Scaling Limit for Compressible Viscoelastic Fluids

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The convergence from a sequence of the unique global solutions to the Cauchy problems for compressible viscoelastic fluids to a unique global solution of the incompressible Navier-Stokes equations without external forces is studied for a wide class of initial data as the Mach number and the elastic coefficient go to zero simultaneously. The proofs are based on a set of conservation laws and a list of estimates which are uniform in the scaling parameter as well as a dispersive estimate for the wave equation.

1. Introduction

A multi-dimensional compressible viscoelastic flow is governed by (see Refs. 5, 7, 8, 13, 14):

\[
\begin{align*}
\rho_t + \text{div}(\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \text{div} \mathbf{u} + \nabla P(\rho) &= \text{div}(\rho \mathbf{F}^T), \\
\mathbf{F}_t + \mathbf{u} \cdot \nabla \mathbf{F} &= \nabla \mathbf{u} \mathbf{F},
\end{align*}
\]

where \( \rho \) stands for the density, \( \mathbf{u} \in \mathbb{R}^N \) \((N = 2, 3)\) the velocity, and \( \mathbf{F} \in M^{N \times N} \) (the set of \( N \times N \) matrices) the deformation gradient. The viscosity coefficients \( \mu, \lambda \) are two constants satisfying \( \mu > 0, 2\mu + N\lambda > 0 \), which ensures that the operator \(-\mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \text{div} \mathbf{u}\) is a strongly elliptic. The pressure \( P(\rho) \) is assumed to be an increasing and convex function of \( \rho \) for \( \rho > 0 \). The symbol \( \otimes \) denotes the Kronecker tensor product, \( \mathbf{F}^T \) means the transpose matrix of \( \mathbf{F} \), and the notation \( \mathbf{u} \cdot \nabla \mathbf{F} \) is understood to be \( (\mathbf{u} \cdot \nabla) \mathbf{F} \). For system (1.1), the corresponding elastic energy is chosen to be
the special form of the Hookean linear elasticity:
\[ W(F) = \frac{1}{2} |F|^2, \]
and for simplicity of our presentation, we assume that \( P'(1) = 1 \). The arguments and results of this paper can be easily applied to more general cases.

A compressible flow, via physics, behaves asymptotically like an incompressible flow when the density is approximately a constant and the velocity is small in a suitable sense. More precisely, if we scale \( \rho, u, \) and \( F \) in the following way:
\[ \rho = \rho^\varepsilon(x, \varepsilon t), \quad u = \varepsilon u^\varepsilon(x, \varepsilon t), \quad F = F^\varepsilon(x, \varepsilon t), \]
and we replace \( \mu \) and \( \lambda \) by the scaled ones as: \( \varepsilon \mu \) and \( \varepsilon \lambda \).

Where \( \varepsilon \in (0, 1) \) is a small parameter and \( \mu > 0, 2\mu + N\lambda > 0 \). Such a scaling ensures in particular that the limit equation as \( \varepsilon \to 0 \) is not an Euler type system. Under these scalings, system (1.1) becomes
\[
\begin{cases}
\rho_t^\varepsilon + \text{div}(\rho^\varepsilon u^\varepsilon) = 0, \\
(\rho^\varepsilon u^\varepsilon)_t + \text{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \mu \Delta u^\varepsilon - (\lambda + \mu) \nabla \text{div} u^\varepsilon + \frac{\text{grad} P(\rho^\varepsilon)}{\varepsilon^2} = \varepsilon \Delta u^\varepsilon + \nabla \text{div} u^\varepsilon + \nabla P(\rho^\varepsilon) / \varepsilon^2, \\
F_t^\varepsilon + u^\varepsilon \cdot \nabla F^\varepsilon = \nabla u^\varepsilon \cdot F^\varepsilon.
\end{cases}
\]

If \( \varepsilon \) is fixed, the global existence and uniqueness of strong solution had been considered in Refs. 7, 13 in the critical spaces (see the definition below) when the fluid variables are close to the equilibrium. Various conservation laws played key roles in the arguments, see for example Proposition 1.1 below. Using conservation laws (quantities) for global existence of strong solution to \((CV^\varepsilon)\) as well as their convergence are, of course, not superficial. They are part of the intrinsic properties of viscoelastic fluids. Formally, we observe that when \( \rho \to 1 \), the first equation in \((CV^\varepsilon)\) leads to: \( \text{div} u = 0 \), which is the incompressibility of a flow, and the first two terms in the second equation of \((CV^\varepsilon)\) induce the term
\[ u_t + \text{div}(u \otimes u) = u_t + u \cdot \nabla u. \]

On the other hand, the incompressible Navier-Stokes equations without external forces is giving by
\[
\begin{cases}
v_t + (v \cdot \nabla)v - \mu \Delta v + \nabla \Pi = 0, \\
\text{div} v = 0.
\end{cases}
\]
Thus one would expect that strong solutions of $(CV^\varepsilon)$ converge in suitable functional spaces to the strong solutions of $(INS)$ when $\rho$ goes to a constant say 1 as $\varepsilon$ goes to 0 provided that one can verify the right hand side of the second equation of $(CV^\varepsilon)$ goes to the gradient of a pressure due to internal elastic deformations. In other words, in such a convergence, one part of the hydrostatic pressure $\Pi$ in $(INS)$ is the “limit” of $(P(\rho)−P(1))/\varepsilon^2$ in $(CV^\varepsilon)$, and the another contribution of the pressure $\Pi$ comes from (the leading part of) the elastic force in this asymptotic limit.

This paper is devoted to the rigorous justification of the above formal reasoning of the scaling limit for global strong solutions of the compressible viscoelastic flows for a more general class of the initial data than the ones considered earlier, e.g., Ref. 9. In fact, we shall assume $\rho^0_\varepsilon = 1 + \varepsilon b^0_\varepsilon$ with $(b^0_\varepsilon, u^0_\varepsilon, F^0_\varepsilon)$ uniformly bounded in a convenient functional space as $\varepsilon$ goes to 0. Furthermore, when

$$
\rho^\varepsilon = 1 + \varepsilon b^\varepsilon,
$$

we are led to study

$$
\begin{aligned}
&b^\varepsilon_t + \frac{\text{div} u^\varepsilon}{2} = \text{div}(b^\varepsilon u^\varepsilon), \\
&(u^\varepsilon_i)_t + u^\varepsilon_i \cdot \nabla u^\varepsilon_i = \frac{\mu_\varepsilon \Delta u^\varepsilon_i + (\lambda_\varepsilon + \mu_\varepsilon) \delta_i^j \text{div} u^\varepsilon}{1 + \varepsilon b^\varepsilon} + \frac{P'(1 + \varepsilon b^\varepsilon)}{1 + \varepsilon b^\varepsilon} \frac{\partial_i}{\varepsilon} \frac{\partial_j}{\varepsilon} F^\varepsilon_{jk}, \\
&F^\varepsilon_t + u^\varepsilon \cdot \nabla F^\varepsilon = \nabla u^\varepsilon F^\varepsilon, \\
&(b^\varepsilon, u^\varepsilon, F^\varepsilon)|_{t=0} = (b^0_\varepsilon, u^0_\varepsilon, F^0_\varepsilon).
\end{aligned}
$$

(1.3)

Notice that, here we have used the fact $\text{div}(\rho^\varepsilon(F^\varepsilon)^\top) = 0$ for all positive time, which is ensured by $(CV^\varepsilon)$ and the assumption $\text{div}(\rho^0_\varepsilon(F^0_\varepsilon)^\top) = 0$ (see Ref. 13 or Lemma 6.1 of Ref. 7). We wish to show $u^\varepsilon$ to tend to $v$ which solves $(INS)$. One of the difficulties is that we need to deal with the propagation of acoustic waves with the speed $\varepsilon^{-1}$.

Despite the fact that we can eventually prove the convergence of $u^\varepsilon$ to $v$, it is not clear that what would be the limiting equation for the third equation in (1.3) when the parameter $\varepsilon$ goes to zero. Though the third equation in (1.3) plays no role in the corresponding $(INS)$ or that it can be derived from the fluid particle trajectory equation as in Ref. 10, it would be interesting to understand in the present case if there are any missing informations coded in possible limiting equations. In this paper, we will focus on the case of the whole space $\mathbb{R}^N$. The periodic domain case may be different due no dispersive estimate for the wave equations and it will be studied later.
A huge literature has been devoted to the existence of solutions for compressible fluids, \((CV^\varepsilon)\), (INS) and to the convergence of \((\rho^\varepsilon, u^\varepsilon, \mathbf{F}^\varepsilon)\) when \(\varepsilon\) goes to zero. Roughly speaking, two different heuristics have been introduced. The first one is for the class of well-prepared data. For example, 
\[
\rho_0^\varepsilon = 1 + O(\varepsilon^2), \quad \text{div} u_0^\varepsilon = O(\varepsilon), \quad \mathbf{F}_0^\varepsilon = \mathbf{F} + O(\varepsilon)
\]
for compressible viscoelastic fluids as in [9]. In this case, the time derivative of the sequence of the density at \(t = 0\) is uniformly bounded and the usual energy method works well.

The problem in this case is of parabolic and dissipative nature. The second heuristics is for a larger class of the initial data which are not so well-prepared. This is the one we considered here. Here certain dispersive estimates for wave equations may play a crucial role. And the problem is of mixed parabolic dissipative and hyperbolic dispersive nature. In this paper, we consider classical strong solutions. Hence we would need to work in critical functional spaces. To explain critical spaces, we observe that \((CV^\varepsilon)\) is invariant under the following transformation

\[
\left\{ \begin{aligned}
(\rho(x), \mathbf{u}_0(x), \mathbf{F}_0(x)) &\rightarrow (\rho(lx), lu_0(lx), \mathbf{F}_0(lx)), \\
(\rho(t, x), \mathbf{u}(t, x), \mathbf{F}(t, x)) &\rightarrow (\rho(l^2t, lx), lu(l^2t, lx), \mathbf{F}(l^2t, lx))
\end{aligned} \right.
\tag{1.4}
\]

for \(l > 0\) and with changes of the pressure law \(P\) into \(l^2P\). This suggests the following definition: A functional space \(\mathfrak{A} \subset \mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)^N \times (\mathcal{S}'(\mathbb{R}^N))^N\) is called a critical space if the associated norm is invariant under the transformation \((\rho, \mathbf{u}, \mathbf{F}) \rightarrow (\rho(l^2 \cdot), l \mathbf{u}(l^2 \cdot), \mathbf{F}(l^2 \cdot))\) (up to a constant independent of \(l\)), where \(\mathcal{S}'\) is the space of tempered distributions, i.e., the dual of the Schwartz space \(\mathcal{S}\). According to this definition, \(B_{-\alpha}^{\frac{N}{2}} \times (B_{-\alpha-1}^{\frac{N}{2}})^N \times B_{-\alpha}^{\frac{N}{2}}\) (see Section 2 for the definition of \(B_{-\alpha}^s := B_{-\alpha}^{s}(\mathbb{R}^N)\)) is a critical space.

