Abstract—A tracking control methodology via time-varying state feedback based on the backstepping technique is proposed for both a kinematic and simplified dynamic model of a two-degrees-of-freedom mobile robot. We first address the local tracking problem where initial tracking errors are sufficiently small. Then, under additional conditions on the desired velocities, we treat the global tracking problem where initial tracking errors are arbitrary. Simulation results are provided to validate and analyse our theoretical results.

1. Introduction
In recent years there has been enormous activity in the study of a class of mechanical control systems called nonholonomic systems. In particular, many kinematic models of physical systems (i.e. systems where velocities are treated as input signals) belong to this category, see the survey by Kolmanovskii and McClamroch (1995) and references cited therein. Controlling such nonholonomic systems turns out to be a nontrivial problem for a number of reasons. Even in the simplest case, which we shall study here, the kinematic model of a two-wheel mobile robot, the stabilization (or parking) problem at a given position requires a nontrivial controller (see e.g. Samson, 1991; Pomet, 1992; Murray et al., 1992; Bloch and Drakunov, 1994; Canudas de Wit et al., 1994; Micali and Samson, 1993; Fierro and Lewis, 1995). In all these papers, basically a local viewpoint in the stabilizing feedback design has been taken by using the Taylor linearization of the corresponding error model. A dynamic feedback linearization approach was proposed in Canudas de Wit et al. (1996, Chapter 8) that allows (local) posture tracking with exponential convergence for restricted mobility robots. Similar results were obtained in Fliess et al. (1995a, b) using time-reparametrization and motion-planning properties of differentially flat systems (systems that have the property that they are linearizable using a dynamic state feedback).

The purpose of the present paper is to use Lyapunov’s direct method for obtaining semiglobal and global results in the tracking problem for the mobile robot. In particular, under our proposed time-varying controllers, the two-degrees-of-freedom mobile robot can globally follow special paths such as straight lines and circles (see Remark 4 below). We do this for both the kinematic model and an ‘integrated’ simplified dynamic model of the mobile robot. In both cases, the design technique to obtain a suitable feedback control law is based upon the integrator backstepping procedure. The latter idea was firstly discovered by Koditschek (1987) and then developed in independent work in the context of nonlinear stabilization (see e.g., Byrnes and Isidori, 1989; Tsinias, 1989) and adaptive nonlinear control (see e.g. Krstic et al., 1995). Applications of the backstepping technique to the adaptive control of nonholonomic systems with unknown parameters and the global stabilization of multi-input chained-form nonholonomic systems were recently considered in Jiang and Pomet (1994, 1995) and Jiang (1996).

The theoretical results obtained in this paper are illustrated by means of simulations using the local (semiglobal) controller and the global controller under changing initial conditions.

The organization of the paper is as follows. We start with basic concepts, stability definitions and preliminary results in Section 2.1. Section 2.2 is devoted to modelling of the tracking configuration for a wheeled mobile robot and the statement of our problems. In Section 3, we first propose time-varying feedback control laws that solve the local tracking problem. Then, under extra (mild) conditions on the desired velocities, we solve the global tracking problem via time-varying state feedback. Along the way, a solution for the tracking problem for the mobile robot described by a simplified dynamic model. Several simulation results are presented in Section 5 to demonstrate our theoretical results. We close with some brief concluding remarks in Section 6.

2. Preliminaries and problem formulation
2.1. Preliminaries. For any bounded function \( \psi: (a, b) \rightarrow \mathbb{R} \), \( \| \psi \|_c \) means its \( L_\infty \) norm, i.e. \( \| \psi \|_c = \sup_{x \in (a, b)} | \psi(x) | \). \( a < x < b \). \( L_p(a, b) \) represents the set of measurable functions \( f \) from \( (a, b) \) to \( \mathbb{R} \) such that \( \| f(x) \|_p \) \( dx < \infty \). For any differentiable function \( \phi: (a, b) \rightarrow \mathbb{R} \), \( \phi(x) \) is the derivative of \( \phi \) at \( x \) (not to be confused with \( \phi(x(t)) \)), which is the time derivative of \( \phi(x(t)) \). We write \( \phi \in C^0 \) if \( \phi \) is a smooth function. For any function \( g: \mathbb{R} \rightarrow \mathbb{R} \), \( \lim_{t \rightarrow \infty} g(t) \) denotes the limit inferior of \( g(t) \) as \( t \rightarrow \infty \), i.e. \( \lim_{t \rightarrow \infty} g(t) = \sup_{t \geq 0} [ \inf_{0 \leq \tau \leq t} g(\tau) ] \).

