, \quad (x-x', k) + C |x-x'|^2 \geq 0, \]

then $k = \partial n(x)$ for some $\theta \geq 0$.

We shall explain this condition below. Let us only repeat for the moment that (1) holds if $\mathcal{O}$ is smooth and $n$ is the unit outward normal.

We now explain precisely what we mean by the solution of the Skorokhod problem: let $w \in C([0, \infty[; \mathbb{R}^d)$ be such that $w(0) \in \mathcal{O}$; we call a couple of functions $(x_t, k_t)$ satisfying

(2) \[ x_t \in C([0, \infty[; \mathcal{O}), \quad k_t \in C([0, \infty[; \mathbb{R}^d), \quad k_t \in BV(0, T) \quad \text{for all } T < \infty \]

and

(3) \[ |k_t| = \int_0^t 1_{(x_s \in \partial \mathcal{O})} d|k_s|, \quad k_t = \int_0^t \xi_s d|k_s| \quad \text{with } \xi_s \in n(x_s), \]

(4) \[ x_t + k_t = w_t \quad \text{for } t \geq 0, \]

a solution of the Skorokhod problem $(w, \mathcal{O}, n)$. Here and everywhere below, we use the notation $|\varphi|_t$ for the total variation of $\varphi$ on $[0, t]$.

We shall need the following assumption:

(5) \[ \exists n \geq 1, \quad \exists \alpha > 0, \quad \exists R > 0; \quad \exists a_1, \cdots, a_n \in \mathbb{R}^d, \quad |a_i| = 1 \quad \forall i, \]
\[ \forall \xi \in \partial \mathcal{O} \cap \bigcup_{i=1}^n B(x_i, R), \forall \xi \in n(x), \quad (\xi, a_i) \geq \alpha > 0. \]

We then have

**Theorem 1.1.** Assume that (1), (5) hold and that, for all $w \in C^\infty([0, \infty[; \mathbb{R}^d)$ (with $w(0) \in \mathcal{O}$), there exists a solution to the Skorokhod problem $(w, \mathcal{O}, n)$. Then for all $w \in C([0, \infty[; \mathbb{R}^d)$ (with $w(0) \in \mathcal{O}$) there exists a unique solution $(x_t, k_t)$ of the Skorokhod problem $(w, \mathcal{O}, n)$. Furthermore, the mapping $(w \mapsto x)$ from $C([0, T], \mathbb{R}^d)$ into itself is Hölder continuous of order $\frac{1}{2}$ on compact sets.

**Remark 1.1.** The remainder of this section is devoted to the proof of this result. In Section 2 we shall show that for "reasonable" domains $\mathcal{O}$ there exists a solution to the problem $(w, \mathcal{O}, n)$ for smooth $w$. Let us also indicate that we could treat unbounded domains as well.
Remark 1.2. Let us now explain condition (1). We claim that if $\mathcal{O}$ is a $C^1$ open set satisfying the “uniform exterior sphere” condition, then (1) holds with $n$ being the unit outward normal.

More precisely, assume that $\mathcal{O}$ is $C^1$ and satisfies
\[ \exists R_0 > 0, \forall x \in \partial \mathcal{O}, \exists y \in \mathbb{R}^d: B(y, R_0) \cap \mathcal{O} = \emptyset, |x - y| = R_0. \]

Then for $C \equiv 1/2R_0$ we have
\[ (n(x), x - x') + C|x - x'|^2 \geq 0, \forall x' \in \mathcal{O}, \]

where $n(x)$ is the unit outward normal. Indeed if $x' \in \mathcal{O}, |x' - y| \geq R_0^2$, we deduce, remarking that $y - x = R_0n(x),
\[ \frac{1}{2R_0}|y - x|^2 - \frac{1}{2}R_0 \geq 0. \]

On the other hand, if $k \in \mathbb{R}^d, x \in \partial \mathcal{O}$ satisfy, for some $C \geq 0$,
\[ \forall x' \in \mathcal{O}, (x - x', k) + C|x - x'|^2 \geq 0, \]

and if $\mathcal{O}$ is of class $C^1$, then we claim that $k = \theta n(x)$ for some $\theta \equiv 0$. Without loss of generality we may assume that $|k| = 1$ and that $C > 0$ and then take $y = x + (1/2C)k$. The same computation as above shows that
\[ B\left(y, \frac{1}{2C}\right) \cap \mathcal{O} = \emptyset, \quad |x - y| = \frac{1}{2C}, \]

and this yields $n(x) = (y - x)|y - x|^{-1} = k$.

Proof of Theorem 1.1: In all arguments below, we shall simply write $n(x_i)$ instead of $\xi_i$ with $\xi_i \in n(x_i)$.

Uniqueness. Let $w \in C([0, \infty[, \mathbb{R}^d)$ with $w(0) \in \mathcal{O}$ and let $(x_t, k_t), (x'_t, k'_t)$ be two solutions of the problem $(w, \mathcal{O}, n)$. Using (1) we have clearly
\[
\exp\{-2C_0(|k_s| + |k'_s|)|x_t - x'_t|^2
\]
\[= \exp\{-2C_0(|k_s| + |k'_s|)|x_t - x'_t|^2
\]
\[= \int_0^t \exp\{-2C_0(|k_s| + |k'_s|)\} \cdot [2(k_s - k'_s) \cdot (n(x_s) d|k_s| - n(x'_s) d|k'_s)]
\]
\[\quad - 2C_0|x_t - x'_t|^2 d(|k_s| + |k'_s|)]
\]

and this shows that $x_t = x'_t$ and thus $k_t = k'_t$ for all $t \equiv 0$. 

Existence. We shall need a few preliminary results. Here and everywhere below we shall use the following notation: \( \| \varphi \|_t = \max_{[0,t]} |\varphi(s)|, \| \varphi \|_s = \max_{[s,t]} |\varphi| \) (and thus \( \| \varphi \|_{0,t} = \| \varphi \|_t \)) for any continuous \( \varphi \).

**Lemma 1.1.** Let \( w, w' \in C^\omega([0,\infty[, \mathbb{R}^d) \) and let \( (x_i, k_i), (x_i', k_i') \) be the solutions of the corresponding problems \( (w, \theta, n), (w', \theta, n) \). Then there exists a constant \( C \) depending only on \( (\theta, n) \) such that, for all \( 0 \leq s \leq t \), we have

\[
|x_i - x_i'|^2 \leq |w_i - w_i'|^2 + 2((k_i - k_i') \cdot (w_i - w_i')) + \exp \left\{ C(|k_i| + |k_i'|) \right\}
\]

\[
|x_i - x_i'|^2 \leq |w_i - w_i'|^2 + 2((k_i - k_i') \cdot (w_i - w_i')) + \exp \left\{ C(|k_i| + |k_i'|) \right\}
\]

Proof: We shall use the following notation: \( a_i = |k_i|, a_i' = |k_i'|, \Delta w = w - w', \Delta k = k - k' \). Choosing \( C \geq 2C_0 \) and using (1) as in the proof of uniqueness, we find

\[
|x_i - x_i'|^2 \exp \left\{ -C(a_i + a_i') \right\} \leq \int_0^t \exp \left\{ -C(a_i + a_i') \cdot [2(x_i - x_i') \cdot (\dot{w}_i - \dot{w}_i')] \right\} ds
\]

\[
+ |x_0 - x_0'|^2
\]

\[
= 2 \int_0^t \exp \left\{ -C(a_i + a_i') \right\} \cdot [(\dot{w}_i - \dot{w}_i')] ds
\]

\[
+ |x_0 - x_0'|^2
\]

\[
-2 \int_0^t \exp \left\{ -C(a_i + a_i') \right\} \cdot [(\dot{w}_i - \dot{w}_i')] ds
\]

\[
\leq \exp \left\{ -C(a_i + a_i') \right\} \cdot [\Delta w_i^2 - \Delta k \cdot \Delta w_i]
\]

\[
+ C \int_0^t \exp \left\{ -C(a_i + a_i') \right\} \cdot [\Delta w_i^2 (da_i + da_i')
\]

\[
- 2 \int_0^t \exp \left\{ -C(a_i + a_i') \right\} \cdot \Delta w_i
\]

\[
\cdot [n(x_i) da_i - n(x_i') da_i'] - C \Delta k \cdot (da_i + da_i')
\]

and we obtain

\[
|x_i - x_i'|^2 \leq [\Delta w_i^2 - \Delta k \cdot \Delta w_i
\]

\[
+ \left( \|\Delta w\|^2 + \frac{2}{C} \|\Delta w\|_s + 2\|\Delta k \cdot \Delta w\|_s \right) \cdot \exp \left\{ C(a_i + a_i') \right\};
\]

this proves (6).
In the same way we find

\[
|x_i - x_j|^2 \exp \{-C(a_i - a_j)\} \leq \int_s^t \exp \{-C(a_u - a_s)\} \cdot [2(x_u - x_s) \cdot \dot{w}_u] \, du
\]

\[
= \exp \{-C(a_i - a_s)\} \cdot |w_i - w_s|^2
\]

\[
+ C \int_s^t \exp \{-C(a_u - a_s)\} \cdot |w_u - w_s|^2 \, da_u
\]

\[
+ 2 \exp \{-C(a_i - a_s)\} \cdot (k_i - k_s) \cdot (w_i - w_s)
\]

\[
- 2 \int_s^t \exp \{-C(a_u - a_s)\} \cdot (w_u - w_s)
\]

\[
\cdot [n(x_u) \, da_u - C(k_i - k_s) \, da_u],
\]

and this yields (7).

