Dynamic analysis of a tethered satellite system with a moving mass

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Abstract This paper presents a dynamic analysis of a tethered satellite system with a moving mass. A dynamic model with four degrees of freedom, i.e., a two-piece dumbbell model, is established for tethered satellites conveying a mass between them along the tether length. This model includes two satellites and a moving mass, treated as particles in a single orbital plane, which are connected by massless, straight tethers. The equations of motion are derived by using Lagrange’s equations. From the equations of motion, the dynamic response of the system when the moving mass travels along the tether connecting the two satellites is computed and analyzed. We investigate the global tendencies of the libration angle difference (between the two sections of tether) with respect to the changes in the system parameters, such as the initial libration angle, size (i.e. mass) of the moving mass, velocity of the moving mass, and tether length. We also present an elliptic orbit case and show that the libration angles and their difference increase as orbital eccentricity increases. Finally, our results show that a one-piece dumbbell model is qualitatively valid for studying the system under certain conditions, such as when the initial libration angles, moving mass velocity, and moving mass size are small, the tether length is large, and the mass ratio of the two satellites is large.

Keywords Nonlinear dynamics · Tethered satellites · Dumbbell model · Moving mass · Libration angle

1 Introduction

Since the concept of a tethered satellite system was introduced by Tsiolkovsky, the Russian pioneer in space technology, several tethered satellite space missions have been flown in space. A classical tethered satellite system consists of two satellites connected by a tether, but recently proposed systems consist of more than two satellites connected by tethers. In tethered satellite systems, the tethers can be classified into the following four categories: gravity gradient stabilization tethers, momentum exchange tethers, electro-dynamic tethers, and space elevator tethers.

Many researchers have studied tethered satellite systems using a one-piece dumbbell model, which is the simplest model for tethered satellites. In the one-piece dumbbell model, two satellites are assumed to be point masses or particles, and are connected by a straight, massless tether. This dumbbell model is reasonable for investigating the behavior of tethered satellites, because the gravity-gradient force maintains a tension in the tether connecting the two satellites.

Optimal control for the deployment and retrieval of a tethered satellite system has been an important area of study. Steindl and Troger [7] studied an optimal control strategy to achieve a force controlled deployment of a tethered subsatellite. Wen et al. [8] presented a nonlinear optimal feedback control for the deployment process of a tethered subsatellite. Jin and Hu [9] also presented an optimal control method for the deployment and retrieval processes of a tethered subsatellite system with three degrees of freedom. Similarly, Williams [10] introduced a methodology for deployment/retrieval optimization of tethered satellite systems.

Recently, the dynamic analysis of tethered space elevators has drawn great attention from researchers in astronautical engineering. A space elevator could enable relatively cheap transportation to geostationary altitude. Cohen and Misra [11] developed a simple planar model of a space elevator, and proposed specific climbing procedures to minimize the adverse effects of climber transit on the tether. Woo and Misra [12] investigated the dynamics of a long tethered space elevator and a partial space elevator. Williams and Ockels [13] also studied the dynamics of a simplified elevator model which accounts for the fundamental libration modes and the motion of the climber mass. Steindl and Troger [14] addressed the question of stability of the radial relative equilibrium of a tapered string reaching from the surface of the Earth to a satellite on a circular geosynchronous orbit around the Earth.

Several researchers have studied the dynamic behavior of tethered satellite systems with a moving mass (i.e. a mass which moves along the tether connecting the two end-bodies). Fujii et al. [15] developed the dynamic equations for a flexible space tether system equipped with a crawler (climber) and studied the behavior of the system via numerical simulation. However, they considered only the case where the crawler mass moves along the tether purely in response to such conservative forces as gravitational and inertial forces. Assuming that the mother satellite is located at a circular orbit, Kojima et al. [16] proposed a two-piece dumbbell model for a tethered satellite system with a single climber to evaluate its fundamental librational motion. They also derived the equations of motion under the assumption that the mother satellite remains a circular orbit. If the mother satellite has much larger mass than the subsatellite, the assumption of the previously mentioned study [16] can be accepted for the dynamic analysis of a tethered satellite system with a moving mass. However, the orbital radius of the mother satellite is a function of time when the mass of the subsatellite is not small compared to that of the mother satellite. In this case, the equations of motion should be derived considering the orbital radius of all the satellites comprising the system.

For our investigation, we construct a two-piece dumbbell model with four degrees of freedom for tethered satellites with a moving mass, considering the orbital radii of all three masses in the system. In this model, the libration angle of the tether connecting one satellite and the moving mass may differ from the libration angle of the tether connecting the other satellite and the moving mass. After deriving the equations of motion for this model using Lagrange’s equations, we compute and analyze the dynamic response of the system. In particular, we examine the global tendencies of the libration angle difference (between the two sections of tether) with respect to the changes in the system parameters, such as the initial libration angle, size (i.e. mass) of the moving mass, velocity of the moving mass, and tether length. We also present an elliptic orbit case and examine the libration angles and their dependence on orbital eccentricity. Finally, we examine the conditions under which a one-piece dumbbell model gives qualitatively valid results when a mass is transported along a tether connecting two satellites.
The position of the satellite \( \dot{V} \) are treated as particles of mass \( m_1 \) and \( m_2 \), and the moving mass \( m \) is also treated as a particle. The satellites and moving mass are assumed to be in a single orbital plane. Furthermore, the tether, which has constant length \( L \), is assumed to be massless and straight between the satellites and the moving mass. The tether length between \( m_1 \) and \( m \) is denoted by \( L_1 \), while the length between \( m_2 \) and \( m \) is denoted by \( L_2 \). The sum of these lengths is the total tether length:

\[
L_1 + L_2 = L
\]  