Next, we recall some basic concepts about stability theory (see e.g. Khalil, 1992; Vidyasagar, 1993). A function \( \gamma: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is of class \( K \) if \( \gamma \) is strictly increasing, continuous and \( \gamma(0) = 0 \). It is of class \( K_\infty \), if furthermore \( \gamma(x) \rightarrow \infty \) as \( x \rightarrow \infty \). A function \( V: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be positive-definite if (i) it is continuous, (ii) \( V(t, 0) = 0 \) \( \forall t \geq 0 \) and (iii) there exists a function \( \gamma_1 \) of class \( K \) such that

\[
\gamma_1(|x|) \leq V(t, x) \quad V(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n.
\]
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V is decrecent if there exists a function $y$, of class $K$ such that

$$V(r, x) = 72(1X1) V(C) E R^+$$ (2)

V is radially unbounded if (1) holds for some continuous function, (not necessarily of class $K$) satisfying $y(r) \to \infty$ as $r \to \infty$.

Definition
(i) The solutions of the system (3) are uniformly bounded if for any $\alpha > 0$ and $t_0 \geq 0$, there exists a $\beta(\alpha) > 0$ such that

$$|x(t_0)| < \alpha, \quad t_0 \geq 0 \Rightarrow |x(t)| < \beta \quad \forall t \geq t_0.$$ (4)

(ii) The zero equilibrium (i.e. $x = 0$) of the system (3) is uniformly stable if, for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that

$$|x(t_0)| < \delta(\epsilon), \quad t_0 \geq 0 \Rightarrow |x(t)| < \epsilon \quad \forall t \geq t_0.$$ (5)

In the following, we give two technical lemmas that are of frequent use in proving our results. Recall that a function $\phi: (a, b) \to \mathbb{R}$ is uniformly continuous if for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that if $|x_1 - x_2| < \delta$, with $x_1, x_2 \in (a, b)$, then $|\phi(x_1) - \phi(x_2)| < \epsilon$.

Lemma 1. (Barbilat). If $\phi: \mathbb{R} \to \mathbb{R}$ is uniformly continuous and if the limit of the integral $\int_0^t \phi(r) \, dr$ exists as $t \to \infty$ and is finite then

$$\lim_{t \to \infty} \phi(t) = 0.$$ (6)


In the same vein, the following lemma can be proved.

Lemma 2. Consider a scalar system

$$\dot{x} = -\alpha + p(t),$$ (7)

where $\alpha > 0$ and $p(t)$ is a bounded and uniformly continuous function. If, for any initial time $t_0 \geq 0$ and any initial condition $x(t_0)$, the solution $x(t)$ is bounded and converges to 0 as $t \to \infty$ then

$$\lim_{t \to \infty} p(t) = 0.$$ (8)


2.2. Problem formulation. The problem we study deals with a wheeled mobile robot with two degrees of freedom. The robot’s dynamics is described by the following differential equations:

$$\dot{x} = v \cos \theta,$$

$$\dot{y} = v \sin \theta,$$

$$\dot{\theta} = \omega,$$ (9)

where $v$ is the linear velocity and $\omega$ is the angular velocity of the mobile robot; $(x, y)$ are the Cartesian coordinates of the center of mass of the vehicle, and $\theta$ is the angle between the heading direction and the $x$ axis (see Fig. 1). Systems like (9), or similar chained systems (see Murray and Sastry, 1993) and further nonholonomic systems have been the subject of much ongoing research; see Kolmanovsky and McClamroch (1995) and references therein.