Our motivations to use the homogeneous Besov space \(B_{-\alpha}^{\frac{N}{2}}\) with the derivative index \(\frac{N}{2}\) including following two: first, \(B_{-\alpha}^{\frac{N}{2}}\) is an algebra embedded in \(L^\infty\), which allows us to control the density and the deformation gradient from below and from above without requiring more regularity on derivatives of \(\rho\) and \(\mathbf{F}\); second, the product is continuous from \(B_{-\alpha}^{\frac{N}{2}} \times B_{-\alpha}^{\frac{N}{2}}\) to \(B_{-\alpha}^{\frac{N}{2}}\) for \(0 \leq \alpha < N\).

In Refs. 7 and 13, authors showed that (1.1) is well-posed for initial data

\[
(b_0, u_0, F_0 - I) \in B_{-\alpha}^{\frac{N}{2}} \times \left(B_{-\alpha-1}^{\frac{N}{2}} \times \left(B_{-\alpha}^{\frac{N}{2}} \times \left(B_{-\alpha-1}^{\frac{N}{2}} \right)^N \right)^N \right),
\]

where \(B_{-\alpha}^{\frac{N}{2}} = B_{-\alpha}^{\frac{N}{2}} \cap B_{-\alpha}^{\frac{N}{2}}\). For convieneces, we summarize the main result of Refs. 7 and 13 as

**Proposition 1.1.** There exists two positive constants \(\gamma\) and \(\Gamma\), such that, if \(\rho_0 - 1 \in B_{-\alpha}^{\frac{N}{2}}, u_0 \in B_{-\alpha}^{\frac{N}{2}}, F_0 - I \in B_{-\alpha}^{\frac{N}{2}}\) satisfy
Theorem 1.1 (H. Fujita and T.Kato). There exists a constant $c > 0$ such that if $v_0 \in \dot{H}^{s-1} \cap L^2$ satisfies $\|v_0\|_{\dot{H}^{s-1}} \leq c$, then (INS) has a unique solution in $L^\infty(\mathbb{R}^+; \dot{H}^{s-1}) \cap L^2(\mathbb{R}^+; \dot{H}^{s})$. 

This paper is organized as follows. In Section 2, we describe a few definitions related to Besov spaces (including homogeneous and hybrid Besov spaces) and recall a couple basic properties that will be needed later on. In Section 3, we state the main convergence result, Theorem 3.1, and give a sketch of the proof of Theorem 3.1. The final Section 4 is devoted to the details of the rigorous proof of Theorem 3.1.
2. Besov Spaces and Basics

Throughout this paper, we use $C$ for a generic constant, and denote $A \leq CB$ by $A \lesssim B$. Also we use $(\alpha_q)_{q \in \mathbb{Z}}$ to denote a sequence such that $\sum_{q \in \mathbb{Z}} \alpha_q \leq 1$. $(f|g)$ denotes the inner product of two functions $f, g$ in $L^2(\mathbb{R}^N)$. The standard summation notation over the repeated indices is adopted.

The definition of homogeneous Besov spaces is built on an homogeneous Littlewood–Paley decomposition. First, we introduce a function $\psi \in C^\infty(\mathbb{R}^N)$, supported in the shell

$$\mathcal{C} = \{ \xi \in \mathbb{R}^N : \frac{5}{6} \leq |\xi| \leq \frac{12}{5} \},$$

such that

$$\sum_{q \in \mathbb{Z}} \psi(2^{-q}\xi) = 1, \text{ if } \xi \neq 0.$$

Denoting by $h := \mathcal{F}^{-1}\psi$ the inverse Fourier transform of $\psi$, we define the dyadic blocks as follows:

$$\Delta_q f = \psi(2^{-q}D)f = 2^{qN} \int_{\mathbb{R}^N} h(2^q y) f(x-y) dy,$$

and

$$S_q f = \sum_{p \leq q-1} \Delta_p f,$$

where $D$ is the first order differential operator. The formal decomposition

$$f = \sum_{q \in \mathbb{Z}} \Delta_q f \quad (2.1)$$

is called the homogeneous Littlewood-Paley decomposition. For $s \in \mathbb{R}$ and $f \in S'(\mathbb{R}^N)$, we denote

$$\|f\|_{B^s} := \sum_{q \in \mathbb{Z}} 2^{sq} \|\Delta_q f\|_{L^2}.$$

The homogeneous Besov spaces are defined as:

**Definition 2.1.** Let $s \in \mathbb{R}$ and $m = -\left[\frac{N}{2} + 1 - s\right]$. If $m < 0$, we set

$$B^s = \left\{ f \in S'(\mathbb{R}^N) : \|f\|_{B^s} < \infty \text{ and } f = \sum_{q \in \mathbb{Z}} \Delta_q f \text{ in } S'(\mathbb{R}^N) \right\}.$$
If $m \geq 0$, we denote by $P_m$ the set of polynomials with $N$ variables of degree $\leq m$ and define

$$B^s = \left\{ f \in S'(\mathbb{R}^N)/P_m : \| f \|_{B^s} < \infty \text{ and } f = \sum_{q \in \mathbb{Z}} \Delta_q f \text{ in } S'(\mathbb{R}^N)/P_m \right\}.$$ 

Functions in $B^s$ have many good properties (for example, see Proposition 2.5 in Ref. 2). A variant of the homogeneous Besov space, so called “hybrid Besov space” was proposed to deal with the different regularity for low and high frequencies (see Refs. 2 and 3). The definition of the hybrid Besov spaces are given as follows (see Definition 2.8 in Refs. 2 or 3).

**Definition 2.2.** Let $s,t \in \mathbb{R}$. We set

$$\| f \|_{\tilde{B}^{s,t}} = \sum_{q \leq 0} 2^{qs} \| \Delta_q f \|_{L^2} + \sum_{q > 0} 2^{qt} \| \Delta_q f \|_{L^2}.$$ 

Denoting $m = - \left[ \frac{N}{2} + 1 - s \right]$, we define

$$\tilde{B}^{s,t} = \left\{ f \in S'(\mathbb{R}^N) : \| f \|_{\tilde{B}^{s,t}} < \infty \right\} \text{ if } m < 0,$$

$$\tilde{B}^{s,t} = \left\{ f \in S'(\mathbb{R}^N)/P_m : \| f \|_{\tilde{B}^{s,t}} < \infty \right\} \text{ if } m \geq 0.$$ 

**Remark 2.1.** Some remarks about the hybrid Besov spaces are in order:

- $\tilde{B}^{s,s} = B^s$;
- If $s \leq t$, then $\tilde{B}^{s,t} = B^s \cap B^t$. Otherwise, $\tilde{B}^{s,t} = B^s + B^t$. In particular, $\tilde{B}^s \hookrightarrow L^\infty$ as $s \leq \frac{N}{2}$;
- The space $\tilde{B}^{0,s}$ coincides with the usual nonhomogeneous Besov space

$$\left\{ f \in S'(\mathbb{R}^N) : \| \chi(D)f \|_{L^2} + \sum_{q \geq 0} 2^{qs} \| \Delta_q f \|_{L^2} < \infty \right\},$$

where $\chi(\xi) = 1 - \sum_{q \geq 0} \phi(2^{-q}\xi)$;
- If $s_1 \leq s_2$ and $t_1 \geq t_2$, then $\tilde{B}^{s_1,t_1} \hookrightarrow \tilde{B}^{s_2,t_2}$.

For products of functions in hybrid Besov spaces, we have (see Proposition 1.4 in Ref. 1 and Proposition 2.10 in Ref. 2):

**Proposition 2.1.** Given $s_1, s_2, t_1, t_2 \in \mathbb{R}$.

- For all $s_1, s_2 > 0$,

$$\| fg \|_{\tilde{B}^{s_1,s_2}} \lesssim \| f \|_{L^\infty} \| g \|_{\tilde{B}^{s_1,s_2}} + \| g \|_{L^\infty} \| f \|_{\tilde{B}^{s_1,s_2}}.$$
Remark 2.2. If $0 = \frac{N}{p_2} - \frac{s_1 + s_2}{2}$ and $s_1 + s_2 \geq 0$, then

$$\|fg\|_{B^{s_1 + s_2 - \frac{N}{p_2}}_{p_1, p_2}} \lesssim \|f\|_{B^s_{p_1}} \|g\|_{B^s_{p_2}}.$$ 

In order to state our existence result, we let $T > 0$, $r \in [0, \infty]$ and $X$ be a Banach space. Denote by $\mathcal{M}(0, T; X)$ the set of measurable functions on $(0, T)$ valued in $X$. For $f \in \mathcal{M}(0, T; X)$, we define

$$\|f\|_{L^r_r(X)} = \left( \int_0^T \|f(t)\|_X^r \, dt \right)^{\frac{1}{r}} \text{ if } r < \infty,$$

$$\|f\|_{L^\infty_r(X)} = \sup_{t \in (0, T)} \|f(t)\|_X.$$ 

Denote $L^r(0, T; X) = \{f \in \mathcal{M}(0, T; X) : \|f\|_{L^r_r(X)} < \infty\}$. If $T = \infty$, we use $L^r_r(\mathbb{R}^r; X)$ and $\|f\|_{L^r_r(X)}$ to denote the corresponding spaces and norms. Also we let $C([0, T], X)$ (or $C(\mathbb{R}^r, X)$) the set of continuous $X$-valued functions on $[0, T]$ (resp. $\mathbb{R}^r$). We shall further use $C_b(\mathbb{R}^r, X)$ for the set of bounded continuous $X$-valued functions.

Remark 2.2. If $f(t, x) \in L^r_r(0, T; B^s_{p, p})$, then the following equivalence holds

$$\|w(\theta)\|_{B^s_p} \approx \theta^{r - \frac{N}{p}} \|w\|_{B^s_p} \text{ for all } \theta > 0,$$  \hspace{10ex} (2.2)$$

$$\|f(\theta^a, \theta^b)\|_{L^r_r(0, \infty; B^s_p)} \approx \theta^{(s - \frac{N}{p}) - \frac{s}{2}} \|f\|_{L^r_r(0, \infty; B^s_p)}.$$ \hspace{10ex} (2.3)$$

3. Main Result and Sketch of Proof

In this section, we state the main convergence result as the parameter tends to zero and outline the sketch of the idea.