Let \( T > 0 \) be fixed.

**Lemma 1.2.** Let \( A \) be a relatively compact set in \( C([0, T], \mathbb{R}^d) \), included in \( C^\infty([0, \infty], \mathbb{R}^d) \) and such that \( w_0 \in \partial \) for all \( w \in A \). Then there exists \( K (= k(A, T)) \) such that, for all \( w \in A, |k|_T \leq K \) if we denote by \((x_i, k_i)\) the solution of the problem \((w, \partial, n)\).

**Proof of Lemma 1.2:** Let \( w \in A \). We use (5) and denote by \( \mathcal{O}_1, \cdots, \mathcal{O}_n \) the open sets \( B(x_1, 2R) \cap \mathcal{O}, \cdots, B(x_n, 2R) \cap \mathcal{O} \). Let \( \partial_0 \) be an open set satisfying \( \partial_0 \subset \mathcal{O}, \partial \subset \bigcup_{i=1}^n (B(x_i, R) \cup \partial_0) \). We then let \( T_i = \inf \{ t \in [0, T], x_t \in \partial_{i,0} \} \), where \( i_0 \) is such that \( w_0 = x_0 \in \partial_{i,0} \). Then either \( x_{T_i} \in \partial_0 \) and we set \( i_1 = 0 \), or \( x_{T_i} \in B(x_i, R) \) for some \( i \). In this way we construct, by induction, \( i_n, T_m \): if \( T_m < T, x_{T_m} \in B(x_{i_0}, R) \) or \( x_{T_m} \in \partial_0, T_{m+1} = \inf \{ t \geq T_m, t \in [0, T], x_t \in \partial_{i_n} \} \).

We want to obtain bounds on \( |k|_{T_{m+1}} - |k|_{T_m} \). Clearly, if \( i_m = 0 \), i.e., \( x_{T_m} \in \partial_0 \), then \( x_m \in \mathcal{O} \) for \( t \in [T_m, T_{m+1}] \) and thus \( |k|_{T_{m+1}} = |k|_{T_m} = 0 \). On the other hand, if \( i_m \geq 1 \), then using (5) we get

\[
(x_{T_{m+1}} - x_{T_m}) \cdot a_{i_m} - (w_{T_{m+1}} - w_{T_m}) \cdot a_{i_m} = (k_{T_{m+1}} - k_{T_m}) \cdot a_{i_m}
\]

\[
= \int_{T_m}^{T_{m+1}} n(x_s) \cdot a_{i_m} \, d|k|_s
\]

\[
\leq a \{ |k|_{T_{m+1}} - |k|_{T_m} \}.
\]

In all cases we obtain \( |k|_{T_{m+1}} - |k|_{T_m} \leq C(1 + \|w\|_T) \). Using (7), this yields, for all \( T_m \leq s \leq t \leq T_{m+1} \),

\[
|x_i - x_j|^2 \leq M(\|w - w_s\|_{T_s}^2 + \|w - w_s\|_{T_s}),
\]

where \( M \) depends only on \( \mathcal{O}, n \) and \( \|w\|_T \).
Thus, if $T_{m+1} < T$ we deduce, remarking that $|x_{T_{m+1}} - x_{T_m}| \geq R$, that

$$\frac{R^2}{M} \leq \mu(T_{m+1} - T_m)^2 + \mu(T_{m+1} - T_m),$$

where $\mu(\cdot)$ is a uniform modulus of continuity on $[0, T]$ of $w \in A$.

This implies $T_{m+1} - T_m \leq \delta = \delta(R, M, \mu)$ and thus $m + 1 \leq T/\delta$. Therefore,

$$k_T = \sum_{n \geq 1} |k|_{T_{m+1}} - |k|_{T_n} \leq \left(\frac{T}{\delta} + 1\right)C(1 + \|w\|_T) = K.$$

We may now complete the proof of Theorem 1.1. We remark first that, in view of (6) and of Lemma 2, the mapping $(w \mapsto x)$ from $C^\omega([0, T], \mathbb{R}^d)$ into $C([0, T], \mathbb{R}^d)$ has a unique extension $\Phi$ to $C([0, T], \mathbb{R}^d)$, which is H"older continuous on compact sets of $C([0, T], \mathbb{R}^d)$. Therefore, it remains only to show that if $w \in C([0, T], \mathbb{R}^d)$ (with $w_0 \in \partial$), then $(\Phi(w), w - \Phi(w))$ is the solution of the problem $(w, \partial, n)$. To this end, we consider $w^m \in C^\omega([0, T], \mathbb{R}^d)$ converging uniformly to $w$ (with $w_0^m = w_0 \in \partial$). Let $(x^m, k^m)$ be the corresponding solution of $(w^m, \partial, n)$. In view of Lemma 1.2, $|k^m|_T$ is bounded independent of $m \geq 1$.

Using (7) this shows that we may assume that $x^m, k^m$ converge uniformly to some $x, k$ which are continuous on $[0, T], x \in \partial$ for $t \in [0, T]$ and $k \in BV(0, T)$. Of course, $w_t = x_t + k_t$ for $t \geq 0$.

We have to check (3): indeed let $\chi \in \mathcal{D}_+(\partial)$, $0 \leq \chi \leq 1$, $\chi = 1$ on a compact set included in $\partial$; then we have

$$0 \equiv \int_0^T \chi(x_t) \, d|k|_s = \lim_m \int_0^T \chi(x^m_t) \, d|k^m|_s = 0,$$

and taking a sequence of such $\chi$ converging (increasingly) to $1_\partial$, we find

$$\int_0^T 1_\partial(x_t) \, d|k|_s = 0 \quad \text{or} \quad |k|_T = \int_0^T 1_\partial(x_t) \, d|k|_s.$$

Next if $\varphi \in C([0, T]), \varphi \equiv 0$, and $\eta \in \partial$, (1) yields

$$\int_0^T \{(x^m_\nu - \eta, n(x^m_\nu)) + C|\eta - x^m_\nu|^2\} \varphi(u) \, d|k^m|_u \geq 0$$

or

$$\int_0^T \varphi(u)(x_\nu - \eta, dk_\nu^m) + C \int_0^T |\eta - x_\nu^m|^2 \varphi(u) \, d|k^m|_u \geq 0.$$

Letting $m \to +\infty$ (taking a subsequence if necessary), the measure $d|k^m|$ converges weakly to some measure $da_u$ and we get

$$\int_0^T \varphi(u)(x_\nu - \eta, dk_\nu) + C \int_0^T |\eta - x_\nu|^2 \varphi(u) \, da_u \geq 0.$$
Since clearly $d|k|_u \leq da_u$, we have $dk_u = h_u \, da_u$, $h_u$ being bounded measurable, and thus for all $\eta \in \mathcal{O}$

$$(x_u - \eta, h_u) + C|y - x_u|^2 \geq 0 \quad \text{a.e.}$$

Then (1) implies that $h_u \in \mathbb{R}_+ n(x_u)$ and hence

$$k_i = \int_0^\cdot n(x_s) \, d|k|_s,$$

and the theorem is proved.

2. Solvability of the Skorokhod Problem

In this section we shall be mainly concerned with the solvability of the Skorokhod problem for smooth domains $\mathcal{O}$ (say $C^2$) thus satisfying (1), (5), $n(x)$ being the unit outward normal, and for smooth $w$.

**Theorem 2.1.** Let $\mathcal{O}$ be a smooth set in $\mathbb{R}^d$ and let $w \in C([0, \infty), \mathbb{R}^d)$ with $w_0 \in \mathcal{O}$. Then there exists a unique solution $(x, k)$ of the Skorokhod problem $(w, \mathcal{O}, n)$. In addition, if $w \in BV(t_1, t_2)$ for some $0 \leq t_1 \leq t_2 < \infty$, then $x \in BV(t_1, t_2)$ and

$$d|k|_i \leq d|w|_i \quad \text{in} \quad (t_1, t_2).$$

**Remark 2.1.** Of course, as in Theorem 1.1, the mapping $(w \to x)$ is Hölder continuous of order $\frac{1}{2}$ on compact sets of $C([0, T], \mathbb{R}^d)$.

Proof of Theorem 2.1: In view of Theorem 1.1, it is enough to show the existence of a solution $(x, k)$ for smooth $w$. We shall do so for $w \in H^1(0, T, \mathbb{R}^d)$ for all $T < \infty$, and then prove (8).

The existence part is proved by a penalty argument somewhat similar to the one introduced in [14]. This type of penalty method is motivated by the analogy between the Skorokhod problem and variational inequalities; this analogy being exact when $\mathcal{O}$ is convex (cf. [21] and [2]).