The problem we consider here is the tracking problem; that is, we wish to find control laws for $v$ and $\omega$ such that the robot follows a reference robot, with position $p_r = (x_r, y_r, \theta_r)^T$ and inputs $v_r$ and $\omega_r$, (see Fig. 1). Denoting the error coordinates by (see Kanayama et al., 1990)

$$\begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r - x \\ y_r - y \\ \theta_r - \theta \end{bmatrix},$$ (10)

the error dynamics are (see Kanayama et al., 1990)

$$\begin{cases} x_e = \omega y_e - v + v_r \cos \theta_r \\ y_e = -\omega x_e + v \sin \theta_r \\ \theta_e = \omega_r - \omega. \end{cases}$$ (11)

In the following sections, we shall examine separately the following two problems.

Local tracking problem. Find appropriate velocity control laws $v$ and $\omega$ of the form

$$v = v(x_e, y_e, \theta_e, v_r, \omega_r, y_r), \quad \omega = \omega(x_e, y_e, \theta_e, v_r, \omega_r, y_r)$$ (12)

such that, for small initial tracking errors $(x_e(0), y_e(0), \theta_e(0))$, the closed-loop trajectories of (11) and (12) are uniformly bounded and converge to zero.

Global tracking problem. Find appropriate velocity control laws $v$ and $\omega$ of the form

$$v = v(x_e, y_e, \theta_e, v_r, \omega_r, y_r), \quad \omega = \omega(x_e, y_e, \theta_e, v_r, \omega_r, y_r)$$ (13)

such that, for arbitrary initial tracking errors $(x_e(0), y_e(0), \theta_e(0))$, the closed-loop trajectories of (11) and (13) are (globally) uniformly bounded and converge to zero.

2. Tracking of the kinematic model

3.1. The local tracking problem. Given any fixed $0 < \epsilon < \pi$, let us introduce a set of functions denoted by $\mathcal{F}^\epsilon$:

$$\mathcal{F}^\epsilon = \{ \varphi: \mathbb{R} \to (\pi - \epsilon, \pi - \epsilon): \varphi \in C^1, \varphi(0) = 0, z \varphi(z) > 0 \forall z \neq 0 \text{ and } \varphi' \text{ is bounded} \}.$$ (14)

Simple examples of functions in $\mathcal{F}^\epsilon$ include $\varphi(z) = \sigma(z)/(1 + z^2)$ for any $0 < \sigma < 2(\pi - \epsilon)$, and $\varphi(z) = \sigma \arctan(\sigma z)$ for all $0 < \sigma < 2(\pi - \epsilon)/\pi$ and $\sigma > 0$.

In the tracking error model (11), $y_e$ is not directly controlled, and to overcome this difficulty we use the idea of integrator backstepping.
Consider the candidate Lyapunov function

\[ V_t(t, x, y, \theta, v) = x^2 \sin \theta + y^2 + \frac{1}{\gamma} \theta_e. \]  

(17)

with \( \gamma > 0 \) and \( \theta_e \) given by (15). As can be directly verified, \( V_t \) is a positive-definite, decrescent and radially unbounded function.

In view of (15) and (16), taking the time derivative of \( V_t \) along solutions of (11) yields

\[ \dot{V}_t(t, x, y, \theta, v) = x(-x + v \cos \theta_e) + y(-\omega x + v \sin \theta_e + \frac{1}{\gamma} \theta_e), \]  

(18)

Noting that

\[ \sin[\varphi(x, y, v) + \theta_e] = \sin[\varphi(x, y, v)] + \frac{d}{dt} \int_0^t \cos[\varphi(x, y, v)] + s \theta_e \, ds, \]  

it follows that (18) implies

\[ \dot{V}_t(t, x, y, \theta, v) = x(-v + v \cos \theta_e) - y \sin \varphi(x, y, v) \]  

\[ + \frac{1}{\gamma} \theta_e \sin \theta_e + \frac{1}{\gamma} \theta_e \sin \theta_e + \frac{1}{\gamma} \theta_e, \]  