(1)

The position of the satellite \( m_1 \) is described by the orbital radius \( r \) and the true anomaly \( \psi \). The unit vectors \( e_r \) and \( e_\psi \) are orthogonal vectors, and \( e_r \) is in the direction from the Earth's center to the satellite \( m_1 \). The angles of the tethers \( L_1 \) and \( L_2 \) relative to the direction of \( e_r \) are described by the libration angles \( \theta_1 \) and \( \theta_2 \), respectively. The angle \( \theta_2 \) is introduced to investigate the effect of the moving mass on the behavior of the tethered satellite system. Note that the libration angles, \( \theta_1 \) and \( \theta_2 \), would be the same if the tethered satellite system was modeled as a one-piece dumbbell model. The speed \( V \) of the moving mass \( m \) relative to the satellite \( m_1 \) is given by differentiating \( L_1 \) with respect to time \( t \):

\[
V = \dot{L}_1 = -\dot{L}_2
\]  

(2)

since \( L_1 = L - L_2 \) and \( \dot{L} = 0 \), where the superposed dot represents differentiation with respect to time. Consequently, the motion of a tethered satellite system with a moving mass can be described by the four generalized coordinates \( r, \psi, \theta_1 \) and \( \theta_2 \), since \( L \) and \( V \) (and hence \( L_1 \) and \( L_2 \)) are specified constants or functions of time.

The position and velocity vectors of the satellites and the moving mass are required to obtain the kinetic and potential energies for the tethered satellite system. The position vectors of the satellites \( m_1 \) and \( m_2 \), \( r_1 \) and \( r_2 \), respectively, are given by

\[
r_1 = r e_r,
\]

\[
r_2 = (r + L_1 \cos \theta_1 + L_2 \cos \theta_2)e_r
+ (L_1 \sin \theta_1 + L_2 \sin \theta_2)e_\psi
\]  

(3)

while the position vector of the moving mass, \( r_m \), is given by

\[
r_m = (r + L_1 \cos \theta_1)e_r + L_1 \sin \theta_1 e_\psi
\]  

(4)

Differentiating the position vectors of the satellites and the moving mass, the velocity vectors of the satellites \( m_1 \) and \( m_2 \), \( v_1 \) and \( v_2 \), and the velocity vector of the moving mass, \( v_m \), can be expressed as follows:

\[
v_1 = \dot{r} e_r + r \dot{\psi} e_\psi
\]  

(5)

\[
v_2 = \left[ \dot{r} + V (\cos \theta_1 - \cos \theta_2) - L_1 (\dot{\psi} + \dot{\theta}_1) \sin \theta_1 
- (L_1 - L_2) (\dot{\theta}_1)^2 \right] e_r 
+ \left[ r \ddot{\psi} + V (\sin \theta_1 - \sin \theta_2) + L_1 (\dot{\psi} + \dot{\theta}_1) \cos \theta_1 
+ (L_1 - L_2) (\dot{\theta}_1)^2 \right] e_\psi
\]  

(6)

\[
v_m = \left[ \dot{r} + V (\cos \theta_1 - \cos \theta_2) - L_1 (\dot{\psi} + \dot{\theta}_1) \sin \theta_1 \right] e_r 
+ \left[ r \ddot{\psi} + V \sin \theta_1 + L_1 (\dot{\psi} + \dot{\theta}_1) \cos \theta_1 \right] e_\psi
\]  

(7)

Since the satellites and the moving mass are regarded as particles, the kinetic energy of a tethered satellite system with a moving mass can be written as

\[
K = \frac{1}{2} m_1 v_1 \cdot v_1 + \frac{1}{2} m_2 v_2 \cdot v_2 + \frac{1}{2} m v_m \cdot v_m
\]  

(8)

The gravitational potential energy is expressed by

\[
P = -GM \frac{m_1}{r} - GM \frac{m_2}{R_2} - GM \frac{m}{R_m}
\]  

(9)
where $G$ is the universal gravitational constant, $M$ is the mass of the Earth, and $R_2$ and $R_m$ are given by

$$R_2 = [r^2 + 2r(L_1 \cos \theta_1 + L_2 \cos \theta_2) + 2L_1L_2 \cos(\theta_1 - \theta_2) + L_1^2 + L_2^2]^{1/2}$$  \hspace{1cm} (10)

$$R_m = (r^2 + 2L_1r \cos \theta_1 + L_1^2)^{1/2}$$  \hspace{1cm} (11)

Note that $R_2$ is the distance from the Earth’s center to the satellite $m_2$, while $R_m$ is the distance from the Earth’s center to the moving mass.