We introduce $p \in C^1(\mathbb{R}^d)$ satisfying $p = 0$ in $\mathcal{O}$, $p > 0$ in $\mathbb{R}^d - \mathcal{O}$, $p = \text{dist} (x, \mathcal{O})^2$ in a neighborhood of $\mathcal{O}$. Of course there exists a solution $x^\varepsilon$ of the following ordinary differential equation:

$$x^\varepsilon_t + \frac{1}{\varepsilon} \nabla p(x^\varepsilon_t) = \dot{w}_t \quad \text{for} \quad t \geq 0, x_0 = w_0.$$

If $w_0 \in \mathcal{O}$, we get

$$p(x^\varepsilon_t) + \frac{1}{\varepsilon} \int_0^t |\nabla p(x^\varepsilon_s)|^2 \, ds = \int_0^t (\dot{w}_s, \nabla p(x^\varepsilon_s)) \, ds$$
and thus
\[
\left( \int_0^t |\nabla p(x^*_s)|^2 \, ds \right)^{1/2} \leq \varepsilon \left( \int_0^t |\dot{w}_s|^2 \, ds \right)^{1/2},
\]
\[
p(x^*_s) \leq \varepsilon \int_0^t |\dot{w}_s|^2 \, ds.
\]

This shows that \( x^*_s, k^*_s = \int_0^t (1/\varepsilon) \nabla p(x^*_s) \, ds \), taking a subsequence if necessary, converge uniformly to \( x_s, k_s \in H^1(0, T; \mathbb{R}^d) \) (\( \forall T < \infty \)) satisfying \( x_s + k_s = w_s \), \( x_s \in \mathcal{C} \forall t \geq 0 \) (since \( p > 0 \) outside \( \mathcal{C} \) and \( x^*_s \) is uniformly bounded on \([0, T])\).

If \( \chi \in \mathcal{D}_+(\mathcal{C}) \), then
\[
\int_0^t \left| \frac{1}{\varepsilon} \nabla p(x^*_s) \right| \chi(x^*_s) \, ds = 0.
\]

And since \( |(1/\varepsilon) \nabla p(x^*_s)| \to h_\varepsilon \) weakly in \( L^2((0, T), \mathbb{R}^d) \) for all \( T < \infty \), we get
\[
0 = \int_0^t h_\varepsilon \chi(x_s) \, ds, \quad |k_s| \leq h_\varepsilon \text{ a.e.; this shows that}
\]
\[
|k_s| = \int_0^t 1_{\{x_s \in \partial \mathcal{C}\}} |d|k_s|.
\]

Next, let \( T < \infty \) be fixed; for \( \varepsilon \) small enough we have
\[
p(x^*_s) = (x^*_s, \mathcal{D})^2 \quad \text{for } t \in [0, T].
\]

Therefore,
\[
k^*_s = \int_0^t \nabla \text{dist} (x^*_s, \mathcal{C})^2 \, ds = \int_0^t n(x^*_s) \, d|k^*_s|.
\]

(recall that in a neighborhood of \( \partial \mathcal{C} \), relative to \( \mathbb{R}^N - \mathcal{C} \), there exists for all \( x \) a unique \( (\xi, t) \in \partial \mathcal{C} \times (0, \infty) \) such that \( x = \xi + t n(\xi) \), \( \nabla \text{dist} (x, \mathcal{C}) = n(\xi) \) and if \( x \to x_0 \in \partial \mathcal{C} \), then \( \xi \to x_0, t \to 0 \); we then set \( n(x) = n(\xi) \)). Letting \( \varepsilon \to 0 \), we deduce
\[
k_s = \int_0^t n(x_s) h_s \, ds
\]
(and we conclude that \( h_\varepsilon = |\dot{k}_s| \) a.e. in \((0, \infty))\).

Next we have to prove (8). We denote \( z(x) = \text{dist} (x, \partial \mathcal{C}) \), \( z \) being smooth in a neighborhood of \( \partial \mathcal{C} \) and \( \nabla z = -n \) on \( \partial \mathcal{C} \). Let \( \chi_\varepsilon(t) = \varepsilon \chi(t/\varepsilon) \) with \( \chi(t) \in C^\infty([0, \infty]) \), \( 0 \leq \chi \leq 1 \), \( \chi(t) = t \) for \( 0 \leq t \leq 1/2 \), \( \chi(t) = \frac{3}{2} \) for \( t \geq 1 \). We denote \( z_\varepsilon = \chi_\varepsilon(z) \). To prove (8), we may assume without loss of generality that \( w \) is smooth; then in view of the above proof, \( x_s, k_s \in H^1(0, T) \) for all \( T < \infty \). We have, for all
We now conclude this section by observing that the estimates obtained above and in Section 1 are good enough to treat the case of more general domains than those of class $C^2$; more precisely, we shall say that a bounded open set $\mathcal{O}$ in $\mathbb{R}^d$ is admissible if the following condition holds: there exists a sequence $(\mathcal{O}_m)_{m=1}^{\infty}$ of bounded smooth open sets in $\mathbb{R}^d$ such that

(i) $0$ and $\partial \mathcal{O}_m$ satisfy (1) (respectively, for a vector field $n$ and the normals $n_m$) constants $C_0$ being uniform, and $\mathcal{O}$ satisfies (5);

(ii) if $x_m, x \in \partial \mathcal{O}_m$, $x_m \to x$, then $x \in \partial \mathcal{O}$;

(iii) if $K$ is compact and $K \subset \mathcal{O}$, then $K \subset \mathcal{O}_m$ for $m$ large enough.

It is then obvious to deduce the following result for an admissible bounded open set $\mathcal{O}$ by using Theorem 2.1. Indeed for smooth $w$ we find a solution of $(w, \mathcal{O}, n)$ by taking the solution $(x^*_m, k^*_m)$ of $(w, \mathcal{O}_m, n_m)$ and using (8); we easily pass to the limit as $m$ goes to $+\infty$ exactly as in the proof of Theorem 1.1 remarking that in view of (i) $C_0$ in (1) is uniform.

**Theorem 2.2.** Let $\mathcal{O}$ be an admissible bounded open set in $\mathbb{R}^d$ and let $w \in C([0, \infty], \mathbb{R}^d)$ with $w_0 \in \partial \mathcal{O}$. Then there exists a unique solution $(x, k)$ of the Skorokhod problem $(w, \mathcal{O}, n)$. In addition the mapping $(w \to x)$ is a Hölder continuous of order $\frac{1}{2}$ on compact sets of $C([0, T], \mathbb{R}^d)$ for all $T < \infty$. Finally, if $w \in BV(t_1, t_2)$ for some $0 \leq t_1 \leq t_2 < \infty$, then $x \in BV(t_1, t_2)$ and we have (8),

$$d|k|, \leq d|w|, \text{ in } (t_1, t_2).$$

**Remark 2.2.** We could obtain similar results for general unbounded domains.

**Remark 2.3.** The case of a convex domain (i.e., the existence and uniqueness statements above) was treated by H. Tanaka in [21].
Remark 2.4. It is quite obvious that if \( \partial \mathcal{O} \) is piecewise smooth with "convex angles", then \( \mathcal{O} \) is admissible. More precisely, if \( \mathcal{O} \) is piecewise smooth and satisfies the uniform exterior sphere condition, then \( \mathcal{O} \) is admissible. Let us also point out that if \( \mathcal{O} \) is convex, then \( \mathcal{O} \) is admissible.

3. Stochastic Differential Equations with Normal Reflecting Boundary Conditions

Let us consider \((\Omega, F, P)\) — a complete probability space with an increasing family of sub \(\sigma\)-fields \((F_t)_{t \geq 0}\) of \(F\). We suppose that each \(F_t\) contains all \(P\)-negligible sets of \(F_t\), that \(F_t = \bigcap_{s > t} F_s\), and that we are given a \(d\)-dimensional \(F_t\)-Brownian motion \((B_t)_{t \geq 0}\). Let \((\mathcal{O}, n(\cdot))\) be admissible (see Section 2) satisfying the condition

\[
\text{there exists a function } \phi \in C^2_\infty(\mathbb{R}^d) \text{ such that:}
\]

\[
\exists \alpha > 0, \forall x \in \partial \mathcal{O}, \forall \xi \in n(x), \quad \nabla \phi(x) \cdot \xi \equiv -\alpha C_0,
\]

where \(C_0\) is given by (1).

Remark 3.1. In case \(\mathcal{O}\) is a smooth (say \(C^3\)) bounded open set in \(\mathbb{R}^d\) and \(n(\cdot)\) is the unit outward normal vector field on \(\partial \mathcal{O}\), it is possible to use an extension \(\phi\) of the function \(d(\cdot, \partial \mathcal{O})\) (defined on the restriction to \(\mathcal{O}\) of a neighborhood of \(\partial \mathcal{O}\)) satisfying \(\nabla \phi = -n\) on \(\partial \mathcal{O}\).

Let us also mention that if \(\mathcal{O}\) is a piecewise smooth bounded open set in \(\mathbb{R}^d\) with a "finite number of convex angles", then \(\mathcal{O}\) is admissible and (9) holds. In particular, if \(\mathcal{O}\) is convex, then we may take \(C_0 = 0\) and \(\Phi = 0\). Finally let us notice that if \(\mathcal{O}\) is admissible, (9) holds locally since (5) implies that, on \(\partial \mathcal{O} \cap B(x, 2R)\),

\[
\nabla \Phi(x) \cdot k \equiv -\alpha < 0, \forall k \in n(x), \quad \text{where } \Phi = \Phi_i = (x - x_i, a_i)
\]

(or more generally \(\Phi = \varphi(\Phi_i)\) and \(\varphi \in C^2(\mathbb{R})\), \(\varphi' > 0\)).