(19)

By choosing the tracking controller \( \omega \) and \( \theta_e \) as

\[ \omega = \left[ 1 + \varphi'(x, y, v) \right]^{-1} [\gamma y \sin \theta_e + \omega_e \sin \theta_e + \varphi'(x, y, v)(v^2 \sin \theta_e + \frac{1}{\gamma} \theta_e)], \]  

(20)

with \( c_1, c_2 > 0 \), we have

\[ \dot{V}_t(t, x, y, \theta, v) = \left[ c_1 - \frac{1}{\gamma} \theta_e \right] - y \sin \varphi(x, y, v) \]  

\[ + \frac{1}{\gamma} \theta_e \sin \theta_e + \frac{1}{\gamma} \theta_e, \]  

(21)

\[ \theta_e \geq 0 \]  

\[ \text{for all } x, y, v, \theta_e \text{ bounded on } [0, \infty). \]

Proposition 1. Assume that \( x, y, v, \omega \) are bounded on \([0, \infty)\). Then there exists a function \( \varphi \in \mathcal{F}_e \) such that the equilibrium point \((x_0, y_0, \theta_0) = (0, 0, 0)\) of the closed-loop system (11), (21), (22) is uniformly stable. Furthermore, if \( v(t) \) does not converge to zero then, for small initial conditions \((x_0(t), y_0(t), \theta_0(t))\), the corresponding solution \((x(t), y(t), \theta(t))\) converges to zero, i.e.

\[ \lim_{t \to \infty} [x(t) + y(t) + \theta(t)] = 0. \]  

(22)

Corollary 1. Under the conditions of Proposition 1, given any reference velocity \( v_r \) with the property that \( \lim_{t \to \infty} x(t) = 0 \), it follows that the zero equilibrium of the closed-loop system (11), (21), (22) is exponentially stable if we select a function \( \varphi \in \mathcal{F}_e \) such that \( \varphi(0) > 0 \).

Proof. By choice of \( \varphi \), \( y_0 v_r(t) \sin \varphi(y, v_r(t)) \) is bounded for all \( v_r \) and all \( t \geq 0 \). Furthermore,

\[ \varphi(y_0 v_r(t)) \int_0^t \varphi'(y_0 v_r(t)) \, dt \]  

\[ \times \int_0^t \cos[y_0 v_r(t) \int_0^t \varphi'(y_0 v_r(t)) \, dt] \, ds \]  

(23)

Note that \( \chi(t, y) = \sup_{t \geq 0} \chi(t, y) \) is a continuous function satisfying \( \chi(0) = \varphi(0) \). Letting \( l_\varphi = \lim_{t \to \infty} v_r(t)^2 \), it
such that \(\{(x_e, y_e, 0.) = R^*\}\) is the largest constant such that
\[
(x_e(t), y_e(t), \theta(t)) \in \Omega \quad \text{for all} \quad t \geq 0.
\]
where \(\frac{d}{dt} x_e(t) = -c_3 \varphi \phi_e(t) - c_5 \psi_e(t)

Therefore, we conclude the local exponential stability of the closed-loop system \((11), (21), (22)\) at the zero equilibrium for all \(t \geq 0\). We can conclude that the zero equilibrium of the closed-loop system \((11), (39), (40)\) tends to zero exponentially after a considerable period of time.

**Remark 3.** Similarly to Corollary 1, we can conclude that, under the additional assumption that \(\lim_{t \to \infty} |\omega(t)| = 0\), the zero equilibrium of the closed-loop system \((11), (39), (40)\) is exponentially stable (for small initial errors). In other words, all the closed-loop trajectories go to zero at an exponential rate after a considerable period of time.