The nonlinear equations of motion of the tethered satellite system with a moving mass are derived by using Lagrange’s equations:

$$\frac{d}{dr} \left( \frac{\partial K}{\partial \dot{q}_k} \right) - \frac{\partial K}{\partial q_k} + \frac{\partial P}{\partial q_k} = Q_k \quad \text{for} \quad q_k = r, \psi, \theta_1, \theta_2$$  \hspace{1cm} (12)

where the $q_k$ are the generalized coordinates, and the $Q_k$ are the non-conservative forces corresponding to the generalized coordinates $q_k$. Note that we are neglecting external forces such as the atmospheric drag force and solar radiation pressure, therefore, the non-conservative forces are zero for the generalized coordinates $r, \psi, \theta_1$ and $\theta_2$. The driving force of the mass, which is also a non-conservative force, determines the speed of the moving mass, but the moving mass speed is a given quantity, and hence is not one of the generalized coordinates. Substituting (8) and (9) into (12), the coupled nonlinear equations of motion for a tethered satellite system with a moving mass are obtained as follows:

$$(m_1 + m_2 + m)(\ddot{r} - r\dot{\psi}^2)$$

$$- (m_2 + m)\left\{ [-\ddot{V} + L_1(\dot{\psi} + \dot{\theta}_1)^2] \cos \theta_1 + [L_1(\dot{\psi} + \dot{\theta}_1) + 2V(\dot{\psi} + \dot{\theta}_1)] \sin \theta_1 \right\}$$

$$- m_2\left\{ \ddot{V} + L_2(\dot{\psi} + \dot{\theta}_2)^2 \cos \theta_2 + [L_2(\dot{\psi} + \dot{\theta}_2) - 2V(\dot{\psi} + \dot{\theta}_2)] \sin \theta_2 \right\}$$

$$+ GMm_1/r^2 + GMm(r + L_1 \cos \theta_1)/R_m^3$$

$$+ GMm_2(r + L_1 \cos \theta_1 + L_2 \cos \theta_2)/R_2^3 = 0$$  \hspace{1cm} (13)

These nonlinear equations of motion can be represented in the following matrix-vector form:

$$M(x)\ddot{x} + N(x, \dot{x}) = 0$$  \hspace{1cm} (17)

where $M$ is the symmetric mass matrix, $N$ is the nonlinear internal force vector, and $x$ is the displacement vector given by

$$x = \{r, \psi, \theta_1, \theta_2\}^T$$  \hspace{1cm} (18)
It should be noted that the mass matrix is a function of the displacement vector, while the internal force vector is a function of the displacement and velocity vectors. The elements of the mass matrix and internal force vector are given in the Appendix.

3 Analysis and discussion

3.1 Time integration

By applying the Newmark time integration method, we can compute the dynamic responses for a tethered satellite system containing a moving mass (which moves along the tether length between the two satellites). To apply the Newmark method, the displacement, velocity, and acceleration vectors at time \( t_n \), \( \mathbf{x}(t_n) \), \( \dot{\mathbf{x}}(t_n) \) and \( \ddot{\mathbf{x}}(t_n) \), must be approximated as \( \mathbf{d}_n \), \( \mathbf{v}_n \) and \( \mathbf{a}_n \), respectively. Furthermore, the equations of motion as given by (17) should be transformed into the following balance equation:

\[
\mathbf{M}(\mathbf{d}_{n+1})\mathbf{a}_{n+1} + \mathbf{N}(\mathbf{d}_{n+1}, \mathbf{v}_{n+1}) = \mathbf{0} \tag{19}
\]

The initial conditions are given by

\[
\mathbf{d}(0) = \begin{bmatrix} r(0), \psi(0), \theta_1(0), \theta_2(0) \end{bmatrix}^T,
\]

\[
\mathbf{v}(0) = \begin{bmatrix} \dot{r}(0), \dot{\psi}(0), \dot{\theta}_1(0), \dot{\theta}_2(0) \end{bmatrix}^T \tag{20}
\]

The initial acceleration vector obtained from (19) can be expressed as

\[
\mathbf{a}_0 = -\left[\mathbf{M}(\mathbf{d}_0)\right]^{-1}\mathbf{N}(\mathbf{d}_0, \mathbf{v}_0) \tag{21}
\]

In order to update the unknown vectors \( \mathbf{d}_{n+1}, \mathbf{v}_{n+1} \), and \( \mathbf{a}_{n+1} \) with the known vectors \( \mathbf{d}_n, \mathbf{v}_n \), and \( \mathbf{a}_n \), the Newmark method defines the updated displacement and velocity vectors as

\[
\mathbf{d}_{n+1} = \mathbf{d}_n + \Delta t \mathbf{v}_n + \left(1/2 - \beta\right)\Delta t^2 \mathbf{a}_n + \beta \Delta t^2 \mathbf{a}_{n+1} \tag{22}
\]

\[
\mathbf{v}_{n+1} = \mathbf{v}_n + \left(1 - \gamma\right)\Delta t \mathbf{a}_n + \gamma \Delta t \mathbf{a}_{n+1} \tag{23}
\]