Let \(x_0 \in \mathcal{O}\), let \(\sigma_i\) and \(b_i\), \(i, j \in [1, d]\), be uniformly bounded real-valued functions on \(\mathbb{R}^d\) satisfying a uniform Lipschitz condition:

\[
\exists K > 0, \forall i, j \in [1, d],
\]

\[
|\sigma_{ij}(x) - \sigma_{ij}(y)| \leq K|x - y|, \quad |b_i(x) - b_i(y)| \leq K|x - y|.
\]

We are now going to study the existence and uniqueness of continuous \(F_t\)-semimartingales \((X_t)_{t \geq 0}\) satisfying:

there exists a continuous bounded variation process \(k_t\) with values in \(\mathbb{R}^d\) such that \(X_t \in \mathcal{O}\) for all \(t \geq 0\) a.s.
Remark 3.2. It is easy to see that (11) is equivalent to saying that, for a.e. \( \omega \in \Omega \), \( X_\omega(\omega) \) is the first component of the solution \((X_\omega, k_\omega)\) of the Skorokhod problem \( (x_0 + \int_0^t \sigma(X_s) \, dB_s + \int_0^t b(X_s) \, ds - k, c, n(\cdot)) \).

We can now state

**Theorem 3.1.** Assume (9), (10), hold; then there exists a unique \( F_t \)-semi-martingale \((X_t)_{t \geq 0}\) satisfying (11).

**Proof:** Define \( H \) to be the Frechet space of continuous adapted processes \( X \) satisfying:

\[
\forall t > 0, \quad E[ \sup_{0 \leq s \leq t} |X_s|^4] < \infty,
\]

equipped with the semi-norms

\[
\|X\|_t = E[ \sup_{0 \leq s \leq t} |X_s|^4]^{1/4},
\]

for \( t \geq 0 \).

Define \( F(\cdot) \) as a map of \( X \in H \) onto the solution \((Y_t)_{t \geq 0} \in H\) of

\[
Y_t = x_0 + \int_0^t \sigma(X_s) \, dB_s + \int_0^t b(X_s) \, ds - k_t,
\]

\[
|k_t| = \int_0^t 1_{\{Y_s \in \sigma\}} \, d|k_s|,
\]

\[
k_t = \int_0^t \xi_s \, d|k_s| \quad \text{with} \quad \xi_s \in n(Y_s).
\]

We are going to prove

**Lemma 3.1.**

\[
\exists C > 0, \forall T \geq 0, \forall X, X' \in H, \quad \|F(X) - F(X')\|^4_T \leq C \int_0^T \|X - X'\|^4 \, ds.
\]
Proof: Let $X, X' \in H$; set $Y = F(X)$ and $Y' = F(X')$. Using (12) and Ito's formula we have

$$
\phi(Y_t) = \phi(x_0) + \int_0^t \nabla \phi(Y_s) \sigma(X_s) \, dB_s + \int_0^t \nabla \phi(Y_s) b(X_s) \, ds
$$

$$
- \int_0^t \nabla \phi(Y_s) \xi_s \, d|k_s| + \frac{1}{2} \int_0^t \text{tr} \left[ \phi''(Y_s)(\sigma \sigma')(X_s) \right] \, ds,
$$

and the same expression for $Y', X', k', \xi'$.

This yields

$$
\exp \left\{ -\frac{1}{\alpha} (\phi(Y_s) + \phi(Y_s')) \right\} \times |Y_s - Y_s'|^2
$$

$$
= 2 \int_0^t \exp \left\{ -\frac{1}{\alpha} (\phi(Y_s) + \phi(Y_s')) \right\} \left[ (Y_s - Y_s') \cdot (\sigma(X_s) - \sigma(X_s')) \right] \, dB_s
$$

$$
+ (Y_s - Y_s') \cdot (b(X_s) - b(X_s')) \, ds - (Y_s - Y_s') \cdot \xi_s \, d|k_s|,
$$

$$
+ (Y_s - Y_s') \cdot \xi_s' \, d|k'|_{s'}
$$

$$
+ \int_0^t \exp \left\{ -\frac{1}{\alpha} (\phi(Y_s) + \phi(Y_s')) \right\} \text{tr} [(\sigma(X_s) - \sigma(X_s')) \cdot (\sigma(X_s) - \sigma(X_s'))'] \, ds
$$

$$
- \frac{1}{\alpha} \int_0^t \exp \left\{ -\frac{1}{\alpha} (\phi(Y_s) + \phi(Y_s')) \right\} |Y_s - Y_s'|^2
$$

$$
\cdot \left[ (\nabla \phi(Y_s) \sigma(X_s) + \nabla \phi(Y_s) \sigma(X_s')) \right] \, dB_s
$$

$$
+ \left( \frac{1}{2} \text{tr} \left[ \phi''(X_s) \sigma(X_s) \sigma'(X_s) + \phi''(X_s') \sigma(X_s') \sigma'(X_s') \right] 
\right.
$$

$$
+ \nabla \phi(Y_s) \cdot b(X_s) + \nabla \phi(Y_s') \cdot b(X_s') \, dS
$$

$$
- \nabla \phi(Y_s) \cdot \xi_s \, d|k_s| - \nabla \phi(Y_s') \cdot \xi_s' \, d|k'|_{s'}
$$

$$
+ \left( \frac{1}{\alpha} \right)^2 \int_0^t \exp \left\{ -\frac{1}{\alpha} (\phi(Y_s) + \phi(Y_s')) \right\} |Y_s - Y_s'|^2
$$

$$
\cdot \left[ \nabla \phi(Y_s) \sigma(X_s) + \nabla \phi(Y_s) \sigma(X_s') \right] \cdot [\sigma'(X_s) \nabla \phi(Y_s) + \sigma'(X_s') \nabla \phi(Y_s')] \, ds
$$

$$
- \frac{2}{\alpha} \int_0^t \exp \left\{ -\frac{1}{\alpha} (\phi(Y_s) + \phi(Y_s')) \right\} i(Y_s - Y_s') \times (\sigma(X_s) - \sigma(X_s'))
$$

$$
\cdot (\sigma'(X_s) \nabla \phi(Y_s) + \sigma'(X_s') \nabla \phi(Y_s')) \, ds.
$$
Now recall that $\phi \in C^2_b$ and
\[
\frac{1}{\alpha} (\nabla \phi(Y_s) \cdot \xi_s) |Y_s - Y_s'|^2 - (Y_s - Y_s') \cdot \xi_s \leq 0, \quad d|k|_s \text{ a.s.,}
\]
\[
\frac{1}{\alpha} (\nabla \phi(Y_s') \cdot \xi_s') |Y_s - Y_s'|^2 - (Y_s' - Y_s) \cdot \xi_s' \leq 0, \quad d|k'|_s \text{ a.s.}
\]

Using Doob's inequality we get
\[
E[\sup_{s \leq t} |Y_s - Y_s'|^4] \leq C_1 \int_0^t E[|X_s - X_s'|^4] \, ds
\]
\[+ C_2 \int_0^t E[|Y_s - Y_s'|^4] \, ds + C_3 \int_0^t E[|Y_s - Y_s'|^2 |X_s - X_s'|^2] \, ds,
\]
where the constants $C_i, \ i = 1, 2, 3,$ depend only on $C_0, \alpha, \sigma, \sigma, b, \phi.$

Since $|Y_s - Y_s'| |X_s - X_s'| \leq \frac{1}{2} |Y_s - Y_s'|^2 + \frac{1}{2} |X_s - X_s'|^2,$ using Gronwall's lemma, we obtain
\[
E[\sup_{s \leq t} |Y_s - Y_s'|^4] \leq C_4 \int_0^t E[\sup_{u \leq s} |X_u - X_u'|^4] \, du
\]
for a constant $C_4$ depending only on $C_0, \alpha, \sigma, \sigma, b, \phi.$ This proves (13).

From (13) we know, using the usual iteration technique, that a fixed point for $F$ in $H$ exists, which gives the existence of a solution of (11). We also know the uniqueness of this fixed point in $H,$ and since $\mathcal{G}$ is bounded, every solution of (11) belongs to $H.$ Thus we have proved existence and uniqueness for the solution of (11).