**Remark 4.** (Path following.) It is of interest to mention that the robot under study can globally follow two particular types of paths: straight lines and circles. Indeed, putting \(v_1 = 0\) and \(v_2 = v_3\), with \(c_4\) a nonzero constant, the reference trajectories are straight lines of the form \(x(t) = x(0) + c_4 \cos \theta(t)\) and \(y(t) = y(0) + c_5 \sin \theta(t)\) in the case where we choose \(\omega_1 = c_4\) and \(\omega_2 = c_5\), with \(c_4\) and \(c_5\) two nonzero constants, the reference trajectories are circles of radius \(c_4\) described by \(x(t) = x(0) + c_4 \cos \theta_4(t)\) and \(y(t) = y(0) + c_5 \sin \theta_5(t)\)

**Proposition 3.** Assume that \(v_1, v_2, \omega_1, \omega_2, \omega_3, \omega_4\) are bounded trajectories of the system \((11)\) in closed loop with the controllers

By choosing the tracking controllers \(v_1\) and \(\omega_2\) as
\[
v_1(t) = \alpha_1 + \gamma_1 \varphi \phi_e(t), 
\]
\[
\omega_2(t) = \alpha_2 + \gamma_2 \varphi \phi_e(t)
\]
with \(c_4, c_5 > 0\), we have
\[
V_2(t, x_e(t), y_e(t), \theta(t)) = -\frac{c_3 \varphi \phi_e(t)}{2} - c_4 \xi_e(t) - c_5 \theta_e(t).
\]
We establish the following result.

**Proposition 2.** Assume that \(v_1, v_2, \omega_1, \omega_2, \omega_3, \omega_4\) are bounded trajectories of the system \((11)\) in closed loop with the controllers

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\]
We establish the following result.
4. Tracking of a simplified dynamic model

In this section, we study the augmented system (11) appended with two integrators, i.e.,

\[
\begin{align*}
\dot{x}_e &= \omega y_e - v + v_1 \cos \theta_e, \\
\dot{y}_e &= -c_3 \omega y_e - \omega z_e + v_1 \sin \theta_e, \\
\dot{\theta}_e &= -c_5 \omega \theta_e - c_6 \omega^2, \\
\dot{\psi} &= u_1, \\
\dot{\omega} &= u_2,
\end{align*}
\]

(47)

where \( u_1 \) and \( u_2 \) may be regarded as torques or generalized force variables of the two-degrees-of-freedom mobile robot. The system (47) is referred to as a simplified dynamic model for the mobile robot. It is well known that consideration of other effects acting on the vehicle are not included. However, we wish to demonstrate that the tracking controllers that were developed for the kinematic model can also be obtained for a simple dynamic model as (47), thereby at least making it plausible that a similar controller could be derived for a ‘complete’ dynamic model. The control objective is to find a control law \( u = (u_1, u_2) \) of the form

\[
\begin{align*}
u_1 &= u_1(x_e, y_e, \theta_e, \psi, \omega, \dot{x}_e, \dot{y}_e, \dot{\theta}_e, \dot{\omega}, \dot{\psi}, \dot{\omega}), \\
u_2 &= u_2(x_e, y_e, \theta_e, \psi, \omega, \dot{x}_e, \dot{y}_e, \dot{\theta}_e, \dot{\omega}, \dot{\psi}, \dot{\omega}),
\end{align*}
\]

(48)

in such a way that local or global tracking is achieved. In other words, \( x_e, y_e \) and \( \theta_e \) are forced to converge to zero.

We discuss in this section how the methodology presented in the previous section can be extended to the system (47). For simplicity, we only look at the global tracking case that extends the local tracking result of Fierro and Lewis (1995). The development for the local case is analogous and is omitted.

Introduce the new variables

\[
\begin{align*}
\bar{v} &= v - \alpha_\psi, \\
\bar{\omega} &= \omega - \alpha_\omega,
\end{align*}
\]

(49)

where \( \alpha_\psi \) and \( \alpha_\omega \) are defined as in (39) and (40) respectively.