where \( \Delta t \) is the size of the time step, i.e., \( \Delta t = t_{n+1} - t_n \); \( \beta \) and \( \gamma \) are the algorithmic parameters given by 1/4 and 1/2, respectively. As shown in (22) and (23), the updated vectors \( \mathbf{d}_{n+1} \) and \( \mathbf{v}_{n+1} \) cannot be determined unless \( \mathbf{a}_{n+1} \) is determined. To determine \( \mathbf{a}_{n+1} \), (22) and (23) are substituted into (19), yielding a nonlinear algebraic vector equation with the unknown vector \( \mathbf{a}_{n+1} \). The Newton-Raphson iteration method is required to solve this equation for \( \mathbf{a}_{n+1} \), which is described by

\[
\mathbf{a}_{n+1}^{(j+1)} = \mathbf{a}_{n+1}^{(j)} + \Delta \mathbf{a}_{n+1}^{(j)} \tag{24}
\]

\[
\mathbf{J}_{n+1}^{(j)} \Delta \mathbf{a}_{n+1}^{(j)} = -\mathbf{M}(\mathbf{d}_{n+1}^{(j)}, \mathbf{a}_{n+1}^{(j)}) - \mathbf{N}(\mathbf{d}_{n+1}^{(j)}, \mathbf{v}_{n+1}^{(j)}) \tag{25}
\]

where \( j \) is the iteration number for each time step \( n \), and \( \mathbf{J}_{n+1}^{(j)} \) is the Jacobian matrix given by

\[
\mathbf{J}_{n+1}^{(j)} = \mathbf{M}(\mathbf{d}_{n+1}^{(j)}) + \beta \Delta t^2 \left[ \frac{\partial \mathbf{M}(\mathbf{d}_{n+1}^{(j)})}{\partial \mathbf{d}_{n+1}^{(j)}} \mathbf{a}_{n+1}^{(j)} + \frac{\partial \mathbf{N}(\mathbf{d}_{n+1}^{(j)}, \mathbf{v}_{n+1}^{(j)})}{\partial \mathbf{d}_{n+1}^{(j)}} \right]
\]

\[
+ \gamma \Delta t \left[ \frac{\partial \mathbf{N}(\mathbf{d}_{n+1}^{(j)}, \mathbf{v}_{n+1}^{(j)})}{\partial \mathbf{v}_{n+1}^{(j)}} \right] \tag{26}
\]

Once the updated acceleration vector \( \mathbf{a}_{n+1} \) is determined, the updated displacement and velocity vectors \( \mathbf{d}_{n+1} \) and \( \mathbf{v}_{n+1} \) can be obtained from (22) and (23).

3.2 Validation of the two-piece dumbbell model

The proposed equations of motion are validated by comparing the dynamic responses computed in this study with those in [12]. As opposed to the present study, the study of [12] considers the mass of tether and assumes a constant distance \( R_c \) from the Earth center to the mass center of the satellite system. Apart from these differences, the same initial conditions and physical parameters as those of [12] are used for validation. The initial conditions for the libration angles are \( \theta_1(0) = \theta_2(0) = \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0 \) and the constant orbital rate of the satellite system around the Earth is \( \dot{\psi} = 7.2923 \times 10^{-5} \) radians/s. The position of the moving mass, \( L_1 \), prescribed in Fig. 3 of [12], is duplicated in Fig. 2. The masses of the satellites and moving mass are \( m_1 = m_2 = 10,000 \) kg and \( m = 100 \) kg, respectively, the total length of the tether is \( L = 10,000 \) km, the cross-sectional area of the tether is \( A = 2 \) mm\(^2\), and the density of the tether is \( \rho = 1,440 \) kg/m\(^3\). In [12], the distance from the Earth center to the mass center of the satellite system is assumed constant and this distance is given by \( R_c = 42,522 \) km. As a first step in performing the validation analysis, a computer program is developed to
compute the in-plane libration angles from the equations presented in [12]. When the tether mass is considered, the libration angles \( \theta_1 \) and \( \theta_2 \) computed by this program are plotted in Fig. 3. Comparing Fig. 3 and Fig. 4(a) of [12], the libration angles of these two figures are seen to be identical. This program will now be used as part of the validation study as described below.

In order to compare the libration angles of the present study and [12] under the same conditions, the constraint of a constant \( R_c \) should be imposed on the equations of motion given by (13)–(16), and the mass of the tether should be neglected in the equations of [12]. The assumption of [12] that \( R_c \) is constant results in a constraint between the orbital radius, \( r \), and the libration angles, \( \theta_1 \) and \( \theta_2 \). The position vector from the Earth center to the mass center of the system is given by

\[
R_c = \frac{m_1 r_1 + m_2 r_2 + m r_m}{m_1 + m_2 + m} \tag{27}
\]

Substituting (3) and (4) into (27), the position vector \( R_c \) becomes a function of \( r \), \( \theta_1 \) and \( \theta_2 \). Therefore, from the constraint that the magnitude of \( R_c \) of (27), \( R_c \), is constant, the orbital radius \( r \) can be expressed as a function of \( \theta_1 \) and \( \theta_2 \). Therefore, the tethered satellite system with a moving mass has three degrees of freedom, and the corresponding generalized coordinates are \( \psi \), \( \theta_1 \) and \( \theta_2 \). If the orbital radius \( r \), which is expressed in terms of \( \theta_1 \) and \( \theta_2 \), is introduced into (14)–(16), the equations of the satellite system without tether mass are obtained under the constraint of \( R_c = \text{const} \). The libration angles computed by the present study and [12], when the tether mass is neglected, are presented in Fig. 4, where the solid line and the square symbol represent the present study and [12], respectively. It is observed in Fig. 4 that the libration angles computed in this study agree with those of [12].