**Remark 3.3.** Suppose we are given a continuous $F_t$-local martingale $(M_t)_{t \geq 0},$ a continuous bounded variation adapted process $(K_t)_{t \geq 0},$ $x_0 \in \mathcal{G},$ and real-valued functions $\sigma(x, t, \omega), b(x, t, \omega)$ on $\mathbb{R}^d \times \mathbb{R} \times \Omega$ satisfying a uniform Lipschitz condition in the $x$-variable which are progressively measurable. We then can use a localization technique with stopping times (see for instance [7] or Dooleans-Dade and Meyer [3]) and the previous proof to obtain the existence and uniqueness of an $F_t$-semimartingale $(X_t)_{t \geq 0}$ satisfying:

there exists a bounded variation $\mathbb{R}^d$-valued process $k_1$ such that $X_t \in \mathcal{G}$ for $t \geq 0$ a.s.,
\[
X_t = x_0 + \int_0^t \sigma(X_s, s, \omega) \, dM_s + \int_0^t b(X_s, s, \omega) \, dK_s - k_t,
\]
\[
|k|_t = \int_0^t 1_{(X_s \in \mathcal{G})} \, d|k|_s, \quad k_t = \int_0^t \xi_s \, d|k|_s, \quad \xi_s \in n(X_s).
\]
We are now going to obtain the existence of weak solutions to (11) in the case \( \sigma_{ij}(\cdot), b_i(\cdot) \) are bounded continuous functions. More precisely, we have

**Theorem 3.2.** We assume that \( \mathcal{O} \) is an admissible open set. Suppose \( \sigma_{ij}, b_i \) are bounded continuous functions on \( \mathbb{R}^d \), and \( x_0 \in \mathcal{O} \); then we can find, on some probability space \( (\Omega, F, F_t, P) \), a \( d \)-dimensional \( F_t \)-Brownian motion \( B \) and a continuous \( F_t \)-semimartingale \( X \) satisfying (11).

**Proof:** Consider a sequence \( \sigma_{ij}^n, b_i^n \) of uniformly bounded and uniformly Lipschitz functions converging uniformly to \( \sigma_{ij}, b_i \) on \( \mathcal{O} \). Define, in view of Theorem 3.1, \( (X_{i}^{n}, k_{i}^{n}) \) to be the solution of (11) with \( \sigma^n \) and \( b^n \). Since the mapping

\[
 w \in C(\mathbb{R}_+, \mathbb{R}^d) \rightarrow (X_{i}^{n}, k_{i}^{n}) \in C(\mathbb{R}_+, \mathbb{R}^d) \times C(\mathbb{R}_+, \mathbb{R}^d)
\]

is continuous, the mapping (see Lemma 1.2) \( w \in C(\mathbb{R}_+, \mathbb{R}^d) \rightarrow |k_i| \in M_+(\mathbb{R}_+) \) maps compact sets into compact sets, and since the laws of the processes

\[
 x_0 + \int_0^t \sigma^n(x^n_s) \, dB_s + \int_0^t b^n(X^n_s) \, ds
\]

are clearly tight, we can choose a subsequence \( n_p \) such that the law \( P^{n_p} \) of \( (X_{i}^{n_p}, k_{i}^{n_p}, d|k_i^n|_i) \) converges weakly to \( P \), and \( P(k \in BV_\mathbb{C}) = 1 \).

It is then routine (see [10]) to show that we can enlarge (if necessary) the probability space \( (C \times BV_\mathbb{C} \times M_+, P) \) endowed with its natural filtration and find a Brownian motion \( \hat{B} \), on the enlarged space such that

\[
 X_t = x_0 + \int_0^t \sigma(X_s) \, d\hat{B}_s + \int_0^t b(X_s) \, ds - k_t,
\]

\[
 |k_i| = \int_0^t 1_{(X_s, \sigma \in \sigma)} |d|k_i^n|_i|, \quad k_t = \int_0^t \xi_s \, |d|k_i^n|_i|, \quad \xi_s \in n(X_s),
\]

which is the result we claimed.

**Remark 3.4.** Using the Markov selection procedure (cf. N. V. Krylov [11], D. W. Stroock and S. R. S. Varadhan [20]), we can build a strong Markov process \( (\Omega, F, F_t, P) \) on \( \mathcal{O} \) corresponding to \( \sigma, b \) under the assumptions of Theorem 3.2. We could also treat the case of a nondegenerate Borel measurable bounded diffusion matrix \( \sigma \).

### 4. Oblique Reflecting Boundary Conditions

We consider now the case of oblique reflecting boundary conditions and we shall first describe the situation when \( \mathcal{O} \) and the direction \( \gamma \) of reflection are

\footnote{\( BV_\mathbb{C} \) denotes the space of bounded variation, continuous \( \mathbb{R}^d \)-valued functions, with value 0 in \( \mathcal{O} \).}
smooth: let \( \mathcal{O} \) be a bounded open smooth domain in \( \mathbb{R}^d \) and let \( \gamma \in C^2_0(\mathbb{R}^d) \) satisfy \( |\gamma(x)| = 1 \) for \( x \in \partial \mathcal{O} \) and

\[
\exists \nu > 0, \quad \forall x \in \partial \mathcal{O}, \quad (\gamma(x), n(x)) \geq \nu.
\]

Let \( w \in C([0, \infty[, \mathbb{R}^d) \) be such that \( w(0) \in \mathcal{O} \); we call a couple of functions \( (x, k) \) satisfying

\[
\begin{align*}
x \in C([0, \infty[, \mathcal{O}), \\
k \in C([0, \infty[, \mathbb{R}^d), k \in BV(0, T) \quad \text{for all} \quad T < \infty
\end{align*}
\]

\[
(3') \quad |k|_s = \int_0^s 1_{(x, t) \in \mathcal{O}} d|k|_s, \quad k = \int_0^s \gamma(x) \, d|k|_s,
\]

\[
x + k = w, \quad \text{for} \quad t \geq 0,
\]

a solution of the Skorokhod problem \( (w, \mathcal{O}, \gamma) \).

Our first result is the following:

**Theorem 4.1.** Let \( \mathcal{O} \) be a bounded open smooth domain in \( \mathbb{R}^d \) and let \( \gamma \) satisfy (16). Then for each \( w \in C([0, \infty[, \mathbb{R}^d) \) with \( w(0) \in \mathcal{O} \) there exists at least one solution \( (x, k) \) of the Skorokhod problem \( (w, \mathcal{O}, \gamma) \). In addition if \( w \) remains in a compact set of \( C([0, T]; \mathbb{R}^d) \) (for any \( T < \infty \)), then the solutions \( (x, k) \) of \( (w, \mathcal{O}, \gamma) \) and \( |k| \) remain in a compact set of \( C([0, T]; \mathbb{R}^d) \) and \( k \) is bounded in \( BV(0, T) \).

Furthermore, if \( w \in BV(0, T) \) (for all \( T < \infty \)), the solution \( (x, k) \) of the Skorokhod problem \( (w, \mathcal{O}, \gamma) \) is unique, \( x \in BV(0, T) \) (for all \( T < \infty \)) and

\[
d|k|_s = 1_{(x, t) \in \mathcal{O}} (dw, n(x)) (n(x), \gamma(x))^{-1}.
\]

**Remark 4.1.** Below we shall give extensions of this result to the case of more general domains \( \mathcal{O} \) or vector fields \( \gamma \). Let us indicate that we could treat unbounded domains as well but we shall skip here and below such an extension.

**Remark 4.2.** Of course the main open question is the uniqueness of \( (x, k) \). The only cases we were able to treat are:

(i) There exist constants \( a_{ij} \) such that \( \sum_{i=1}^d a_{ij} \gamma_j = n_i \forall i, \det (a) > 0 \). Then the uniqueness follows from the results of Section 1.

(ii) If \( \mathcal{O} \) is a half-space, \( \mathcal{O} = \{ x \in \mathbb{R}^d : x_i > 0 \} \). We point out that in this case \( (x(t), k(t)) \) is the solution of the one-dimensional Skorokhod problem \( \{ x_i > 0 \}, w_i(t, n) \), where \( d\tilde{k} = +\gamma_i(x) \, d|k| \). Therefore, if \( (x(t), k(t)), (x'(t), k'(t)) \) are two solutions of \( (w, \mathcal{O}, \gamma) \), we have necessarily \( x = x', k = k' \) for \( t \geq 0 \). This
yields \( d|k_i| = \gamma_1(x_i') \gamma_1(x_i)^{-1} d|k|_s \), and we see that

\[
x_i - x_i' = \int_0^t \gamma(x_s') d|k'|_s - \gamma(x_s) d|k|_s,
\]

\[
|x_i - x_i'| \leq \int_0^t |\gamma(x_s') - \gamma(x_s)\gamma_1(x_s')\gamma_1(x_s)^{-1}| d|k'|_s,
\]

\[
\leq C \int_0^t |x_i - x_i'| d|k'|_s.
\]

We conclude using Gronwall's lemma.

Proof of Theorem 4.1: We shall only present the main new features of the proof since most of the details are analogous to arguments made in the preceding sections.