To begin with, our main convergence result as $\varepsilon \to 0$ can be described as

Theorem 3.1. Let $\gamma$ and $\Gamma$ be same as those in Proposition 1.1. Assume that $\rho_0 = 1 + \varepsilon b_0$ with $b_0 \in \dot{B}^{-1}_2$, $\rho_0 \in \dot{B}^{-1}_2$, and $\frac{p_0 - 1}{\varepsilon} \in \dot{B}^{-1}_2$ satisfy that for all $0 < \varepsilon$

$$C_0^\varepsilon \text{def=} \|b_0\|_{\dot{B}^{-1}_1} + \varepsilon \|b_0\|_{\dot{B}^{-1}_1} + \|u_0\|_{\dot{B}^{-1}_1} + \left\| \frac{\mathcal{F}_0^\varepsilon - I}{\varepsilon} \right\|_{\dot{B}^{-1}_1} + \varepsilon \left\| \frac{\mathcal{F}_0^\varepsilon - I}{\varepsilon} \right\|_{\dot{B}^{-1}_1} \leq C_0 \varepsilon,$$ 

$$\leq 1,$$
and $Pu_0^\varepsilon$ converges to $v_0$ with $\text{div} v_0 = 0$ and $\|v_0\|_{B^{\frac{N}{2}-1}} \leq \gamma$. Then there is a unique global solution $(\rho^\varepsilon, u^\varepsilon, F^\varepsilon)$ of $(CV^\varepsilon)$ such that

$$C^\varepsilon \overset{def}{=} \|b^\varepsilon\|_{B^{\frac{N}{2}-1}} + \varepsilon\|b^\varepsilon\|_{B^{\frac{N}{2}-1}} + \|\frac{F^\varepsilon - I}{\varepsilon}\|_{B^{\frac{N}{2}-1}} + \varepsilon\|\frac{F^\varepsilon - I}{\varepsilon}\|_{B^{\frac{N}{2}-1}} \leq \Gamma \gamma.$$

Moreover the density $\rho^\varepsilon$, the velocity $u^\varepsilon$, and the variance of the deformation gradient $F^\varepsilon - I$ converge to 1, $v$ and 0, respectively, where $v$ is the unique solution to (INS) with initial data $v_0$.

**Remark 3.1.** We note that if $(b, \pi, F)$ is a solution of (1.1), then $b^\varepsilon = \varepsilon^{-1}b(x/\varepsilon, t/\varepsilon^2)$, $u^\varepsilon = \varepsilon^{-1}\pi(x/\varepsilon, t/\varepsilon^2)$ and $F^\varepsilon = F(x/\varepsilon, t/\varepsilon^2)$ is a solution of $(CV^\varepsilon)$. Moreover, the norms $C_0^\varepsilon$ and $C^\varepsilon$ correspond exactly to the norm defined in Prop. 1.1. Hence the global existence described in the theorem 3.1 follows from Prop. 1.1, see also section 4.1 below for details.

Theorem 3.1 implies that the first two equations in $(CV^\varepsilon)$ will converge to their counterparts in incompressible Navier-Stokes equations, while the variance of the deformation gradient from the identity matrix will converge to zero. However, the kind of convergences we can establish here does not ensure what would be the limit of the third equation in $(CV^\varepsilon)$. It is partially due to the high singular term $\frac{\nabla u^\varepsilon}{\varepsilon}$ in the equations and the lack of *a priori* estimates on the time derivative of $\frac{F^\varepsilon - I}{\varepsilon}$.

**Remark 3.2.** In the case of initial-boundary value problems, the rigorously verification will depend on the estimates near the spatial boundary that will be discussed elsewhere. One can also obtain various convergences and verify a similar result as Theorem 3.1 in the case of periodic domain, except that we can only obtain the weak convergence of the incompressible part of the velocity since we can not apply the dispersive estimate for wave equations, see also Ref. 1.

**Remark 3.3.** The system $(CV^\varepsilon)$ is similar to the singular limit of compressible magnetohydrodynamic fluids (MHD) with zero magnetic diffusivity in Ref. 12 as the Mach number and the Alfvén number go to zero simultaneously. As discussed in Ref. 12, when the Alfvén number goes to zero, one can obtain a uniform bound for the Lorentz forces, but can not guarantee *a priori* convergence of the Lorentz forces. Here we are able to verify the convergence of the elastic forces as the parameter goes to zero. What has played key roles in our arguments is a set of local conservation
laws (quantities), see Proposition 3.1. The extension of our result to MHD may become possible now because of the recent global well-posedness result for the incompressible MHD equations with zero magnetic diffusivity and small initial data established in Ref. 11.

We now outline the proofs of the Theorem 3.1. First of all, we recall conservation laws or local conserved quantities for compressible viscoelastic fluids (see, for example, Refs. 7 and 13).

**Proposition 3.1.** Assume that \((\rho^\varepsilon, \mathbf{u}^\varepsilon, F^\varepsilon)\) is a solution of the system \((CV^\varepsilon)\). Then the following identities
\[
\text{div}(\rho^\varepsilon (F^\varepsilon)^\top) = 0, \tag{3.1}
\]
and
\[
(F^\varepsilon)_{ik} \nabla_l (F^\varepsilon)_{ij} = (F^\varepsilon)_{lj} \nabla_l (F^\varepsilon)_{ik} \tag{3.2}
\]
hold for all time \(t > 0\) if they are true initially.

Thanks to Proposition 3.1, we shall reformulate \((CV^\varepsilon)\). First, one notices that
\[
F^\varepsilon_{jk} \partial_j F^\varepsilon_{ik} = \text{div} F^\varepsilon + (\delta^\varepsilon_{jk} - \delta_{jk}) \nabla_j F^\varepsilon_{ik} \tag{3.3}
\]
where
\[
\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j; \\ 1, & \text{if } i = j. \end{cases}
\]
Since \text{div curl } f = 0 for any vector-valued function \(f\), one has
\[
(-\Delta)^{-1} \text{div curl } (F^\varepsilon - I) = 0.
\]

On the other hand, we have
\[
\begin{align*}
(-\Delta)^{-1} \text{div curl } (F^\varepsilon - I) &= (-\Delta)^{-1} \text{div} (\partial_{x_k} \partial_{x_j} (F^\varepsilon_{ij} - \delta_{ij}) - \partial_{x_j} \partial_{x_i} (F^\varepsilon_{kj} - \delta_{kj})) \\
&\overset{\text{(3.2)}}{=} (-\Delta)^{-1} \text{div} (\partial_{x_j} \partial_{x_i} (F^\varepsilon_{ik} - \delta_{ik}) - \partial_{x_i} \partial_{x_j} (F^\varepsilon_{ki} - \delta_{ki})) \\
&\quad + (-\Delta)^{-1} \text{div} \partial_{x_j} \left( F^\varepsilon_{ij} - \delta_{ij} \right) \nabla_l F^\varepsilon_{lk} - (F^\varepsilon_{ik} - \delta_{ik}) \nabla_l F^\varepsilon_{lj} \\
&\quad - (F^\varepsilon_{ij} - \delta_{ij}) \nabla_l F^\varepsilon_{ki} + (F^\varepsilon_{li} - \delta_{li}) \nabla_l F^\varepsilon_{kj} \\
&= \text{div}(F^\varepsilon - I) - \text{div}((F^\varepsilon)^\top - I) + \mathcal{N}^\varepsilon
\end{align*}
\]
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Thus we obtain
\[ \nabla \cdot (\mathbf{F}^\varepsilon - I) = \nabla \cdot (\mathbf{F}^\varepsilon) - \nabla \cdot \mathbf{N}^\varepsilon. \] (3.4)

By (3.1), one has
\[ \nabla \cdot (\mathbf{F}^\varepsilon) - I) = -\varepsilon \nabla \mathbf{b}^\varepsilon - \varepsilon \nabla \cdot (\mathbf{F}^\varepsilon) - I) \mathbf{b}^\varepsilon. \] (3.5)

Thanks to (3.3), (3.4), and (3.5), we can rewrite (1.3) as
\[
\begin{align*}
\frac{\partial \mathbf{b}^\varepsilon}{\partial t} + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{b}^\varepsilon = & \quad -\nabla \cdot (\mathbf{b}^\varepsilon \mathbf{u}^\varepsilon), \\
\left( \mathbf{u}^\varepsilon \right)_t + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon = & \quad \mu \Delta \mathbf{u}^\varepsilon + \frac{\lambda + \mu}{\varepsilon} \partial_i \text{div} \mathbf{u}^\varepsilon + \left( 1 + \frac{\varepsilon (1 + \varepsilon \mathbf{b}^\varepsilon)}{1 + \varepsilon \mathbf{b}^\varepsilon} \right) \frac{\partial \mathbf{b}^\varepsilon}{\partial t} = \mathcal{M}^\varepsilon, \\
\mathbf{F}^\varepsilon_t + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{F}^\varepsilon = & \quad \nabla \mathbf{u}^\varepsilon \mathbf{F}^\varepsilon, \\
(\mathbf{b}^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{F}^\varepsilon)|_{\varepsilon = 0} = & \quad (\mathbf{b}_0^\varepsilon, \mathbf{u}_0^\varepsilon, \mathbf{F}_0^\varepsilon)
\end{align*}
\] (3.6)

with
\[
\mathcal{M}^\varepsilon = \frac{1}{\varepsilon^2} \left( (\mathbf{F}^\varepsilon - \delta_{ij}) \nabla_i \mathbf{F}^\varepsilon_{jk} - \varepsilon \nabla \cdot (\mathbf{F}^\varepsilon - I) \mathbf{b}^\varepsilon \right) - \mathcal{N}^\varepsilon.
\]