Following the notation used in Section 3 (see in particular (34)), in the new coordinates \((\bar{x}_e, \bar{y}_e, \bar{\theta}_e, \bar{\psi}, \bar{\omega})\) the system (47) is transformed into

\[
\begin{align*}
\dot{\bar{x}}_e &= \omega \bar{y}_e - c_4 \bar{x}_e - \bar{v}, \\
\dot{\bar{y}}_e &= -c_3 \omega \bar{y}_e - \omega \bar{z}_e + c_1 \psi \sin \theta_e, \\
\dot{\bar{\theta}}_e &= -c_5 \omega \bar{\theta}_e - c_6 \bar{\psi} \int_0^1 \cos (s \theta_e) \, ds - \bar{\omega}, \\
\dot{\bar{\psi}} &= u_1 - \bar{\psi}, \\
\dot{\bar{\omega}} &= \bar{v} - \bar{\omega}.
\end{align*}
\]

(50)

where \( \bar{\alpha}_\psi \) and \( \bar{\alpha}_\omega \) are given by

\[
\begin{align*}
\bar{\alpha}_\psi &= (\cos \theta_e - c_5 \psi \sin \theta_e) \psi - c_1 \bar{y}_e \bar{u}_2 \\
&+ (c_3 \omega^2 + c_4) (\omega \bar{y}_e - v + v_1 \cos \theta_e) \\
&- (c_5 \bar{u}_2 + c_6 \psi) (-\omega \bar{z}_e + \psi \sin \theta_e) \\
&- (v, \sin \theta_e + c_3 \psi \bar{v}_1 \cos \theta_e) (\omega_1 - \omega) \\
&+ (2c_2 \omega \bar{z}_e - c_5 \bar{y}_e \sin \theta_e - c_6 c_4 \bar{y}_e \bar{u}_2), \\
\bar{\alpha}_\omega &= \bar{\omega} + \gamma (\bar{y}_e \bar{v} - \omega \bar{x}_e + v_1 \psi \sin \theta_e) \int_0^1 \cos (s \theta_e) \, ds \\
&- c_6 \gamma (\omega_1 - \omega) \int_0^1 \sin (s \theta_e) \, ds \\
&+ c_1 \gamma (\omega_1 - \omega).
\end{align*}
\]

(51)

(52)

It can be directly checked that \( \bar{U} \) is a positive-definite, decrescent and radially unbounded function.

According to the calculation performed in Section 3.2, and in particular (41), the time derivative of \( \bar{U} \) along solutions of (50) satisfies

\[
\begin{align*}
\dot{\bar{U}}(t, x_e, y_e, \theta_e, \psi, \omega) &= -c_1 \omega^2 \bar{y}_e^2 + c_4 \bar{x}_e^2 - c_5 \bar{\theta}_e^2 \\
&- \bar{x}_e \bar{\psi} - \bar{\theta}_e \bar{\omega} + \bar{v}(u_1 - \bar{\psi}) \\
&+ \bar{\omega}(u_2 - \bar{\omega}).
\end{align*}
\]

(54)

Applying the feedback controllers

\[
\begin{align*}
u_1 &= \bar{x}_e + \bar{\psi} - c_6 \psi, \\
u_2 &= \bar{\theta}_e + \bar{\omega} - c_5 \bar{\omega},
\end{align*}
\]

(55)

with \( c_6, c_7 > 0 \), we arrive at

\[
\begin{align*}
\dot{\bar{U}}(t, x_e, y_e, \theta_e, \psi, \omega) \\
&= -c_1 \omega^2 \bar{y}_e^2 - c_4 \bar{x}_e^2 - c_5 \bar{\theta}_e^2 - c_6 c_7 \bar{\omega}^2.
\end{align*}
\]

(57)

We have the following proposition.