First of all we prove the existence for smooth \( w \), i.e., for \( w \in H^1(0, T; \mathbb{R}^d) \) (for all \( T < \infty \)), by a penalty argument similar to the one used in the proof of Theorem 2.1. Let \( q \in W^{1,\infty}(\mathbb{R}^d) \), \( q = 0 \) in \( \overline{\mathcal{D}} \), \( q = \text{dist}(x, \mathcal{D}) \) in a neighborhood of \( \overline{\mathcal{D}} \), \( q > 0 \) in \( \mathbb{R}^d - \overline{\mathcal{D}} \). We consider the \( x^\varepsilon \) solution of

\[
\dot{x}^\varepsilon + \frac{1}{\varepsilon} \gamma(x^\varepsilon) q(x^\varepsilon) = \dot{w}, \quad \text{for} \quad t \geq 0, \quad x_0 = w_0 = x \in \overline{\mathcal{D}}.
\]

Without loss of generality we may assume that

\[
q^2 \in C^1(\mathbb{R}^d), \quad (\nabla q(x), \gamma(x)) \geq \frac{1}{2} \nu \forall x \in \mathbb{R}^d - \overline{\mathcal{D}},
\]

and we find

\[
q^2(x^\varepsilon_t) + \frac{\nu}{\varepsilon} \int_0^t q^2(x^\varepsilon_s) \, ds \equiv C_T \left( \int_0^t q^2(x^\varepsilon_s) \, ds \right)^{1/2} \quad \text{if} \quad t \leq T.
\]

This shows that

\[
x^\varepsilon_t, k^\varepsilon_t = \int_0^t \frac{1}{\varepsilon} \gamma(x^\varepsilon_s) q(x^\varepsilon_s) \, ds
\]

(taking a subsequence if necessary) converge uniformly to \( x_t, k_t \in H^1(0, T; \mathbb{R}^d) \) (for all \( T < \infty \)) satisfying \( x_t + k_t = w_t \) for \( t \geq 0 \), \( x_t \in \overline{\mathcal{D}} \) for \( t \geq 0 \). The remainder of the argument is similar to the one given in the proof of Theorem 2.1. Relation (17) is proved exactly as \((8')\) was proved in that theorem.

To complete the proof of Theorem 4.1 we shall use the following lemma (introduced in [12]):
Lemma 4.1. Let \( \gamma \in C_b(\mathbb{R}^d; \mathbb{R}^d) \) satisfy (16); then there exists a \( d \times d \) symmetric matrix-valued function \((a_{ij}(x))\) satisfying

\[
(a_{ij}(x)) \equiv \nu' I_d \quad \text{for some} \quad \nu' > 0, \quad a_{ii} \in C_b(\mathbb{R}^d);
\]

\[
\sum_{i=1}^{d} a_{ij}(x) \gamma_j(x) = n_i(x) \quad \text{for} \quad 1 \leq i \leq d, \forall x \in \partial \mathcal{O}.
\]

In particular, there exists \( C_0 \geq 0 \) such that

\[
C_0 |x - x'|^2 + \sum_{i=1}^{d} a_{ij}(x)(x_i - x'_i)(\gamma_i(x)) \geq 0
\]

for all \( x \in \partial \mathcal{O}, x' \in \partial \mathcal{O} \).

In addition, if \( \gamma \in C_b^1 \) (respectively \( W^{1,\infty} \)), then \((a_{ij}) \in C_b^1\) (respectively \( W^{1,\infty} \)), and if \( \gamma \in C_b^2 \) (respectively \( W^{2,\infty} \)), then \((a_{ij}) \in C_b^2\) (respectively \( W^{2,\infty} \)).

Proof of Lemma 4.1: Clearly, (19) is a consequence of (18) and of the properties of \( \mathcal{O} \) proved in Section 1.

To prove (18), consider \( p(x, z) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) and \( \lambda(x, z) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) defined by

\[
\forall (x, z) \in \mathbb{R}^d \times \mathbb{R}^d \left\{ \begin{array}{l}
z = p(x, z) + \lambda(x, z) \gamma(x), \\
p(x, z) \in n(x)^{\perp},
\end{array} \right.
\]

\( (p, z) \) is the projection on \( n(x)^{\perp} \) parallel to \( \gamma(x) \); we can see easily that \( \lambda(x, z) = z \cdot n(x)/\gamma(x) \cdot n(x) \) and it is clear that

\[
a_{ij}(x) = n(x) \cdot \gamma(x)[p(x, e_i) \cdot p(x, e_j) + \lambda(x, e_i) \cdot \lambda(x, e_j)]
\]

satisfies (18) and the regularity conditions claimed.

We prove now the uniqueness if \( w \) lies in \( BV(0, T) \) for all \( t < \infty \). Let \((x, k), (x', k)\) be two solutions of the Skorokhod problem \((w, \mathcal{O}, \gamma)\); to this end we consider \( \phi(t) = (a_{ij}(x_t) + a_{ij}(x'_t))(x_i(t) - x'_i(t))(x_j(t) - x'_j(t)) \). We then have, using the Lipschitz character of \( a(x) \),

\[
\phi(t) = \int_{0}^{t} d\phi(s)
\]

\[
\leq -2 \int_{0}^{t} (a_{ij}(x_t) + a_{ij}(x'_t))(x_i(t) - x'_i(t))(\gamma_j(x_t) dA_i - \gamma_j(x'_t) dA'_i)
\]

\[
+ C_l \left( |x_t - x'_t|^2 d|x_t| + d|x'_t| \right),
\]

where \( A_i = |k|, A'_i = |k'|. \)
Using (17), (18) and (19), we deduce that
\[ \phi(t) \leq C_2 \int_0^t |x_s - x'_s|^2 \, dw_s. \]
Using (18) again, this yields
\[ |x_t - x'_t|^2 \leq C_3 \int_0^t |x_s - x'_s|^2 \, dw_s, \]
and we conclude that \( x_t = x'_t \) for \( t \geq 0 \).

The only remaining point to prove is the existence of a priori estimates on \( |k|_T \) and on the modulus of continuity of \( x \) over \([0,T]\), where \( T < \infty \) is arbitrary and \((x,k)\) is a solution of \((w,\sigma,\gamma)\).

To this end we remark that the analogue of (5) holds:
\[ \exists n \geq 1, \ \exists \alpha > 0, \ \exists R > 0, \ \exists a_1, \ldots, a_n \in \mathbb{R}^d, \ |a_i| = 1 \ \forall i, \]
\[ \forall \alpha \ (x_1, \ldots, x_n) \in \partial Q \text{ such that} \]
\[ \partial Q \subset \bigcup_{i=1}^n B(x_i, R), \forall i, \forall x \in \partial Q \cap B(x_i, 2R) \ (\gamma(x), a_i) \geq \alpha > 0. \]

We introduce \( Q, T \) exactly as in the proof of Lemma 1.2 and we obtain in a similar way the following bound:
\[ |k_s - k|_s \leq \frac{1}{\alpha} \{ |x_t - x_s| + \mu(t-s) \} \text{ if } T_m \leq s \leq t \leq T \land T_{m+1}, \]
where \( T > 0 \) is fixed and \( \mu \) is the modulus of continuity of \( w \) over \([0,T]\). If we consider \( s \) such that \( T_m \leq s \leq T \land T_{m+1} \), we set
\[ \psi(t) = a_{ij}(x_i)(x_i(t) - x_i(s))(x_j(t) - x_j(s)) \text{ for } s \leq t \leq T \land T_{m+1}. \]

We then compute as in Lemma 1.1. We set \( A_s = |k_s|_s \),
\[ \psi(t) = \int_s^t d\psi(\lambda) = 2 \int_s^t a_{ij}(x_i)(x_i(\lambda) - x_i(s))(dw_i(\lambda) - \gamma_i(x_\lambda)) \, dA_\lambda \]
\[ = 2 \int_s^t a_{ij}(x_i)(w_i(\lambda) - w_i(s) - (k_i(\lambda) - k_i(s))) \, dw_i(\lambda) \]
\[ - 2 \int_s^t a_{ij}(x_\lambda)(x_\lambda(\lambda) - x_i(s)) \gamma_i(x_\lambda) \, dA_\lambda \]
\[ + 2 \int_s^t \{a_{ij}(x_\lambda) - a_{ij}(x_i)(x_i(\lambda) - x_i(s)) \gamma_i(x_\lambda) \, dA_\lambda. \]
Integrating by parts in the first term, using (18) in the second term, and denoting by \( \varphi \) the modulus of continuity of \( a_{ij} \) and by \( \mu_x \) the modulus of continuity of \( x \) on \([T_m, T \land T_{m+1}]\), we finally find

\[
|x_t - x_s|^2 \leq C\{\mu(t-s)^2 + \mu(t-s)(A_t - A_s) + \varphi(\mu_x(t-s))\mu_x(t-s)(A_t - A_s)\}
+ C' \int_s^t |x_t - x_s|^2 \, dA,
\]

where here and below \( C \) and \( C' \) denote various constants depending only upon \( \mathcal{O}, \gamma, (a_{ij}) \). By Gronwall's lemma, this yields, for \( s \leq t \leq T \land T_{m+1} \),

\[
|x_t - x_s|^2 \leq \exp \left\{ C' (t-s)^2 \varphi(\mu(t-s))\mu_x(t-s^2) \right\}.
\]

Since \( \varphi(\lambda) \to 0 \) as \( \lambda \to 0^+ \), this implies that we have obtained a modulus of continuity of \( x \) (and thus of \( k \), \(|k| \) by (21)) on the interval \([T_m, T \land T_{m+1}]\) which depends only on the modulus of continuity of \( w \), and on \( \mathcal{O}, \gamma, (a_{ij}) \). This then enables us to conclude, exactly as in Lemma 1.2, that Lemma 1.1 holds.