To obtain uniform estimates, let us split the velocity into a divergence-free part \( \mathcal{P} \mathbf{u}^\varepsilon \) and a gradient part \( \mathcal{Q} \mathbf{u}^\varepsilon \). The operator \( \mathcal{P} \) is the projection to the divergence-free vectors, and the operator \( \mathcal{Q} \) is the projection to its orthogonal compliment, that is, the curl-free vectors. They can be defined as
\[ \mathcal{P} = I - \nabla \Delta^{-1} \text{div}, \quad \mathcal{Q} = I - \mathcal{P}, \]
where \( I \) is the identity operator. The equation \((\mathcal{C} \mathcal{V}^\varepsilon)\) can then be split according to this decomposition of the velocity:
\[
\begin{align*}
\partial_t \mathbf{b}^\varepsilon + \frac{\text{div} \mathbf{Q} \mathbf{u}^\varepsilon}{\varepsilon} = & \quad -\text{div}(\mathbf{b}^\varepsilon \mathbf{u}^\varepsilon), \\
\partial_t \mathbf{Q} \mathbf{u}^\varepsilon - \nu \Delta \mathbf{Q} \mathbf{u}^\varepsilon + \frac{2 \nabla \mathbf{b}^\varepsilon}{\varepsilon} = & \quad \mathcal{Q} \left( \mathcal{M} - \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon - \frac{\varepsilon b^\varepsilon}{1 + \varepsilon b^\varepsilon} \mathbf{A} \mathbf{u}^\varepsilon - K(\varepsilon b^\varepsilon) \frac{\nabla b^\varepsilon}{\varepsilon} \right), \\
\partial_t \mathcal{P} \mathbf{u}^\varepsilon - \nu \Delta \mathcal{P} \mathbf{u}^\varepsilon = & \quad \mathcal{P} \left( \mathcal{M} - \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon - \frac{\varepsilon b^\varepsilon}{1 + \varepsilon b^\varepsilon} \mathbf{A} \mathbf{u}^\varepsilon \right), \\
\mathbf{F}^\varepsilon_t + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{F}^\varepsilon = & \quad \nabla \mathbf{u}^\varepsilon \mathbf{F}^\varepsilon, \\
(\mathbf{b}^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{F}^\varepsilon)|_{\varepsilon = 0} = & \quad (\mathbf{b}_0^\varepsilon, \mathbf{u}_0^\varepsilon, \mathbf{F}_0^\varepsilon)
\end{align*}
\] (3.7a-3.7e)
where the vector $M$ and the function $K$ are defined as

$$M^\varepsilon \overset{\text{def}}{=} (M_1^\varepsilon, M_2^\varepsilon, M_3^\varepsilon),$$

$$K(z) \overset{\text{def}}{=} \frac{P'(1 + z)}{1 + z} - 1$$

with $K(0) = 0$ and $A = \mu \Delta + (\lambda + \mu) \nabla \div$. Here the implicit summation over repeated indices is assumed throughout this paper.

We note that in the equations for the incompressible part $\mathcal{P}u^\varepsilon$, there is no linear coupling with other three equations in the system (3.7), and hence it is reasonable to believe that one may obtain a control on $\mathcal{P}u^\varepsilon$ in view of the standard theory for parabolic equations. On the other hand, there is a strong coupling between other three equations in the system (3.7). To overcome the difficulties, we will use in particular some dispersive estimates for the wave equations (the so-called Strichartz estimates, see Ref. 1 and references therein). More precisely, the following lemma will be needed in our arguments. One may find its proof in Proposition 7.1 in Ref. 1.

**Lemma 3.1 (Strichartz estimates).** Let $(b, d)$ be a solution of the following system

$$\begin{aligned}
\partial_t b + \frac{\Delta d}{\varepsilon} &= J, \\
\partial_t d - \frac{\Delta b}{\varepsilon} &= K.
\end{aligned}$$

Then for any $s \in \mathbb{R}$ and positive time $T$ (possible infinite), the following estimate holds

$$\|(b, d)\|_{L^r_t(B^{s-N(1/p-1/2)+1/r}_p)} \lesssim \varepsilon^{1/r} \left( \|(b_0, d_0)\|_{B^s_2} + \|(J, K)\|_{L^r_t(B^s_2)} \right)$$

with $p \geq 2$, $2/r \leq \min(1, (N - 1)(1/2 - 1/p))$ and $(r, p, N) \neq (2, \infty, 3)$.

Strichartz estimates for wave equations along with some estimates for parabolic equations will give us a list of uniform estimates on $(b^\varepsilon, u^\varepsilon, F^\varepsilon)$. The latter would be important to derive convergences of various quantities. Indeed our main result will follow from the following Proposition.

**Proposition 3.2.** Under assumptions as in Theorem 3.1, the following results hold:

- **Existence:** For all $0 < \varepsilon$, system $(C^\varepsilon)$ has a unique solution $(b^\varepsilon, u^\varepsilon, F^\varepsilon)$ in $\mathcal{B}^{\frac{3-N}{2}}_2$ such that The norm as defined in Theorem 3.1,
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$C^\varepsilon$ is bounded by $\Gamma\gamma$. (INS) has a unique solution $v \in \mathfrak{B}_p^{\frac{3}{p}}$ such that

$$
\|v\|_{L^1(B_{p}^{\frac{3}{p}+1})} + \|v\|_{L^\infty(B_{p}^{\frac{3}{p}-1})} \leq \Gamma\|v_0\|_{B_{p}^{\frac{3}{p}-1}}.
$$

- **Convergence:** If

$$
\begin{align*}
\|\mathcal{P}u^\varepsilon_0 - v_0\|_{\hat{B}_{p}^{\frac{3}{p}-1, \frac{3}{p} - \frac{1}{q}}} & \lesssim \varepsilon, \quad \text{as} \quad N = 3, \\
\|\mathcal{P}u^\varepsilon_0 - v_0\|_{\hat{B}_{p}^{\frac{3}{p}-1, \frac{3}{p} - \frac{1}{q}}} & \lesssim \varepsilon, \quad \text{as} \quad N = 2,
\end{align*}
$$

then

- If $N = 3$, for all $p \in [2, \infty)$ and $q \in (1, 2)$,

$$
\|\mathcal{P}u^\varepsilon - v\|_{L^1(\hat{B}_{p}^{\frac{3}{p}+1, \frac{3}{p} + \frac{1}{q}})} + \|\mathcal{P}u^\varepsilon - v\|_{L^\infty(\hat{B}_{p}^{\frac{3}{p}+1, \frac{3}{p} - \frac{1}{q}})}
$$

$$
+ \|(b^\varepsilon, Qu^\varepsilon)\|_{L^\infty(\hat{B}_{p}^{\frac{3}{p}+1, \frac{3}{p} - \frac{1}{q}})} \lesssim \varepsilon^{\frac{1}{2} - \frac{1}{p}},
$$

and

$$
\frac{\|F^\varepsilon - I\|_{L_{t,x}^2(\hat{B}_{p}^{\frac{3}{p}+1, \frac{3}{p} + \frac{1}{q}})}}{\varepsilon} \lesssim \varepsilon^\min(\frac{2-q}{4}, \frac{3}{2}-\frac{1}{q}).
$$

- If $N = 2$, for all $p \in [2, 6]$ and $q \in (1, 2)$,

$$
\|\mathcal{P}u^\varepsilon - v\|_{L^1(\hat{B}_{p}^{\frac{3}{p}+1, \frac{3}{p} + \frac{1}{q}})} + \|\mathcal{P}u^\varepsilon - v\|_{L^\infty(\hat{B}_{p}^{\frac{3}{p}+1, \frac{3}{p} - \frac{1}{q}})}
$$

$$
+ \|(b^\varepsilon, Qu^\varepsilon)\|_{L^\infty(\hat{B}_{p}^{\frac{3}{p}+1, \frac{3}{p} - \frac{1}{q}})} \lesssim \varepsilon^{\frac{1}{2} - \frac{1}{p}},
$$

and

$$
\frac{\|F^\varepsilon - I\|_{L_{t,x}^2(\hat{B}_{p}^{\frac{3}{p}+1, \frac{3}{p} + \frac{1}{q}})}}{\varepsilon} \lesssim \varepsilon^\min(\frac{2-q}{4}, \frac{3}{2}-\frac{1}{q}).
$$

To prove Proposition 3.2, we first use an appropriate change of variables which enables us to apply Proposition 1.1. Under the smallness assumptions, we get in particular global solutions $(b^\varepsilon, u^\varepsilon, F^\varepsilon)$ in $\mathfrak{B}_p^{\frac{3}{p}}$ with a list of estimates which are uniform in $\varepsilon$. Next, as these uniform estimates will lead to uniform bounds in $L^1(\mathbb{R}^+; B_{p}^{\frac{3}{p}-1})$ for the right-hand side of the first two equations in (3.7), and for $\nu\Delta Qu^\varepsilon$, then we use Lemma 3.1 to conclude that $(b^\varepsilon, Qu^\varepsilon)$ converges to zero. Also, these uniform estimates enable us to prove the nonlinear part $M$ will converge to 0 as $\varepsilon \to 0$. Finally, the estimates for heat equations allow us to verify that $\mathcal{P}u^\varepsilon \to v$ in a suitable functional space. Then the strong convergence will be followed by a classical interpolation method with the aid of these uniform estimates.
4. Proof of Proposition 3.2

The proof will be divided into several steps.

4.1. Existence of global solution for \((CV^\varepsilon)\) and uniform estimates

To this end, let us introduce the following change of unknowns and change of variables:
\[
\begin{align*}
\overline{\rho} & \overset{def}{=} \varepsilon b(\varepsilon^2 t, \varepsilon x), \\
\overline{u} & \overset{def}{=} \varepsilon u(\varepsilon^2 t, \varepsilon x) \quad \text{and} \quad \overline{F} \overset{def}{=} F(\varepsilon^2 t, \varepsilon x).
\end{align*}
\]

Then \((b, u, F)\) solves \((1.3)\) if and only if \((\overline{\rho}, \overline{u}, \overline{F})\) solves
\[
\begin{align*}
\overline{\rho}_t + \text{div}\overline{u} &= -\text{div}(\overline{\rho}\overline{u}), \\
(\overline{u}_t) &+ \overline{u} \cdot \nabla \overline{u} - \frac{\mu \Delta \overline{u}}{1+\varepsilon^2} \text{div} \overline{u} + \frac{F'(b) - 1}{1+\varepsilon^2} \partial_i \overline{\rho} = aF_j k \overline{F}_{ik}, \\
\overline{F}_t + \overline{u} \cdot \nabla \overline{F} &= \nabla \overline{u} \overline{F}, \\
(\overline{\rho}, \overline{u}, \overline{F})|_{t=0} &= (\overline{\rho}_0, \overline{u}_0, \overline{F}_0).
\end{align*}
\]

(4.1)