Another validation is performed through comparison of the dynamic responses of this study with those of [16]. As mentioned in Introduction, the main difference between the models of this study and [16] is whether the orbital radius of the mother satellite is a
function of time or not. The orbital radius of [16] is assumed to be constant while the radius of this study is allowed to vary according to the system dynamics, and hence is a function of time. Since the model of [16] assumes the orbital radius of the mother ship is held constant at 6600 km, a large value is imposed of \( r(0) = 6600 \text{ km} \), and hence is a function of time. Since the model is allowed to vary according to the system dynamics, we can assume the orbital radius to be nearly the same as the constant value of 6600 km used from which it is observed that the orbital radius is a function of time or not. The orbital radius of [16] is assumed to be constant while the radius of this study is assumed to be constant while the orbital radius of the mother ship is held constant at 6600 km, a large value is imposed on the mass \( m_1 \) for our study in order to try and replicate this situation; therefore the mass of the satellite \( m_1 \) is assumed to be \( m_1 = 10^6 \text{ kg} \) for this part of our study. To facilitate comparison, the initial conditions and relevant physical properties are the same as those given in [16]. The initial conditions are given by \( r(0) = 6600 \text{ km}, \quad \psi(0) = \pi/2 \text{ radians}, \quad \theta_1(0) = \pi \text{ radians}, \quad \theta_2(0) = \pi \text{ radians}, \quad L_1(0) = 10 \text{ km}, \quad \dot{r}(0) = 0, \quad \dot{\psi}(0) = 1.2 \times 10^{-3} \text{ radians/s}, \quad \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0 \). The physical parameters are \( L = 100 \text{ km} \) and \( m = m_2 = 1000 \text{ kg} \). The computed orbital radius \( r \) of the satellite \( m_1 \) for the orbital time is plotted in Fig. 5, from which it is observed that the orbital radius is nearly the same as the constant value of 6600 km used for the study performed in [16], as described above. The in-plane libration angles \( \theta_1 \) and \( \theta_2 \) are plotted in Fig. 6, where the solid line and the square symbols represent the responses of this study and [16], respectively. As shown in Fig. 6, the libration angles computed in this study are in good agreement with those computed in [16].

3.3 Analysis using the two-piece Dumbbell model

The dynamic responses of the tethered satellites with a moving mass are analyzed when the moving mass travels from the satellite \( m_1 \) to the satellite \( m_2 \). Unless noted otherwise, the following initial conditions and physical properties are used for the computations in this paper: \( r(0) = 6600 \text{ km}, \quad \psi(0) = \theta_1(0) = \theta_2(0) = 0, \quad \dot{r}(0) = 0, \quad \dot{\psi}(0) = 1.178 \times 10^{-3} \text{ radians/s}, \quad \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0, \quad L = 10 \text{ km}, \quad m_1 = m_2 = 10^3 \text{ kg}, \quad m = 100 \text{ kg}, \quad V = 1 \text{ m/s}, \quad \dot{V} = 0 \). The above initial conditions for \( r \) and \( \psi \) lead to a circular orbit for the system. Figure 7 shows the orbital radius and in-plane libration angles as functions of true anomaly, \( \psi \), which are computed using the initial conditions listed above. The solid and dashed lines of Fig. 7 represent the results obtained from the present study and [16], respectively. As the figure shows, large differences in the orbital radius and libration angles can be seen when comparing the results of the present study and [16] when \( m_1 \) is similar in size to \( m_2 \). This implies that the variation of the orbital radius should be considered when \( m_1 \) is similar in size to \( m_2 \). On the other hand, since the difference between \( \theta_1 \) and \( \theta_2 \) is very small, the tethered satellite system with a moving mass can be simplified as a one-piece dumbbell model when the initial libration angles are zero; in other words, the tether between the satellites \( m_1 \) and \( m_2 \) can be regarded as a straight tether because \( \theta_1 \) remains nearly the same as \( \theta_2 \) for the case where the initial libration angles are zero.

We now investigate the effects of the initial libration angles on the dynamic response of a tethered satellite system with a moving mass. For various values of the initial libration angles, the time his-
Fig. 7 Comparison of the dynamic response as a function of true anomaly: present study (solid line) and [16] (dashed line)

dynamics of the orbital radius and libration angles are computed while the mass \( m \) moves from the satellite \( m_1 \) to the satellite \( m_2 \). These responses are shown in Fig. 8, where Fig. 8(a) shows the orbital radius, \( r \); Fig. 8(b) shows the libration angle, \( \theta_1 \), of tether \( L_1 \); Fig. 8(c) shows the libration angle, \( \theta_2 \), of tether \( L_2 \); and Fig. 8(d) shows the difference between \( \theta_1 \) and \( \theta_2 \). The abscissas in Fig. 8 represent the dimensionless position of the moving mass. If the moving mass \( m \) is located at the same position as satellite \( m_1 \), the value of \( L_1/L \) is zero. However, when \( m \) is located at the same position as \( m_2 \), the value of \( L_1/L \) becomes one. The solid, dashed, and dotted lines of Fig. 8 represent the responses for initial libration angles \( \theta_1(0) = \theta_2(0) = 0 \), \( \pi/18 \), and \( \pi/9 \) radians, respectively. As shown in Fig. 8(a), the oscillation amplitude of the orbital radius variation decreases as the initial libration angles are increased. However, Figs. 8(b)–8(d) show that the amplitudes of the libration angles, as well as their differences, become large as the initial libration angles are increased.