Looking carefully at the above proof, we see that we may extend the above results to more general domains \( \mathcal{O} \) and vector fields \( \gamma \): let \( \gamma \) be a multi-valued vector field on \( \mathbb{R}^d \) satisfying

\[
\forall x \in \partial \mathcal{O}, \forall \xi \in \gamma(x), \ |\xi| = 1.
\]

We shall say that \( \mathcal{O} \) is \textit{admissible} if the following conditions hold:

1. \( \exists n \geq 1, \ \exists \alpha > 0, \ \exists R > 0; \ \exists a_1, \cdots, a_n \in \mathbb{R}^d, \ |a_i| = 1 \ \forall i, \ \exists x_1, \cdots, x_n \in \partial \mathcal{O} \) such that

   \[
   (20') \quad \partial \mathcal{O} \subset \bigcup_{i=1}^n B(x_i, R), \ \forall i, \ \forall x \in \partial \mathcal{O} \cap B(x_i, 2R), \ \forall \xi \in \gamma(x),
   \]

   \[
   (\xi, a_i) \equiv \alpha > 0.
   \]

2. There exist smooth open bounded sets \( \mathcal{O}_m \) in \( \mathbb{R}^d \) and \( a_m(x) \equiv a_{ij}(x) \) in \( C_b(\mathbb{R}^d) \), \( 1 \leq i, j \leq d \), such that

   \[\text{(i) if } x_m \in \bar{\mathcal{O}}_m, x_m \to x, \text{ then } x \in \bar{\mathcal{O}};\]

   \[\text{(ii) if } K \text{ is a compact set of } \mathcal{O}, \text{ then } K \subset \mathcal{O}_m \text{ for } m \text{ large enough};\]

   \[\text{(iii) } \exists \nu' > 0, \ \forall x \in \mathbb{R}^d \quad (a_{ij}(x)) \equiv \nu' I_d;\]
(iv) there exists $C_0 \geq 0$ such that, for all $m \geq 1$,

$$C_0|\mathbf{x} - \mathbf{x}'|^2 + (n_m(x), x - x') \geq 0, \forall \mathbf{x} \in \partial \Sigma_m, \forall \mathbf{x}' \in \partial \Sigma_m,$$

$$C_0|\mathbf{x} - \mathbf{x}'|^2 + \sum_{i,j=1}^d a_{ij}(x)(x_i - x'_i)\xi_j \geq 0, \forall \mathbf{x} \in \partial \Sigma,$$

$$\forall \xi \in \partial \gamma(x);$$

(v) let $\xi \in \mathbb{R}^d, x \in \partial \Sigma$; if for some $C_0 \geq 0$,

$$C_0|\mathbf{x} - \mathbf{x}'|^2 + \sum_{i,j=1}^d a_{ij}(x)(x_i - x'_i)\xi_j \geq 0, \forall \mathbf{x}' \in \partial \Sigma,$$

then $\xi \in \partial \gamma(x)$ for some $\theta \not\equiv 0$.

We denote by $n_m$ the unit outward normal to $\partial \Sigma_m$.

**Remark 4.3.** If we set $n(x) = \{a(x)^{-1} \cdot \xi/\xi \in \gamma(x)\}$ for $x \in \partial \Sigma$, then we see that the above conditions are essentially equivalent to the notion of an admissible open set (for $n$) given in Section 2. The only difference lies in (20'), where $\xi \in \gamma(x)$ replaces $\xi \in n(x)$. For example, (iv)-(v) are equivalent to (1) and to the fact that $\partial \Sigma_m$ satisfies (1) uniformly in $m$. Therefore, if $\Sigma$ is a piecewise smooth open set with convex angles and if $a_{ij} = a_{ji} \in C_b(\mathbb{R}^d)$ satisfies (iii), then (22) holds for $\gamma = a \cdot n$. In particular, if $\Sigma$ is convex and $a_{ij}(x) = a_{ji}(x) \in C_b(\mathbb{R}^d)$, then (22) holds for $\gamma = a \cdot n$.

**Theorem 4.2.** Let $\Sigma$ be a bounded admissible open set in $\mathbb{R}^d$. Then for each $w \in C([0, \infty]; \mathbb{R}^d)$ with $w(0) \in \partial \Sigma$ there exists at least one solution $(x, k)$ of the Skorokhod problem $(w, \Sigma, \gamma)$. In addition, if $w$ remains in a compact set of $C([0, T]; \mathbb{R}^d)$ (for any $T < \infty$), then the solutions $(x, k)$ of $(w, \Sigma, \gamma)$ and $|k|$ remain in a compact set of $C([0, T]; \mathbb{R}^d)$.

Furthermore, if $w \in BV(0, T)$ (for all $T < \infty$), then $x \in BV(0, T)$ (for all $T < \infty$) and we have $|x|_t \leq C_0 |w|_t$, where $C_0$ depends only on $\Sigma, \gamma, a$.

Finally, if $w \in BV(0, T)$ (for all $T < \infty$) and if $a_{ij}$ is Lipschitz on $\partial \Sigma$ (for $1 \leq i, j \leq d$), then the solution $(x, k)$ of $(w, \Sigma, \gamma)$ is unique.

**Remark 4.4.** Even in the case of a smooth open set $\Sigma$, one has to make assumptions concerning the regularity of the vector field $\gamma(\cdot)$ in order to be able to obtain uniqueness results in the Skorokhod problem. For instance, take $\Sigma = \{(x, y) \in \mathbb{R}^2/x > 0\}$,

$$\gamma((0, y)) = a = 2^{-1/2}(-1, -1), \quad y < 0,$$

$$\gamma((0, y)) = b = 2^{-1/2}(-1, 1), \quad y > 0,$$

$$\gamma((0, 0)) = \{(1 + \alpha^2)^{-1/2}(-1, \alpha), \alpha \in [-1, +1]\},$$

and $w(t) = (-t, 0) \in BV(0, T), \forall T > 0$. 
The problem \((w, \mathcal{O}, \gamma)\) admits the two solutions \(x^1(t) = 0, k^1(t) = (-t, 0)\) and \(x^2(t) = (0, -t), k^2(t) = (-t, t)\).

**Remark 4.5.** Along the same lines, take \(\mathcal{O}\) to be the positive quadrant in \(\mathbb{R}^2, \theta \in ]0, \frac{\pi}{2}[\), \(\gamma((x, 0)) = a = -(\cos \theta, \sin \theta), x > 0, \gamma((0, y)) = b = -(\sin \theta, \cos \theta), \gamma((0, 0)) = \{-(\cos \beta, \sin \beta), \beta \in [\theta, \frac{\pi}{2} - \theta]\}, w(t) = t \cdot b\).

Then \(x^1(t) = 0, k^1(t) = t \cdot b\) and \(x^2(t) = t\mu(1, 0), k^2(t) = t\lambda a\) (with \(b = \lambda a + \mu e_t\)) are two solutions of the problem \((w, \mathcal{O}, \gamma)\).

In this case, the assumptions (22)(i), (ii), (iv), (v) are satisfied with a constant symmetric matrix

\[
\begin{pmatrix}
\sin \theta & \cos \theta \\
\cos \theta & \sin \theta
\end{pmatrix},
\]

however, we do not have uniqueness for data \(w \in \text{BV}(0, T)\) (because \(a_{ij}\) is not positive definite).

Let us notice moreover that if we would have chosen \(\hat{\gamma} = \gamma \circ \sigma, \sigma\) being the symmetry with respect to the first bisectrix instead of \(\gamma\), we would have been in the scope of Theorem 4.2.

Let us finally remark that such counterexamples can be adapted to yield a counterexample in the case of \(\gamma(x) = n(x)\) and a nonadmissible domain \(\mathcal{O}\); consider for example an open set in \(\mathbb{R}^2\) smooth except for an interior angle. Then one can easily construct a nonuniqueness example with such geometries by arguments which are similar to those given above.

Of course, by solution of \((w, \mathcal{O}, \gamma)\) we mean a couple \((x, k)\) satisfying (2), (3') with \(\gamma(x_i)\) replaced by \(\xi \in \gamma(x_i)\), and (4).

We now turn to stochastic differential equations with reflection along \(\gamma\). In order to keep the new ideas clear we shall first consider the case when \(\mathcal{O}, \gamma\) are smooth and \(\gamma\) satisfies (16) and then extend the results to more general situations without proofs. Let us first consider the problem we want to solve; we keep the notations of Section 3 concerning the probability space \((\Omega, F, F, \mathbb{P}, \mathbb{F})\) and the coefficients \(\sigma_{ij}, b_i\) satisfying (10). We want to study the existence and uniqueness of continuous \(F_i\)-semimartingales \((X_t)_{t \geq 0}\) satisfying:

there exists a continuous bounded variation process \(k\), with values in \(\mathbb{R}^d\) such that \(X_t \in \mathcal{D}\) for \(t \geq 0\) a.s.,

\[
X_t = x_0 + \int_0^t \sigma(X_s) \ dW_s + \int_0^t b(X_s) \ ds - k_t \quad \text{for} \quad t \geq 0,
\]

\[
|k_t| = \int_0^t 1_{X_s \in \mathcal{O}} \ d|k|_s, \quad k_t = \int_0^t \gamma(X_s) \ d|k|_s \quad \text{for} \quad t \geq 0.
\]

Now if we want to mimick the proof given in Section 3 we immediately see that, since we do not have the uniqueness of the solution of the Skorokhod problem
for general continuous data, it is not clear how to construct for a given continuous $F_t$-adapted process $Z_t$ (with values in $\mathcal{O}$) continuous $F_t$-adapted processes $(X_t, k_t)$ such that

$$X_t = Y_t - k_t, \quad X_t \in \mathcal{O} \quad \text{for} \quad t \geq 0 \ \text{a.s.},$$