According to Proposition 1.1, there exist two positive constants
\[
\gamma = \gamma(N, \mu, \lambda, P) \quad \text{and} \quad \Gamma = \Gamma(N, \mu, \lambda, P)
\]
such that \((4.1)\) has a unique solution in \(\mathfrak{B}^{\overline{\varepsilon}}\) satisfying
\[
\|\overline{\rho}, \overline{u}, \overline{F} \|_{\mathfrak{B}^{\overline{\varepsilon}}} \leq \Gamma \|(\overline{\rho}_0, \overline{u}_0, \overline{F}_0)\|_{\mathfrak{B}^{\overline{\varepsilon}}},
\]

(4.2)

provided that
\[
C_0 \overset{def}{=} \|\overline{\rho}_0\|_{\mathfrak{B}^{\overline{\varepsilon}-1}} + \|\overline{u}_0\|_{\mathfrak{B}^{\overline{\varepsilon}-1}} + \|\overline{F}_0 - I\|_{\mathfrak{B}^{\overline{\varepsilon}-1}} \leq \gamma.
\]

(4.3)

Due to the scaling properties \((2.2), (2.3)\), we easily obtain
\[
\begin{align*}
\|\overline{\rho}_0\|_{\mathfrak{B}^{\overline{\varepsilon}-1}} + \|\overline{u}_0\|_{\mathfrak{B}^{\overline{\varepsilon}-1}} + \|\overline{F}_0 - I\|_{\mathfrak{B}^{\overline{\varepsilon}-1}} &= \|b_0\|_{\mathfrak{B}^{\overline{\varepsilon}-1}} + \varepsilon \|b_0\|_{\mathfrak{B}^{\overline{\varepsilon}} - \varepsilon} + \|u_0\|_{\mathfrak{B}^{\overline{\varepsilon}-1}} + \varepsilon \|F_0 - I\|_{\mathfrak{B}^{\overline{\varepsilon}}} + \varepsilon \|\overline{u}_0\|_{\mathfrak{B}^{\overline{\varepsilon}}} + \varepsilon \|\overline{F}_0 - I\|_{\mathfrak{B}^{\overline{\varepsilon}}},
\end{align*}
\]

(4.4)

and
\[
\begin{align*}
\|\overline{\rho}, \overline{u}, \overline{F} \|_{\mathfrak{B}^{\overline{\varepsilon}}} &\geq \varepsilon \|b^\varepsilon\|_{L^\infty(0, \infty; \mathfrak{B}^{\overline{\varepsilon}})} + \|b^\varepsilon\|_{L^\infty(0, \infty; \mathfrak{B}^{\overline{\varepsilon}-1})} + \varepsilon \|F^\varepsilon - I\|_{L^\infty(0, \infty; \mathfrak{B}^{\overline{\varepsilon}})} \nonumber \\
&+ \|\overline{u}_0\|_{\mathfrak{B}^{\overline{\varepsilon}}} + \|u_0\|_{\mathfrak{B}^{\overline{\varepsilon}}} + \|\overline{F}_0 - I\|_{\mathfrak{B}^{\overline{\varepsilon}}},
\end{align*}
\]

(4.5)
Moreover, by the interpolation
\[ \|w\|_{B^s} \leq \|w\|_{B^{s-1}}^{\frac{1}{2}} \|w\|_{B^{s+1}}^{\frac{1}{2}}, \]
on one has
\[ \|w\|_{L^2(0,\infty;B^{s})} \leq \|w\|_{L^\infty(0,\infty;\tilde{B}^{s-1})}^{\frac{1}{2}} \|w\|_{L^1(0,\infty;\tilde{B}^{s+1})}^{\frac{1}{2}}. \]
Hence, from (4.2) and the definition of \( B^{N} \), we obtain that
\[ \|\rho\|_{L^2(0,\infty;B^{N})} + \|F - I\|_{L^2(0,\infty;B^{N})} \leq C\Gamma \gamma. \]

Based on these uniform estimates, one can show that

**Lemma 4.1.** For these solutions, \( M^\varepsilon \) are uniformly bounded in \( L^1(B^{N-1}) \). More precisely, we have
\[ \|M^\varepsilon\|_{L^1(B^{N-1})} \leq C\varepsilon^2 \gamma^2. \]

**Proof.** Recall that
\[ M^\varepsilon = \frac{1}{\varepsilon^2} \left( (F^\varepsilon_{jk} - \delta_{jk}) \nabla_j F^\varepsilon_{ik} - \varepsilon \text{div}((F^\varepsilon)^T - I)b^\varepsilon - N^\varepsilon \right). \]
To prove this lemma, we estimate \( M^\varepsilon \) term by term. First, for
\[ \frac{1}{\varepsilon^2} \left( (F^\varepsilon_{jk} - \delta_{jk}) \nabla_j F^\varepsilon_{ik} \right), \]
we have, by (4.6)
\[ \| \frac{1}{\varepsilon^2} \left( (F^\varepsilon_{jk} - \delta_{jk}) \nabla_j F^\varepsilon_{ik} \right) \|_{L^1(B^{N-1})} \leq \frac{1}{\varepsilon^2} \left( (F^\varepsilon_{jk} - \delta_{jk}) \nabla_j F^\varepsilon_{ik} - \delta_{ik} \right) \|_{L^1(B^{N-1})} \leq \frac{1}{\varepsilon^2} \left( (F^\varepsilon - I) \varepsilon \right) \|_{L^1(B^{N-1})} \leq \frac{1}{\varepsilon^2} \left( (F^\varepsilon - I) \varepsilon \right) \|_{L^1(B^{N-1})} \leq CT^2 \gamma^2. \]
Secondly, for the term \( \frac{1}{\varepsilon} \text{div}((F^\varepsilon)^\top - I)b^\varepsilon \), we observe that, by (4.6)
\[
\left\| \frac{1}{\varepsilon} \text{div}((F^\varepsilon)^\top - I)b^\varepsilon \right\|_{L^1(B_1^{1/2} - 1)} \lesssim \left\| \frac{F^\varepsilon - I}{\varepsilon} b^\varepsilon \right\|_{L^1(B_1^{1/2})} \\
\lesssim \left\| \frac{F^\varepsilon - I}{\varepsilon} \right\|_{L^2(B_1^{1/2})} \left\| b^\varepsilon \right\|_{L^2(B_1^{1/2})} \leq C\varepsilon^2 \gamma^2.
\]
(4.8)

Finally, for \( \frac{1}{\varepsilon^2} N^\varepsilon \), we have, by (4.6)
\[
\left\| \frac{1}{\varepsilon^2} N^\varepsilon \right\|_{L^1(B_1^{1/2} - 1)} \lesssim \left\| \frac{F^\varepsilon - I}{\varepsilon} \nabla \left( \frac{F^\varepsilon - I}{\varepsilon} \right) \right\|_{L^1(B_1^{1/2} - 1)} \\
\lesssim \left\| \frac{F^\varepsilon - I}{\varepsilon} \right\|_{L^2(B_1^{1/2})} \left\| \nabla \left( \frac{F^\varepsilon - I}{\varepsilon} \right) \right\|_{L^2(B_1^{1/2} - 1)} \lesssim \left\| \frac{F^\varepsilon - I}{\varepsilon} \right\|_{L^2(B_1^{1/2})}^2 \leq C\varepsilon^2 \gamma^2.
\]
(4.9)
This completes the proof.

From the bounds of \( M \) and \( u^\varepsilon \) in \( L^1(B_1^{1/2} - 1) \), equation (3.7c) implies that \( \partial_t P u^\varepsilon \) is also bounded in \( L^1(B_1^{1/2} - 1) \). Also, as an easy consequence of the uniform bound on \( N^\varepsilon \), one obtains the uniform bound on the divergece free part of \( \frac{1}{\varepsilon} \text{div}(F^\varepsilon - I), \ P \frac{1}{\varepsilon} \text{div}(F^\varepsilon - I), \) in \( L^1(B_1^{1/2} - 1) \). More precisely, we have

**Corollary 4.1.** For the sequence \( \{F^\varepsilon\}_{\varepsilon > 0} \), the following bound holds
\[
\left\| \frac{1}{\varepsilon} P \text{div} \left( \frac{F^\varepsilon - I}{\varepsilon} \right) \right\|_{L^1(B_1^{1/2} - 1)} \leq C\varepsilon^2 \gamma^2.
\]

**Proof.** Indeed, from (3.4) and (3.5), one deduces that
\[
\text{div}(F^\varepsilon - I) = -\varepsilon \nabla b^\varepsilon - \varepsilon \text{div}(((F^\varepsilon)^\top - I)b^\varepsilon) - N^\varepsilon,
\]
and hence
\[
\frac{1}{\varepsilon} P \text{div} \left( \frac{F^\varepsilon - I}{\varepsilon} \right) = -P \text{div} \left( \frac{(F^\varepsilon)^\top - I}{\varepsilon} b^\varepsilon \right) - P \frac{N^\varepsilon}{\varepsilon^2}.
\]

Inequalities (4.8) and (4.9) implies that the right-hand side of the above identity is bounded by \( C\varepsilon^2 \gamma^2 \) in \( L^1(B_1^{1/2} - 1) \), and thus
\[
\left\| \frac{1}{\varepsilon} P \text{div} \left( \frac{F^\varepsilon - I}{\varepsilon} \right) \right\|_{L^1(B_1^{1/2} - 1)} \leq C\varepsilon^2 \gamma^2.
\]
This finishes the proof.
4.2. Convergence of $b^\varepsilon$ and $Qu^\varepsilon$

The convergence of $b^\varepsilon$ and $Qu^\varepsilon$ are based on the following lemma. In the rest of this paper, the space $B^r_p$ will be denoted by $B^r_p$ for convenience.