Fig. 8 Dynamic response as a function of moving mass position with variation of the initial libration angles: \( \theta_1(0) = \theta_2(0) = 0 \) (solid line), \( \theta_1(0) = \theta_2(0) = \pi/18 \) radians (dashed line), and \( \theta_1(0) = \theta_2(0) = \pi/9 \) radians (dotted line)

Next, the dynamic behavior of the tethered satellite system is investigated, examining the effect of varying the direction of the moving mass. The outward and inward moving directions are both considered: the outward direction is from the satellite \( m_1 \) to the satellite \( m_2 \), while the inward direction is from the satellite \( m_2 \) to the satellite \( m_1 \). Moving mass velocities of \( V = 1 \) and \( -1 \) m/s are assumed for the outward and inward moving directions, respectively. The dynamic responses for \( V = 1 \) and \( -1 \) m/s are represented in Fig. 9 as solid and dashed lines, respectively. Figure 9(a) shows that the orbital radius of the system...
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Fig. 9 Dynamic response as a function of moving mass position with variation of the direction of the moving mass: $V = 1 \text{ m/s}$ (solid line) and $V = -1 \text{ m/s}$ (dashed line)

Fig. 10 Configurations of the tethered satellites with a moving mass (a) when the moving mass travels from $m_1$ to $m_2$ and (b) when the mass travels from $m_2$ to $m_1$

when the moving mass is moving outward ($V = 1 \text{ m/s}$) is less than the orbital radius of the system when the moving mass is moving inward ($V = -1 \text{ m/s}$). Figure 9(b) and 9(c) show that $\theta_1$ is negative and $\theta_2$ is positive when $V = 1 \text{ m/s}$, while $\theta_1$ is positive and $\theta_2$ is negative when $V = -1 \text{ m/s}$. Therefore, the difference between the libration angles, $\theta_2 - \theta_1$, is greater than zero when the moving mass is moving outward and less than zero when the moving mass is moving inward, as shown in Fig. 9(d). The configuration of the tethered satellite system when the moving mass travels from $m_1$ to $m_2$ is plotted in Fig. 10(a), and the configuration when the mass travels from $m_2$ to $m_1$ is plotted in Fig. 10(b).

The velocity magnitude of the moving mass also affects the dynamic behavior of the tethered satellite. In order to analyze the effect of the velocity of the moving mass, the dynamic responses are computed for moving mass velocities $V = 1$, 2, and 4 m/s. The solid, dashed, and dotted lines in Fig. 11 represent the responses for $V = 1$, 2, and 4 m/s, respectively. As shown in Fig. 11(a), the oscillation amplitude of the orbital radius variation does not change even though the moving mass velocity is increased. Figure 11(b) and 11(c) show that the oscillation amplitudes of the libration angles become larger as the moving mass velocity increases. Finally, Fig. 11(d) shows that the difference between the libration angles also increases as the moving mass velocity increases.

We now investigate the effects of the moving mass size (i.e. mass) on the dynamic behavior of the tethered satellite system. Figure 12 shows the computed dynamic responses of the satellites for moving masses of 100, 200, and 300 kg, which are plotted with solid, dashed, and dotted lines, respectively. The oscillation amplitude of the orbital radius variation decreases as
the moving mass size increases, as shown in Fig. 12(a). However, Figs. 12(b)–12(d) show that the amplitudes of $\theta_1$, $\theta_2$, and $\theta_2 - \theta_1$ increase as the moving mass size is increased. In addition, Fig. 12(d) shows that the value of $\theta_2 - \theta_1$ is approximately proportional to the size of the moving mass. The values of $\theta_2 - \theta_1$, are found to be approximately 0.01, 0.02, and 0.03 radians, for moving mass sizes of 100, 200, and 300 kg, respectively.

The influence of tether length on the dynamic behavior is now analyzed. Figure 13 shows the computed dynamic responses of the tethered satellites with a moving mass, where the solid, dashed, and dotted lines represent tether lengths of $L = 1$, 10, and 100 km, respectively. Figure 13(a) shows that the oscillation amplitude of the orbital radius variation increases with increasing tether length. The libration angles $\theta_1$ and $\theta_2$ do not show an apparent tendency to increase or decrease with tether length, as shown in Figs. 13(b) and 13(c). However, the difference between the libration angles, $\theta_2 - \theta_1$, decreases as the tether length increases.