(24)

$k_t$ has bounded variations, $|k_t| = \int_0^t 1_{(X_s \in \partial \mathcal{O})} \, d|k_s|$, $k_t = \int_0^t \gamma(X_s) \, d|k_s|$ for $t \geq 0$,

where

$$Y_t = x_0 + \int_0^t \sigma(Z_s) \, dB_s + \int_0^t b(Z_s) \, ds.$$

However when the continuous data $w$ is given by the trajectories of a “nice" semimartingale $Y$, the following result (which will be improved later on) shows that it is possible to improve Theorem 4.1 slightly:

**Proposition 4.1.** Let $Y_t$ be a continuous $F_t$-adapted semimartingale such that $Y_0 = x_0 \in \partial \mathcal{O}$ a.s.; $Y_t = M_t + K_t$ and $M_t$ is a continuous $F_t$-martingale such that $d(M_t) \leq Ct$, $K_t$ is a continuous bounded variation $F_t$-adapted process such that $d|K_t| \leq Ct$ for some constant $C \geq 0$. Let $\mathcal{O}$ be a bounded smooth open set in $\mathbb{R}^d$ and let $\gamma \in C^2_b(\mathbb{R}^d)$ satisfy (16). Then there exists a unique couple of $F_t$-adapted processes $(X_t, k_t)$ satisfying (24).

Proof: Let us first prove the uniqueness part; recall that, in view of Lemma 4.1, $a_j(x) \in C^2_b(\mathbb{R}^d)$. In addition, in view of (16) and Remark 3.1, there exists $\Phi \in C^2_b(\mathbb{R}^d)$ satisfying

$$\exists \alpha > 0, \forall x \in \partial \mathcal{O}, \quad \frac{\partial \Phi}{\partial \gamma}(x) \leq -\alpha < 0.$$  

Then if $(X_t, k_t), (X'_t, k'_t)$ are two solutions of (24), denoting by $\lambda$ a constant greater than 0 to be determined, we consider the quantity

$$\phi(t) = \exp \{-\lambda (\Phi(X_t) + \Phi(X'_t))\} \{a_j(X_t) + \{a_j(X'_t)\} \\
\cdot (X_t(t) - X'_t(t)) \cdot (X_j(t) - X'_j(t)) \}.$$

Using (24) and Itô's formula, we obtain, via tedious computations quite similar to those performed in the proof of Theorem 3.1,

$$E[\phi(t)] \leq C_1 E \int_0^t |X_s - X'_s|^2 \exp \{-\lambda (\Phi(X_s) + \Phi(X'_s))\} \, ds$$

$$+ C_2 E \int_0^t |X_s - X'_s|^2 \exp \{-\lambda (\Phi(X_s) + \Phi(X'_s))\} \, (d|k_s| + d|k'_s|)$$
Next, to prove the existence of $(X_t, k_t)$ we approximate $M_t$ by a sequence $M^n_t$ of continuous bounded variation $F_t$-adapted processes such that $M^n_t$ converges uniformly on bounded intervals to $M_t$, a.s., the law of $M^n_t$ converges weakly to the one of $M_t$. We denote $Y^t := M^n_t + k^n_t$. In view of Theorem 4.1, there exists a unique continuous bounded variation process $(X^n_t, k^n_t)$ satisfying (24) with $Y^n_t$ replaced by $Y^n_t$, and $X^n_t, k^n_t$ are $F_t$-adapted. Moreover, using again Theorem 4.1, we see that the joint law $P^n$ of $(X^n_t, Y^n_t, k^n_t, |k^n_t|)$ is relatively compact and, extracting a subsequence if necessary, converges weakly to a probability measure $P$ on $C(R^+, R^d) \times C(W^+, R^d) \times \text{BV}(R^+ \times W^+)$; the existence in law is proved. But in view of the pathwise uniqueness proved above, a standard argument (due to T. Yamada and S. Watanabe [23]) enables us to conclude, proving the existence of a strong solution $(X_t, k_t)$ of (24).

Actually, using again the pathwise uniqueness and considering the joint law of $(X^n_t, X^n_t, Y^n_t, k^n_t, |k^n_t|)$, we see that $X^n_t$ converges uniformly to $X_t$ on bounded intervals of $R^+$ almost surely.

We may now state our main result concerning stochastic differential equations with reflection along $\gamma$.

**Theorem 4.3.** Let $\mathcal{O}$ be a bounded smooth open set in $R^d$, let $\gamma \in C^2(\mathcal{O})$ satisfy (16) and let (10) hold. Then there exists a unique $F_t$-semimartingale $(X_t)_{t \geq 0}$ satisfying (23).

**Proof:** We may now copy the proof of Theorem 3.1 and define $F, H$ as we did before. Then for $X, X' \in \mathcal{H}$, we set $Y = F(X), Y' = F(X')$. With the above choice of $\lambda$ we find easily that

$$E \left[ \sup_{0 \leq s \leq t} |Y_s - Y'_s|^4 \right] \leq CE \left( \int_0^t |X_s - X'_s|^4 + |Y_s - Y'_s|^4 \, dx \right)$$

which easily concludes our proof.

$$-\alpha \nu' E \int_0^t |X_s - X'_s|^2 \exp \{-\lambda (\Phi(X_s) + \Phi(X'_s))\} \, ds$$

$$-2E \int_0^t \exp \{-\lambda (\Phi(X_s) + \Phi(X'_s))\} \{a_{ij}(X_s) + a_{ij}(X'_s)\}$$

$$\cdot (X_i(s) - X'_i(s))(\gamma(X_s) \, ds - \gamma(X'_s) \, ds),$$

where $C_2$ does not depend on $\lambda$ and where we used (25) and the properties of $a_{ij}(x)$. Then using (19) and choosing $\lambda = (1/\alpha \nu')(C_2 + 2C_0)$, we finally deduce that

$$E[\phi(t)] \leq C_1 E \int_0^t |X_s - X'_s|^2 \exp \{-\lambda (\Phi(X_s) + \Phi(X'_s))\} \, ds,$$

or

$$E[|X_t - X'_t|^2] \leq C_1 E \int_0^t |X_s - X'_s|^2 \, ds,$$

and we conclude easily that the pathwise uniqueness is proved.
This result, with the same proof, can be extended to the following situation:

**Theorem 4.4.** Let \( \mathcal{O} \) be a bounded admissible open set in \( \mathbb{R}^d \).

(i) We assume that there exists \( \Phi \in C^2_b(\mathbb{R}^d) \) such that

\[
(25') \quad \exists \alpha > 0, \quad \forall x \in \partial \mathcal{O}, \quad \forall \xi \in \gamma(x), \quad \frac{\partial \Phi}{\partial \xi}(x) \equiv - \alpha C_0,
\]

where \( C_0 \) is given in (22). If (10) holds and if \( a_{ij} \in W^{2,\infty}(\mathbb{R}^d) \) (for \( 1 \leq i, j \leq d \)), then there exists a unique \( F_t \)-semimartingale \( (X_t)_{t \geq 0} \) satisfying (23).

(ii) If \( \sigma_{ij}, b_i \in C_b(\mathbb{R}^d) \) (for \( 1 \leq i, j \leq d \)), then we can find a probability space \((\Omega, F, F, P)\) and an \( F_t \)-Brownian motion \( B_t \) such that

\[
(23) \quad \text{has at least one solution.}
\]

**Remark 4.6.** Exactly as in the preceding sections it is possible to extend the above results in many directions. Let us just indicate that the above result (part (i)) still holds if we consider a continuous \( F_t \)-local martingale \((M_t)_{t \geq 0}\), a continuous bounded variation process \( K_t \), \( x_0 \in \partial \mathcal{O} \), and functions \( \sigma_{ij}(x, t, \omega), b_i(x, t, \omega) \) satisfying Lipschitz conditions which are uniform in \((t, \omega)\) and progressively measurable for fixed \( x \). In this case, if \( \mathcal{O} \) is admissible, \((25')\) holds and \( a_{ij} \in W^{2,\infty}(\mathbb{R}^d) \), there exists a unique continuous \( F_t \)-semimartingale \( (X_t)_{t \geq 0} \) satisfying: there exists a continuous bounded variation adapted process \( k_t \) such that \( X_t \in \partial \mathcal{O} \) for \( t \geq 0 \) a.s.,

\[
X_t = x_0 + \int_0^t \sigma(X_s, s, \omega) \cdot dM_s + \int_0^t b(X_s, s, \omega) \cdot dK_s - k_t \quad \text{for} \quad t \geq 0,
\]

\[
|k|_t = \int_0^t 1_{(X_s \notin \partial \mathcal{O})} d|k|_s, \quad k_t = \int_0^t \xi_s d|k|_s, \quad \xi_s \in \gamma(X_s).
\]

**Remark 4.7.** It is possible (but we shall not exploit this fact here) to extend the results of Theorem 4.4 concerning S.D.E. with oblique reflection conditions, to more general unsmooth \((\mathcal{O}, \gamma(\cdot))\) by the use of diffeomorphisms. In this case, one should also notice that the Stratonovitch formulation is more adequate.

**Bibliography**


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