**Lemma 4.2.** Let $C^0_\varepsilon$ is defined as (4.3). The solutions obtained in the previous subsection satisfy

- If $N = 3$: For all $2 \leq p < \infty$,
  \[
  \| (b^\varepsilon, Qu^\varepsilon) \|_{L^{2p\frac{2}{p}-1}((B^r_p)^2)} \lesssim \varepsilon^{\frac{1}{p} - \frac{1}{2}};
  \]

- If $N = 2$: For all $2 \leq p \leq \infty$,
  \[
  \| (b^\varepsilon, Qu^\varepsilon) \|_{L^{4p\frac{3}{p}-1}((B^r_p)^2)} \lesssim \varepsilon^{\frac{1}{4p} - \frac{1}{2}}.
  \]

**Proof.** Let

\[
 d^\varepsilon \text{ def } = \Lambda^{-1} \text{div} Qu^\varepsilon,
\]

we have, according to (3.7)

\[
\begin{cases}
  \partial_t b^\varepsilon + \frac{\Delta d^\varepsilon}{\varepsilon} = J, \\
  \partial_t d^\varepsilon - 2\Lambda b^\varepsilon = K,
\end{cases}
\]

(4.10)

with

\[
 J \text{ def } = -\text{div}(b^\varepsilon u^\varepsilon),
\]

and

\[
 K \text{ def } = \nu \Delta d^\varepsilon - \Lambda^{-1} \text{div} \left( u^\varepsilon \cdot \nabla u^\varepsilon + \frac{\varepsilon b^\varepsilon}{1+\varepsilon b^\varepsilon} A u^\varepsilon + \frac{K(\varepsilon b^\varepsilon) \nabla b^\varepsilon}{\varepsilon} - \mathcal{M}^\varepsilon \right).
\]

One notices that $Qu^\varepsilon = -\nabla \Lambda^{-1} d^\varepsilon$, and hence estimations for $Qu^\varepsilon$ or $d^\varepsilon$ are equivalent (up to harmless constants). Applying Lemma 3.1 to (4.10) with $s = \frac{N}{p} - 1$, $2 \leq p < \infty$ and $r = \frac{2p}{p-2}$ if $N = 3$ or $2 \leq p \leq \infty$ and $r = \frac{4p}{p-2}$ if $N = 2$, we readily obtain estimates in Lemma 4.2, whenever

\[
 \| (J, K) \|_{L^1((B^r_p)^{-1})} \lesssim C \eta^2 \gamma^2,
\]

(4.11)

is valid. Here we have observed that by (4.4), one has

\[
 \|(b^\varepsilon_0, d^\varepsilon_0)\|_{B^{\frac{N}{p} - 1}} \lesssim C \eta^2 \gamma^2.
\]
To verify (4.11), we need to use uniform estimates obtained in the previous subsection. Indeed, one can derive by using (4.4)–(4.6) that
\[\|\hat{J}\|_{L^1(B^{\frac{2}{3}-1})} \lesssim \|b^c\|_{L^1(B^{\frac{2}{3}})} \lesssim \|b^c\|_{L^2(B^{\frac{2}{3}})} \lesssim \|u^c\|_{L^2(B^{\frac{2}{3}})} \lesssim \|u^c\|_{L^\infty(B^{\frac{2}{3}-1})} \lesssim \|u^c\|_{L^1(B^{\frac{2}{3}-1})} \lesssim \|u^c\|_{L^2(B^{\frac{2}{3}+1})} \lesssim CT^2\gamma^2,\]
\[\|\Delta d^c\|_{L^1(B^{\frac{2}{3}-1})} \lesssim \|d^c\|_{L^1(B^{\frac{2}{3}+1})} \lesssim \|u^c\|_{L^1(B^{\frac{2}{3}+1})} \lesssim CT^2\gamma^2,\]
\[\|u^c \cdot \nabla u^c\|_{L^1(B^{\frac{2}{3}-1})} \lesssim \|u^c\|_{L^2(B^{\frac{2}{3}})} \|\nabla u^c\|_{L^2(B^{\frac{2}{3}-1})} \lesssim CT^2\gamma^2,\]
\[\left\| \frac{\varepsilon b^c}{1 + \varepsilon b^c} Au^c \right\|_{L^1(B^{\frac{2}{3}-1})} \lesssim \varepsilon \|b^c\|_{L^\infty(B^{\frac{2}{3}})} \|Au^c\|_{L^1(B^{\frac{2}{3}-1})} \lesssim \varepsilon \|b^c\|_{L^\infty(B^{\frac{2}{3}})} \|u^c\|_{L^1(B^{\frac{2}{3}+1})} \lesssim CT^2\gamma^2.\]
Combining above estimates with Lemma 4.1, one concludes (4.11). This completes the proof. \[\square\]

### 4.3. Convergence of \(\frac{F^\varepsilon - I}{\varepsilon}\)

Based on the previous bounds on \(b^c\) and estimates (4.8), (4.9) and (3.2), one can obtain the following bound for \(\frac{F^\varepsilon - I}{\varepsilon}\).

**Lemma 4.3.** Let \(\Gamma, \gamma\) be defined as in Theorem 3.1. For \(1 < q < 2\) and \(p \geq 2\), the solution obtained in the previous subsection satisfies that if \(N = 3\)
\[\|F^\varepsilon - I\|_{L^q(B^{\frac{2}{3}+\frac{3}{q}})} \lesssim \varepsilon^\min\{\frac{2-n}{q}, \frac{1}{2} - \frac{1}{p}\},\]
and if \(N = 2\)
\[\|F^\varepsilon - I\|_{L^q(B^{\frac{2}{3}+\frac{3}{q}})} \lesssim \varepsilon^\min\{\frac{2-n}{q}, \frac{1}{2} - \frac{1}{p}\}.\]
Therefore, \(\frac{F^\varepsilon - I}{\varepsilon}\) converges to 0 in corresponding spaces as \(\varepsilon \to 0\).
Proof. In fact, from (3.4) and (3.5), one has
\[
\text{div} \left( \frac{F^\varepsilon - I}{\varepsilon} \right) = -\nabla b^\varepsilon - \varepsilon \text{div} \left( \frac{(F^\varepsilon)^T - I}{\varepsilon} b^\varepsilon \right) - \frac{1}{\varepsilon} N^\varepsilon.
\]
Notice that estimates (4.8) and (4.9) imply
\[
\varepsilon \left\| \text{div} \left( \frac{(F^\varepsilon)^T - I}{\varepsilon} b^\varepsilon \right) \right\|_{L^1(B^{1-1})} \lesssim \varepsilon \Gamma^2 \gamma^2,
\]
and
\[
\left\| \frac{1}{\varepsilon} N^\varepsilon \right\|_{L^1(B^{1-1})} \lesssim \varepsilon \Gamma^2 \gamma^2.
\]
Hence, we have
\[
\left\| \text{div} \left( \frac{F^\varepsilon - I}{\varepsilon} + b^\varepsilon I \right) \right\|_{L^1(B^{1-1})} \lesssim \varepsilon. \tag{4.12}
\]
Notice that if $2 \leq p < \infty$, one has, see Proposition 1.3.5 in Ref. 4
\[
\begin{cases}
B^p \hookrightarrow B^{\frac{p}{p-1}}_p \hookrightarrow B^{\frac{p}{p-1} - \frac{q}{q-2}}_p, & \text{if } N = 3; \\
B^0 \hookrightarrow B^{\frac{p}{p-1}}_p \hookrightarrow B^{\frac{p}{p-1} - \frac{q}{q-2}}_p, & \text{if } N = 2.
\end{cases}
\]
Therefore, if $N = 3$, using (4.6) and (4.12), we have, as $1 < q < 2$
\[
\left\| \text{div} \left( \frac{F^\varepsilon - I}{\varepsilon} + b^\varepsilon I \right) \right\|_{L^q(B^{\frac{q}{q-1} - \frac{q}{q-2}}_p)} \lesssim \left\| \text{div} \left( \frac{F^\varepsilon - I}{\varepsilon} + b^\varepsilon I \right) \right\|_{L^1(B^{1-1})} \lesssim \varepsilon^{\frac{2-q}{q}}
\]
Similarly, if $N = 2$, we have, as $1 < q < 2$
\[
\left\| \text{div} \left( \frac{F^\varepsilon - I}{\varepsilon} + b^\varepsilon I \right) \right\|_{L^q(B^{\frac{q}{q-1} - \frac{q}{q-2}}_p)} \lesssim \varepsilon^{\frac{2-q}{q}}.
\]
If $2 < p < \infty$, Proposition 1.3.5 in Ref. 4 implies
\[
\begin{cases}
B^\frac{\alpha}{\beta} \ni \frac{\alpha}{2} & \Leftrightarrow B^{\frac{\alpha}{\beta} - \frac{1}{p} - \frac{1}{q}} \ni \frac{\alpha}{2}, & \text{if } N = 3; \\
B^\frac{\alpha}{\beta} \ni \frac{\alpha}{2} & \Leftrightarrow B^{\frac{\alpha}{\beta} - \frac{1}{p} - \frac{1}{q}} \ni \frac{\alpha}{2}, & \text{if } N = 2.
\end{cases}
\]
Hence, combined with Lemma 4.2, we have if $N = 3$
\[
\left\| \text{div} \left( \frac{F^\varepsilon - I}{\varepsilon} \right) \right\|_{L^q_{\text{loc}}(B^{\frac{\alpha}{\beta} - \frac{1}{p} - \frac{1}{q}})} 
\leq \left\| \text{div} \left( \frac{F^\varepsilon - I}{\varepsilon} + b\varepsilon I \right) \right\|_{L^q(B^{\frac{\alpha}{\beta} - \frac{1}{p} - \frac{1}{q}})} + \| \nabla \varepsilon^\varepsilon \|_{L^q_{\text{loc}}(B^{\frac{\alpha}{\beta} - \frac{1}{p} - \frac{1}{q}})}
\lesssim \varepsilon^{\frac{2-q}{p}} + \| b\varepsilon \|_{L^\infty(B^{\frac{\alpha}{\beta} - \frac{1}{p}})}
\lesssim \varepsilon^{\frac{2-q}{p}} + \varepsilon^{\frac{1}{2} - \frac{1}{p}} \lesssim \varepsilon^{\min\left(\frac{2-q}{p}, \frac{1}{2} - \frac{1}{p}\right)},
\]
and similarly if $N = 2$,
\[
\left\| \text{div} \left( \frac{F^\varepsilon - I}{\varepsilon} \right) \right\|_{L^q_{\text{loc}}(B^{\frac{\alpha}{\beta} - \frac{1}{p} - \frac{1}{q}})} \lesssim \varepsilon^{\min\left(\frac{2-q}{p}, \frac{1}{2} - \frac{1}{p}\right)}. \tag{4.14}
\]
On the other hand, from (3.2), we have
\[
\text{curl} \left( \frac{F^\varepsilon - I}{\varepsilon} \right)
= \varepsilon \left( \frac{F^\varepsilon - \delta_{ij}}{\varepsilon} \right) \nabla_l \left( \frac{F^\varepsilon - \delta_{ik}}{\varepsilon} \right) - \left( \frac{F^\varepsilon - \delta_{lk}}{\varepsilon} \right) \nabla_l \left( \frac{F^\varepsilon - \delta_{lj}}{\varepsilon} \right).
\]
As for (4.7), it is easy to check that the right hand side of the above identity is bounded by $CT^2 \gamma^2 \varepsilon$ in $L^1(B^{\frac{\alpha}{\beta} - 1})$. Hence, one further concludes
\[
\left\| \text{curl} \left( \frac{F^\varepsilon - I}{\varepsilon} \right) \right\|_{L^1(B^{\frac{\alpha}{\beta} - 1})} \lesssim \varepsilon. \tag{4.15}
\]
Therefore, we have if $N = 3$

$$\left\| \text{curl} \left( \frac{F^\varepsilon - I}{\varepsilon} \right) \right\|_{L^2(B^{d-1}_r \setminus B^{d-1}_r)} \lesssim \varepsilon^{\frac{2(d-2)}{q}}.$$  \hspace{1cm} (4.16)

and similarly if $N = 2$

$$\left\| \text{curl} \left( \frac{F^\varepsilon - I}{\varepsilon} \right) \right\|_{L^2(B^{d-1}_r \setminus B^{d-1}_r)} \lesssim \varepsilon^{\frac{2(d-2)}{q}}.$$  \hspace{1cm} (4.17)