Next, we analyze the effect of the relative mass ratio of the satellites ($m_1/m_2$) on the dynamic responses
of a tethered satellite system with a moving mass. The dynamic responses are computed for $m_1 = 10^3$, $10^4$, and $10^5$ kg with a fixed value of $m_2 = 10^3$ kg; i.e. the ratios of $m_1/m_2$ for the three cases are 1, 10, and 100. Figure 14 shows that as the mass ratio increases, the oscillation amplitudes of the orbital radius variation and libration angles decreases, and the magnitude of the angle difference decreases as well.

It is interesting to observe the global tendencies of the libration angle difference, $\theta_2 - \theta_1$, with respect to the initial conditions and physical parameter changes. So as to facilitate comparisons of $\theta_2 - \theta_1$ for different parameter values and initial conditions, the libration angle differences are computed when the moving mass is midway between $m_1$ and $m_2$, i.e. when $L_1/L = 0.5$. Figure 15 shows the computed libration angles for the changes of the initial libration angles, moving velocity, moving mass and tether length. As shown in Fig. 15, the libration angle differences increase if the initial libration angles, moving velocity and moving mass are increased, while the angle differences decrease if the tether length is increased. It is also observed in Fig. 15 that the mass ratio of the satellites, $m_1/m_2$, has an influence on the difference between the libration an-
Fig. 15 Libration angle differences when \( L_1 = 0.5L \) for the changes of the physical parameters: (a) the initial libration angles, (b) moving velocity, (c) moving mass, and (d) tether length.

angles, i.e. the difference between the libration angles becomes large as the mass ratio \( m_1/m_2 \) decreases.

In order to show that the proposed model can be used for an elliptic orbit case, the dynamic responses of the tethered satellite system are now investigated for cases where the system is in an elliptic orbit. Except for \( \dot{\psi}(0) \), all the initial conditions and physical parameters for the elliptic orbit cases are the same as those for the circular orbit studies. The initial condition of \( \dot{\psi} \) to obtain an elliptic orbit is given by

\[
\dot{\psi}(0) = \sqrt{\frac{GM(1 + e)}{R^3}}
\]

where \( e \) is the orbital eccentricity. The dynamic responses for orbits with \( e = 0, 0.1 \) and \( 0.3 \) are now computed. Figure 16 presents the dynamic responses for the circular and elliptic orbits computed when the moving mass travels from the satellite \( m_1 \) to the satellite \( m_2 \). The solid, dashed, and dotted lines of Fig. 16 represent the circular orbit, elliptic orbit with \( e = 0.1 \), and elliptic orbit with \( e = 0.3 \), respectively. As shown in Fig. 16, the libration angles and their difference increase with as the orbital eccentricity is increased. Since the libration angle difference for \( e = 0.3 \) is large, we see that a one-piece dumbbell model cannot be used for the tethered satellite system with a moving mass if the system is in a highly elliptic orbit.

4 Conclusions

This study analyzes the dynamic behavior of a tethered satellite system with a moving mass which travels between the two end-body satellites. We establish a two-
piece dumbbell model with four degrees of freedom, where the equations of motion are derived using Lagrange’s equations. Using these equations of motion, the dynamic responses when the moving mass travels between the two satellites are computed and analyzed. A key feature of our model is that we do not assume that one of the satellites necessarily remains in a circular orbit of constant radius, and instead allow all of the orbital radii of the satellites comprising the system to vary with time. We investigate the global tendencies of the libration angle difference (between the two sections of tether) with respect to the changes in the system parameters, such as the initial libration angle, size of the moving mass, velocity of the moving mass, and tether length. We also present an elliptic orbit case. Finally, we examine the conditions under which a one-piece dumbbell model can be used to produce qualitatively valid results when a mass is transported along a tether connecting two satellites.

According to our study, the dynamic characteristics of a tethered satellite system when a moving mass travels along the tether can be summarized as follows:

1. The amplitudes of the libration angles and their difference become large as the initial libration angles increase.
2. The difference between the libration angles depends on the direction of moving mass. The difference of $\theta_2 - \theta_1$ is positive for the outward moving direction and negative for the inward moving direction.
3. The oscillation amplitudes of the libration angles, as well as the difference between the libration angles, become large as the moving mass velocity increases.
4. As the size (i.e. mass) of the moving mass increases, the oscillation amplitude of the orbital radius variation decreases, but the amplitudes of the libration angles and their difference increase with the size of the moving mass.
5. As the tether length increases, the oscillation amplitude of the orbital radius variation increases; however, the difference between the libration angles decreases as the tether length is increased.
6. As the relative mass ratio of the satellites ($m_1/m_2$) increases, the oscillation amplitudes of the orbital radius variation and libration angles decrease, as well as the magnitude of the angle difference.
7. A one-piece dumbbell model can only be used (to obtain qualitative results) for tethered satellites with a moving mass when the initial libration angles, moving mass velocity and moving mass size are small, the tether length is large, and the relative mass ratio ($m_1/m_2$) of the two satellites is large.
8. A one-piece dumbbell model cannot be used for a tethered satellite system with a moving mass which is in a highly elliptic orbit, because the libration angle difference becomes large when the tethered system is in a highly elliptic orbit.