Proposition 4. Under the conditions of Proposition 2, if \( \bar{v}_1 \) and \( \bar{\omega} \) are bounded then all the trajectories of the resulting system composed of (47), (55) and (56) are globally uniformly bounded. Furthermore, if \( v_1(t) \) does not converge to zero, or if \( v_1(t) \) converges to zero but \( \omega(t) \) does not converge to zero, then

\[
\lim_{t \to \infty} \left[ \|x_e(t)\| + \|y_e(t)\| + |\theta_e(t)| + |v_1(t) - v_1(t)| + |\omega(t) - \alpha_\omega(t)| \right] = 0.
\]

(58)

Proof. This follows the same reasoning as the proof of Proposition 2.

5. Discussion and simulation results

With the purpose of illustrating the tracking controllers derived in this paper, a number of simulations have been done. The simulations were carried out using MATLAB, with the following choice for the parameters in the controllers (21), (22), (39) and (40) and the reference velocities

\[
\begin{align*}
c_1 &= c_5 = c_6 = c_7 = 1, \\
c_2 &= c_4 = 2, \quad v_1 = 1, \quad \omega_1 = 0.
\end{align*}
\]

The simulations not only illustrate the effectiveness of the tracking controllers but are also used for obtaining an insight into the difference between the usefulness of the global versus the local controller under changing initial conditions. Clearly, the local controller (21), (22) assures, by Proposition 1, that the tracking errors converge to zero provided that the initial errors are sufficiently small, but no explicit estimate of how small these errors should be was given. On the other hand, the global controller (39), (40) can be used for arbitrary (large) initial errors (see Proposition 2), but the price will be a relatively slow convergence of the tracking errors. We demonstrate these effects as follows. In Figs 2 and 3, the local controller (21), (22) is applied with initial tracking errors \((x_e(0), y_e(0), \theta_e(0)) = (-0.5, 0.5, -1))\). Similarly, in Figs 4 and 5, the global controller (39), (40) is applied with the same initial tracking errors as in Fig. 2 (respectively Fig. 3). One can clearly see the difference between the local and global controller under the changing initial tracking errors.
Fig. 2. Local tracking of the kinematic model, with initial errors \((x_0(0), y_0(0), \theta_0(0)) = (-0.5, 0.5, 1)\).

To quantify the difference between the four simulations, one may use the following error measure over the time period \([0, T]\):

\[
P = \frac{1}{T} \int_0^T [x_0(t)^2 + y_0(t)^2 + \theta_0(t)^2] \, dt.
\]

(60)

corresponding to the simulations described in Figs 2–5, we find the values

\[
P_2 = 0.1249, \quad P_3 = 15.6502,
\]

\[
P_4 = 0.1367, \quad P_5 = 7.7944.
\]

(61)

Indeed, the above outcomes agree with our expectations in that the local controller performs better for small initial tracking errors, but for large initial tracking errors the global controller (39), (40) is preferable.

6. Conclusions

The mobile robot kinematic model, or its simplified dynamic model, serves, as has been shown, as an excellent 'test-bed' for using the backstepping technique in the tracking control problem. Both the local and global tracking problems with exponential convergence have then solved. Our theoretical results have been confirmed by means of a number of simulations together with an analysis of the performance of these controllers. The backstepping tracking control method presented in this paper was recently extended to the more general class of nonholonomic chained systems (Jiang and Nijmeijer 1997).

As in most previous work on the study of nonholonomic systems, our results are heavily based on a 'nonholonomic' assumption of the form \(x \sin \theta - y \cos \theta = 0\). It should be mentioned that this condition is an idealization of real situations, and is never satisfied by real physical control systems. In d'Andréa-Novel et al. (1995), the authors proposed a singular perturbation approach to point-tracking control for nonlinear mechanical systems that do not satisfy ideal velocity constraints. There is still no general answer for the tracking control problem if common velocity constraints are not satisfied by the class of nonholonomic mechanical systems under consideration.

Acknowledgements—Most of this work was done when the first author was with the Department of Systems Engineering, Australian National University, and the second author was visiting the Faculty of Engineering and Information Technology, Australian National University under a research fellowship. The authors acknowledge funding support from the Centre for Robust and Adaptive Systems established under the Cooperative Research Centre Program of the Commonwealth of Australia.

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