Hence, estimates (4.13) (4.14), (4.16), (4.17) and Bernstein’s inequality imply that if $N = 3$

$$\left\| \frac{F^\varepsilon - I}{\varepsilon} \right\|_{L^4(B^{d-1}_r \setminus B^{d-1}_r)} \lesssim \varepsilon^{\min\left\{\frac{2(d-2)}{q}, \frac{d}{2}\right\}},$$

and if $N = 2$

$$\left\| \frac{F^\varepsilon - I}{\varepsilon} \right\|_{L^4(B^{d-1}_r \setminus B^{d-1}_r)} \lesssim \varepsilon^{\min\left\{\frac{2(d-2)}{q}, \frac{d}{2}\right\}}.$$  \hspace{1cm} (4.18)

This finishes the proof. \hfill \Box

**Remark 4.1.** There is an alternative proof of the convergence of $\frac{F^\varepsilon - I}{\varepsilon}$ to 0. Indeed, from Corollary 4.1, we know that

$$\left\| \mathcal{P} \text{div} \left( \frac{F^\varepsilon - I}{\varepsilon} \right) \right\|_{L^1(B^{d-1}_r \setminus B^{d-1}_r)} \lesssim \varepsilon.$$  \hspace{1cm}

From (3.4) and (3.5), one deduces that

$$\mathcal{Q} \text{div} \left( \frac{F^\varepsilon - I}{\varepsilon} \right) = -\nabla b^\varepsilon - \mathcal{Q} \text{div}((F^\varepsilon)^\top - I)b^\varepsilon - \mathcal{Q} \frac{N^\varepsilon}{\varepsilon}.$$  \hspace{1cm}

It can be easily shown that

$$\left\| -\mathcal{Q} \text{div}((F^\varepsilon)^\top - I)b^\varepsilon - \mathcal{Q} \frac{N^\varepsilon}{\varepsilon} \right\|_{L^1(B^{d-1}_r \setminus B^{d-1}_r)} \lesssim \varepsilon,$$
and hence in view of Lemma 4.2, the quantity $Q\div\left(\frac{E-L}{\epsilon}\right)$ is bounded in a suitable space by a positive power of $\epsilon$. Therefore, the quantity $\div\left(\frac{E-L}{\epsilon}\right)$ is bounded in a suitable space by a positive power of $\epsilon$.

Also, as in (4.15), we know that $\curl\left(\frac{E-L}{\epsilon}\right)$ is bounded by $\epsilon$ in $L^1(B^{\frac{N}{2}-1})$. Thus, the quantity $\frac{E-L}{\epsilon}$ is bounded in a suitable space by a positive power of $\epsilon$, and thus converges to 0 in that space as $\epsilon \to 0$.

The strong convergence in Lemma 4.3, the uniform bound in $L^\infty(\mathbb{R}^+, B^{\frac{N}{2}-1})$, and the interpolations imply that $\frac{E-L}{\epsilon}$ converges to 0 in the intermediate spaces between $L^\infty(\mathbb{R}^+, B^{\frac{N}{2}-1})$ and these spaces in Lemma 4.3. Moreover, as a consequence of the convergence of $\frac{E-L}{\epsilon}$, the nonlinear terms $\mathcal{M}^\epsilon$ and $\mathcal{N}^\epsilon$ also converge to zero. In fact, we have

**Lemma 4.4.** Let $\mathcal{M}^\epsilon$ and $\mathcal{N}^\epsilon$ be defined as before. As $\epsilon \to 0$, for any $1 < q < 2$, if $N = 3$,  
\[ \mathcal{M}^\epsilon, \mathcal{N}^\epsilon \to 0 \text{ in } L^q_{\text{loc}}(B^{\frac{N}{2}-1, \frac{3}{2} - \frac{2}{q}}), \]

while if $N = 2$  
\[ \mathcal{M}^\epsilon, \mathcal{N}^\epsilon \to 0 \text{ in } L^q_{\text{loc}}(B^{\frac{N}{2}-1, \frac{3}{2} - \frac{2}{q}}). \]

**Proof.** We only prove the case as $N = 3$, and the case as $N = 2$ can be treated similarly. By the definitions of $\mathcal{M}^\epsilon$ and $\mathcal{N}^\epsilon$, it suffices to prove  
\[ \frac{1}{\epsilon^2} \left( (F^\epsilon_{jk} - \delta_{jk}) \nabla_j (F^\epsilon_{ik} - \delta_{ik}) \right) \to 0, \quad (4.18) \]

and  
\[ \frac{1}{\epsilon} \div((F^\epsilon)^\top - I) b^\epsilon) \to 0 \quad (4.19) \]

in $L^q_{\text{loc}}(B^{\frac{N}{2}-1, \frac{3}{2} - \frac{2}{q}})$ as $\epsilon \to 0$.

To see (4.18), we note that, using Proposition 2.1 and Lemma 4.3

\[
\left\| \frac{1}{\epsilon^2} \left( (F^\epsilon_{jk} - \delta_{jk}) \nabla_j (F^\epsilon_{ik} - \delta_{ik}) \right) \right\|_{L^q_{\text{loc}}(B^{\frac{N}{2}-1, \frac{3}{2} - \frac{2}{q}})} \\
\leq \left\| \frac{F^\epsilon - I}{\epsilon} \right\|_{L^{2^N-2}_{\text{loc}}(B^{\frac{N}{2}-1, \frac{3}{2} - \frac{2}{q}})} \left\| \nabla \frac{F^\epsilon - I}{\epsilon} \right\|_{L^2(B^{\frac{N}{2}})} \\
\leq \epsilon^{\min\left(\frac{3-q}{2}, \frac{3}{2} - \frac{2}{q}\right)} \left\| \frac{F^\epsilon - I}{\epsilon} \right\|_{L^2(B^{\frac{N}{2}})} \\
\leq \epsilon^{\min\left(\frac{3-q}{2}, \frac{3}{2} - \frac{2}{q}\right)},
\]
and hence
\[
\frac{1}{\varepsilon} \left( (\mathcal{F}_{jk}^\varepsilon - \delta_{jk}) \nabla_j (\mathcal{F}_{ik}^\varepsilon - \delta_{ik}) \right) \to 0 \text{ in } L_{loc}^q(B_p^{\frac{2}{q} - 1, \frac{2}{q' - 1}})
\]
as \varepsilon \to 0.

Similarly, for (4.19), we have, using Proposition 2.1 and Lemma 4.3
\[
\left\| \frac{1}{\varepsilon} \text{div}((\mathcal{F}^\varepsilon) - I) b^\varepsilon) \right\|_{L_{loc}^q(B_p^{\frac{2}{q} - 1, \frac{2}{q' - 1}})} \lesssim \left\| \frac{1}{\varepsilon} ((\mathcal{F}^\varepsilon) - I) b^\varepsilon \right\|_{L_{loc}^q(B_p^{\frac{2}{q} - 1, \frac{2}{q' - 1}})}
\]
\[
\lesssim \left\| \frac{\mathcal{F}^\varepsilon - I}{\varepsilon} \right\|_{L_{loc}^{q-2}(B_p^{\frac{2}{q} - 1, \frac{2}{q' - 1}})} \left\| b^\varepsilon \right\|_{L^2(B_p^{\frac{2}{q} - 1, \frac{2}{q' - 1}})}
\]
\[
\lesssim \varepsilon \min\left( \frac{\varepsilon}{\beta_1}, \frac{1}{\alpha_1}, \frac{1}{\lambda_1}, \frac{1}{\varepsilon} \right),
\]
and hence
\[
\frac{1}{\varepsilon} \text{div}((\mathcal{F}^\varepsilon) - I) b^\varepsilon \to 0 \text{ in } L_{loc}^q(B_p^{\frac{2}{q} - 1, \frac{2}{q' - 1}})
\]
as \varepsilon \to 0.

This finishes the proof. \qed

**Remark 4.2.** Actually, from Lemma 4.4, the quantities \( \frac{1}{\varepsilon} \text{div} \left( \frac{\mathcal{F}^\varepsilon - I}{\varepsilon} \right) \) are not only bounded in \( L^1(B_p^{\frac{2}{q} - 1}) \), but also converge to zero in \( L_{loc}^q(B_p^{\frac{2}{q} - 1, \frac{2}{q' - 1}}) \) if \( N = 3 \), and \( L_{loc}^q(B_p^{\frac{2}{q} - 1, \frac{2}{q' - 1}}) \) if \( N = 2 \) as \( \varepsilon \to 0 \).