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The mass matrix of (17) is a 4 × 4 matrix given by

\[ M = [m_{ij}] \quad \text{for } i, j = 1, 2, 3, 4 \] (29)

where

\[ m_{11} = m_1 + m_2 + m, \]
\[ m_{12} = m_{21} = -(m_2 + m)L_1 \sin \theta_1 - m_2 L_2 \sin \theta_2, \]
\[ m_{14} = m_{41} = -m_2 L_2 \sin \theta_2, \]
\[ m_{22} = (m_1 + m_2 + m)^2 \]
\[ + (m_2 + m)L_1(L_1 + 2r \cos \theta_1) \]
\[ + m_2 L_2[L_2 + 2r \cos \theta_2 + 2L_1 \cos(\theta_2 - \theta_1)], \]
\[ m_{23} = m_{32} \]
\[ = (m_2 + m)L_1(L_1 + r \cos \theta_1) \]
\[ + m_2 L_1 L_2 \cos(\theta_2 - \theta_1), \]
\[ m_{24} = m_{42} \]
\[ = m_2 L_2[L_2 + r \cos \theta_2 + L_1 \cos(\theta_2 - \theta_1)], \]
\[ m_{33} = (m_2 + m)L_1^2, \]
\[ m_{34} = m_{43} = m_2 L_1 L_2 \cos(\theta_2 - \theta_1), \]
\[ m_{44} = m_2 L_2^2 \]

The nonlinear internal force vector of (17) is a 4 × 1 column vector given by

\[ N = \{N_i\}^T \quad \text{for } i = 1, 2, 3, 4 \] (31)

where

\[ N_1 = -(m_1 + m_2 + m)r \dot{\theta}^2 \]
\[ - (m_2 + m)\left[\left[-\ddot{V} + L_1(\dot{\psi} + \dot{\theta}_1)^2\right] \cos \theta_1 \right] \]
\[ + 2V(\dot{\psi} + \dot{\theta}_1) \sin \theta_1 \]
\[ - m_2\left[\ddot{V} + L_2(\dot{\psi} + \dot{\theta}_2)^2\right] \cos \theta_2 \]
\[ - 2V(\dot{\psi} + \dot{\theta}_2) \sin \theta_2 \] \[ + GMm (r + L_1 \cos \theta_1) / R_m^3 \]
\[ + GMm_2 (r + L_1 \cos \theta_1 + L_2 \cos \theta_2) / R_2^3 \] (32)

\[ N_2 = 2(m_1 + m_2 + m)r \dot{\psi} \]
\[ + (m_2 + m)(2L_1 V(\dot{\psi} + \dot{\theta}_1) \]
\[ + 2L_1 \dot{\psi} Vr(\dot{\psi} + \dot{\theta}_1)^2 \cos \theta_1 \]
\[ + \left[\ddot{V} r - L_1 (2r \dot{\psi} \dot{\theta}_1 + r \dot{\theta}_1^2)\right] \sin \theta_1 \]
\[ + m_2\left[-2L_2 V(\dot{\psi} + \dot{\theta}_2) \right] \]
\[ - 2Vr(\dot{\psi} + \dot{\theta}_2) - 2L_2 r \dot{\psi} \cos \theta_2 \]
\[ - \left[\ddot{V} r + L_2 (2r \dot{\psi} \dot{\theta}_2 + r \dot{\theta}_2^2)\right] \sin \theta_2 \]
\[ + 2L_2 V(\dot{\psi} + \dot{\theta}_1) \]
\[ \times \cos(\theta_2 - \theta_1) \]
\[ - \left[L \ddot{V} + L_1 L_2 (\dot{\theta}_2 - \dot{\theta}_1)(2\dot{\psi} + \dot{\theta}_1 + \dot{\theta}_2)\right] \]
\[ \times \sin(\theta_2 - \theta_1) \] (33)

\[ N_3 = (m_2 + m)L_1\left[2V(\dot{\psi} + \dot{\theta}_1) \]
\[ + 2r \dot{\psi} \cos \theta_1 + r \dot{\psi}^2 \sin \theta_1 \]
\[ + m_2 L_1 \left[\left[-2V(\dot{\psi} + \dot{\theta}_2)\right] \cos(\theta_2 - \theta_1) \right] \]
\[ - \left[\ddot{V} + L_2 (\dot{\psi} + \dot{\theta}_2)^2\right] \sin(\theta_2 - \theta_1) \]
\[ - GMmL_1 r \sin \theta_1 / R_m^3 \]
\[ - GMm_2 L_1 \left[r \sin \theta_1 - L_2 \sin(\theta_2 - \theta_1)\right] / R_2^3 \] (34)

\[ N_4 = m_2 L_2\left[-2V(\dot{\psi} + \dot{\theta}_2) + 2r \dot{\psi} \cos \theta_2 \right] \]
\[ + r \dot{\psi}^2 \sin \theta_2 + 2V(\dot{\psi} + \dot{\theta}_1) \cos(\theta_2 - \theta_1) \]
\[ - \left[\ddot{V} - L_1 (\dot{\psi} + \dot{\theta}_1)^2\right] \sin(\theta_2 - \theta_1) \]
\[ - GMm_2 L_2 \left[r \sin \theta_2 + L_1 \sin(\theta_2 - \theta_1)\right] / R_2^3 \] (35)

References

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