### 4.4. Convergence of \( \mathcal{P} u^\varepsilon \)

To this end, we denote
\[
w^\varepsilon = \mathcal{P} u^\varepsilon - v.
\]
Applying Leray Projector \( \mathcal{P} \) to the second equation of \( (CV^\varepsilon) \) and subtracting (INS) from it yields a heat equation for \( w^\varepsilon \)
\[
\begin{cases}
\partial_t w^\varepsilon - \mu \Delta w^\varepsilon = H^\varepsilon, \\
w^\varepsilon |_{t=0} = 0,
\end{cases}
\]
(4.20)
where \( H^\varepsilon \) is given by
\[
H^\varepsilon \overset{def}{=} -\mathcal{P} \left( w^\varepsilon \cdot \nabla v + u^\varepsilon \cdot \nabla w^\varepsilon + Q u^\varepsilon \cdot \nabla v + u^\varepsilon \cdot \nabla Q u^\varepsilon 
+ \frac{\varepsilon}{1 + \varepsilon b^\varepsilon} A u^\varepsilon + M^\varepsilon \right).
\]

**Case 1:** \( N = 3 \). For \( 2 \leq p < \infty \), we have, by interpolation
\[
L_2^2 \left( 0, \infty; B_p \frac{2}{2 + \frac{2}{q}} \right) = \left[ L^1(0, \infty; B_p^{\frac{2}{2 + \frac{2}{q}}}); L_{loc}^q(0, \infty; B_p^{\frac{2}{q} - 1}) \right]_{\frac{2}{q} - 1}.
\]
Due to (4.5) and Lemma 4.2, we obtain
\[ \| Q_u \|_{L^2(B_{p}^{\frac{3}{p} - \frac{1}{2}})} \lesssim \| Q u \|_{L^2(B_{p}^{\frac{3}{p} - \frac{1}{2}})} \lesssim \epsilon^{\frac{1}{2} - \frac{1}{p}} \text{ for all } 2 \leq p < \infty. \] (4.21)

Next, we claim that \( w^c \) converges to 0 in \( L^p_{loc}(B_{p}^{\frac{3}{p} + 1, \frac{3}{p} + \frac{1}{2}}) \cap C \left( 0, \infty; B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}} \right) \) \( (2 \leq p < \infty) \). In fact, we only need to prove
\[ Y_p \overset{def}{=} \| w^c \|_{L^1(0, \infty; B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})} + \| w^c \|_{L^\infty(0, \infty; B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})} \lesssim \epsilon^{\frac{1}{2} - \frac{1}{p}}. \] (4.22)

In order to verify (4.22), we will apply Proposition 7.3 in Ref. 1 and estimates in two previous subsections. Actually, we have
\[
\| \mathcal{P}(w^c \cdot \nabla v) \|_{L^1(B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})} \lesssim \| \nabla v \|_{L^2(B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})} \| w^c \|_{L^2(B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})};
\]
\[
\| \mathcal{P}(u^c \cdot \nabla w^c) \|_{L^1(B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})} \lesssim \| u^c \|_{L^2(B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})} \| \nabla w^c \|_{L^2(B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})};
\]
\[
\| \mathcal{P}(Q u^c \cdot \nabla v) \|_{L^1(B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})} \lesssim \| \nabla v \|_{L^2(B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})} \| Q u^c \|_{L^2(B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})} \lesssim \epsilon^{\frac{1}{2} - \frac{1}{p}},
\]
\[
\| \mathcal{P}(u^c \cdot \nabla Q u^c) \|_{L^1(B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})} \lesssim \| u^c \|_{L^2(B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})} \| \nabla Q u^c \|_{L^2(B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})} \lesssim \epsilon^{\frac{1}{2} - \frac{1}{p}},
\]
\[
\| \mathcal{P}M \|_{L^1(B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})} \lesssim \epsilon^{\frac{1}{2} - \frac{1}{p}}.
\]

Thanks to the embedding \( B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}} \hookrightarrow B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}} \) and
\[
L^\frac{np}{n+2}(B_{p}^{\frac{3}{p} + 2}) = \left[ L^1(B_{p}^\frac{3}{p} + 1, \infty(B_{p}^{\frac{3}{p} + 1})) \right]_{\frac{2n+2}{np}},
\]
we further have, using (4.5)
\[
\| \mathcal{P} \left( \frac{\epsilon b^c}{1 + \epsilon b^c} A u^c \right) \|_{L^1(B_{p}^{\frac{3}{p} - 1, \frac{3}{p} - \frac{1}{2}})} \lesssim \| \epsilon b^c \|_{L^{\frac{n}{2}}(B_{p}^{\frac{3}{p} + 1, \frac{3}{p} + \frac{1}{2}})} \| A u^c \|_{L^{\frac{n}{2}}(B_{p}^{\frac{3}{p} + 1, \frac{3}{p} + \frac{1}{2}})} \lesssim \| \epsilon b^c \|_{L^{\frac{n}{2}}(B_{p}^{\frac{3}{p} + 1, \frac{3}{p} + \frac{1}{2}})} \| A u^c \|_{L^{\frac{n}{2}}(B_{p}^{\frac{3}{p} + 1, \frac{3}{p} + \frac{1}{2}})} \lesssim \| b^c \|_{L^\frac{np}{n+2}(B_{p}^{\frac{3}{p} + 2})} \| u^c \|_{L^\frac{np}{n+2}(B_{p}^{\frac{3}{p} + 2})} \lesssim \epsilon \left( \frac{1}{\epsilon} \right)^{\frac{1}{2} + \frac{1}{p}} \lesssim \epsilon^{\frac{1}{2} - \frac{1}{p}}.
\]
Combining these estimates together, we get
\[ Y_p \leq C \varepsilon^{\frac{3}{4} - \frac{1}{p}} + C \gamma \|w\|_{L^1(B_\rho^k \frac{4}{7} + \frac{4}{k})} \]
\[ \leq C \varepsilon^{\frac{1}{2} - \frac{1}{p}} + C \gamma Y_p, \]
and hence (4.22) follows by choosing \( \gamma \) sufficiently small.

**Case 2:** \( N = 2 \). In this case, again by the interpolation, we have
\[ L^2 \left( 0, \infty; B^{\frac{4}{7} + \frac{1}{p}}_{2p} \right) = \left[ L^1(0, \infty; B^2); L^{\frac{4}{7} + \frac{4}{k}}(0, \infty; B_k^{\frac{4}{7} + \frac{1}{p}}) \right]_{\frac{k+2}{2p}}. \]
Hence, by Lemma 4.2, we obtain
\[ \| \mathcal{Q}u^x \|_{L^2(B_\rho^k \frac{4}{7} + \frac{4}{k})} \lesssim \| \mathcal{Q}u^x \|_{L^2(B_\rho^{\frac{4}{7} + \frac{1}{p}} \frac{4}{7} + \frac{4}{k})} \lesssim \varepsilon^{\frac{1}{2} - \frac{1}{p}} \]
for all \( 2 \leq p \leq 6 \). (4.23)

Now we claim that
\[ Y_p \overset{\text{def}}{=} \|u\|_{L^1(B_\rho^{\frac{4}{7} + \frac{1}{p}} \frac{4}{7} + \frac{4}{k})} + \|w\|_{L^\infty(B_\rho^{\frac{4}{7} + \frac{1}{p}} \frac{4}{7} + \frac{4}{k})} \lesssim \varepsilon^{\frac{1}{2} - \frac{1}{p}}. \]
By the estimates for the product, we have
\[ \| \mathcal{P}(w^x \cdot \nabla v) \|_{L^1(B_\rho^{\frac{4}{7} + \frac{1}{p}} \frac{4}{7} + \frac{4}{k})} \lesssim \| \nabla v \|_{L^2(B^0)} \| w^x \|_{L^2(B_\rho^{\frac{4}{7} + \frac{1}{p}} \frac{4}{7} + \frac{4}{k})}, \]
\[ \| \mathcal{P}(u^x \cdot \nabla w^x) \|_{L^1(B_\rho^{\frac{4}{7} + \frac{1}{p}} \frac{4}{7} + \frac{4}{k})} \lesssim \| u^x \|_{L^2(B^1)} \| \nabla w^x \|_{L^2(B_\rho^{\frac{4}{7} + \frac{1}{p}} \frac{4}{7} + \frac{4}{k})}, \]
\[ \| \mathcal{P}(\mathcal{Q}u^x \cdot \nabla v) \|_{L^1(B_\rho^{\frac{4}{7} + \frac{1}{p}} \frac{4}{7} + \frac{4}{k})} \lesssim \| \nabla v \|_{L^2(B^0)} \| \mathcal{Q}u^x \|_{L^2(B_\rho^{\frac{4}{7} + \frac{1}{p}} \frac{4}{7} + \frac{4}{k})} \]
\[ \lesssim \varepsilon^{\frac{1}{2} - \frac{1}{p}}, \]
\[ \| \mathcal{P}(u^x \cdot \nabla \mathcal{Q}u^x) \|_{L^1(B_\rho^{\frac{4}{7} + \frac{1}{p}} \frac{4}{7} + \frac{4}{k})} \lesssim \| u^x \|_{L^2(B^1)} \| \nabla \mathcal{Q}u^x \|_{L^2(B_\rho^{\frac{4}{7} + \frac{1}{p}} \frac{4}{7} + \frac{4}{k})} \]
\[ \lesssim \varepsilon^{\frac{1}{2} - \frac{1}{p}}, \]
and
\[ \| \mathcal{P}\mathcal{M} \|_{L^1(B_\rho^{\frac{4}{7} + \frac{1}{p}} \frac{4}{7} + \frac{4}{k})} \lesssim \varepsilon^{\frac{1}{2} - \frac{1}{p}}. \]
On the other hand, note that as \( p \leq 6 \), \( \frac{4}{7} - \frac{1}{p} > 0 \). Moreover, thanks to the embedding \( B^{\frac{4}{7} + \frac{1}{p}}_{\frac{4}{7} + \frac{4}{k}} \leftrightarrow B^{\frac{4}{7} + \frac{1}{p}}_{\frac{4}{7} + \frac{4}{k}} \) and
\[ L^{\frac{4}{7} + \frac{4}{k}}(B^{\frac{4}{7} + \frac{1}{p}}_{\frac{4}{7} + \frac{4}{k}}) = [L^1(B^2), L^\infty(B^0)]_{\frac{k+2}{2p}}. \]
we also have
\[ \left\| \frac{\varepsilon b^\varepsilon}{1 + \varepsilon b^\varepsilon} A u \right\|_{L^1(B^{\frac{3}{p} - 1}_{p, \infty}, \frac{7}{p} + \frac{2}{p} \cdot \frac{3}{2})} \lesssim \| b^\varepsilon \|_{L^{\frac{3}{p}}(B^{\frac{3}{p} - 1}_{p, \infty}, \frac{7}{p} + \frac{2}{p} \cdot \frac{3}{2})} \| A u \|_{L^\infty(B^{\frac{1}{p} + \frac{2}{p} \cdot \frac{3}{2}})} \lesssim \varepsilon \| b^\varepsilon \|_{L^\infty(B^{\frac{3}{p} - 1}_{p, \infty}, \frac{7}{p} + \frac{2}{p} \cdot \frac{3}{2})} \| u^\varepsilon \|_{L^\infty(B^{\frac{1}{p} + \frac{2}{p} \cdot \frac{3}{2}})} \lesssim \varepsilon \left( \frac{1}{\varepsilon} \right)^{\frac{3}{p} + \frac{2}{p} \cdot \frac{3}{2}} \lesssim \varepsilon^{\frac{1}{p}} \varepsilon^{\frac{2}{p}}. \]

As in the case \( N = 3 \), we can also obtain the desired inequality.

References

