A Compressive Multi-Frequency Linear Sampling Method for Underwater Acoustic Imaging

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Abstract—This paper investigates the use of a qualitative inverse scattering method known as the Linear Sampling Method (LSM) for imaging underwater scenes using limited aperture receiver configurations. The LSM is based on solving a set of unstable integral equations known as the far-field equations and whose stability breaks down even further for under-sampled observation aperture data. Based on the results of a recent study concerning multi-frequency LSM imaging, we propose an iterative inversion method that is founded upon a compressive sensing framework. In particular, we leverage multi-frequency diversity in the data by imposing a partial frequency variation prior on the solution which we show is justified when the frequency bandwidth is sampled finely enough. We formulate an Alternating Direction Method of Multiplier (ADMM) approach to minimize the proposed cost function. Proof of concept is evaluated of underwater structures, and deep sea search/rescue recovery, unexploded ordnance (UXO) disposal, non-destructive military applications. Some examples include sea mine discovering, unexploded ordnance (UXO) disposal, non-destructive evaluation of underwater structures, and deep sea search/rescue operations. In addition to an accurate visual reconstruction of a particular scene, increasingly the term imaging in many applications includes automatic detection and/or recognition of objects within the scene. Advances in supervised machine learning in recent years has significantly improved the feasibility of such automated systems. However, accurate and robust acoustic imaging in underwater environments remains a difficult feat, particularly due to the presence of natural and man-made clutter [1], [2]. As is known in the machine learning community, the performance of a learning algorithm is heavily dependent on the use of salient features in conjunction with rich training data sets which encompass examples representative of the type of objects to be identified. In a given Region of Interest (ROI), knowledge of the acoustic profile (the index of refraction) within the ROI can potentially yield a wealth of information needed for feature extraction; in particular features that can effectively distinguish clutter from a potential target. In this sense faithful reconstruction of an ROI’s acoustic profile can serve as a basis for “acoustic vision” applications, especially if reconstruction can be completed in near real-time.

The particular problem of reconstructing an ROI’s acoustic profile from a set of scattered measurements is known as the inverse scattering problem. Solving acoustic inverse scattering problems in an accurate, robust, and speedy manner still remains a challenge; primarily due to the ill-posed and non-linear mathematical nature of the inverse problem. It is important to emphasize that the inverse relationship between the scattered acoustic field and its index of refraction is non-linear even when the corresponding forward relationship is linear [3]. Ill-posedness or instability, if not treated effectively, can lead to reconstructions that are easily perturbed by a nominal amount of additive noise; often bearing little resemblance to the true acoustic profile of the ROI. Ill-posedness reflects the fact that the inverse scattering problem is effectively under-determined and requires some over-redundancy in the data to ensure stable inversion. Recently the use of multi-static sensing configurations, where the scene is interrogated from a number of different directions for which the corresponding scattered responses is then observed at a number of distinct spatial locations, has been employed as a means to introduce such over-redundancy. Theoretically, when coupled with a sufficiently wide bandwidth, information rich multi-static data provides a sufficient basis for a unique identification of inhomogeneities in the ROI [4], [5]. In practice however, effective exploitation of multi-static data is limited by the particular inversion algorithm employed.

Although a comprehensive overview of various inverse scattering approaches is beyond the scope of this article, a brief discussion of some general inverse scattering approaches is
helpful for motivating the use of qualitative methods. Broadly speaking, approaches to solving the inverse scattering problem can be classified into three categories: iterative non-linear approaches, linearized approaches, and qualitative approaches. Non-linear approaches such as Newton’s method [6] and the Level-Set Method [7] attempt to reconstruct the acoustic profile through a sequence of modifications to the predicted acoustic profile of a ROI. On each iteration, the current solution is updated according to the error between the given scattering data and a that of a scattering response predicted through a forward scattering model. When the two quantities sufficiently agree the iterative process is terminated. Under the right conditions, non-linear approaches can be quite effective at accurately reconstructing the acoustic profile of a scene, even for a small number of interrogating directions [5]. However, non-linear approaches in general are difficult/impractical to apply for a number of reasons. Firstly, they are prone to model mismatch since they rely on computing a forward model and assuming certain geometrical constraints on the scatterer (star-like targets, parametric surfaces, etc...). Secondly, reliance on computing a forward model on each iteration is quite cumbersome and results in an extremely slow reconstruction process. For high-resolution imaging where the number of elements needed for meshing the surfaces/volumes can be quite large, a non-linear approach may simply be infeasible. Lastly, non-linear formulations are non-convex, and therefore the initial estimate plays a critical role. A good initial estimate will not only ensure a small number of iterations till convergence, but can also help to ensure the terminating point is a global minimum. This has motivated the use of so-called hybrid approaches, where a linear inverse scattering method provides a suitable initialization to a non-linear method.

The more common approach to handling the non-linearity of the inverse scattering problem is to employ weak scattering approximations [3]. This results in a linear albeit ill-posed relationship between the acoustic profile and the data. With appropriate filtering the resulting inversion often yields good results which in many cases can be efficiently computed using fast Fourier transform (FFT) operators. Therefore fast on-the-spot reconstructions are quite feasible for linearized approaches. However, weak scattering assumptions such as the Born approximation, only model first order scattering effects and thus fail to account for important non-linear wave behavior including multi-bounce events and elasticity effects. In addition to introducing unwanted artifacts and distortions in the solution (see section 5 in [8] for example), linearization also discards critical information that may be relevant for target identification. For example, the work in [9] demonstrated how physics-based features exploiting a target’s elasticity could boost classification performance of buried UXO targets against clutter.

Qualitative inverse scattering methods [10] are a more recently developed class of inverse scattering methods partly motivated as a compromise between linearized and non-linear approaches. Qualitative methods aim to reconstruct the shape of inhomogeneities in the ROI rather than the quantitative values of the acoustic profile. This has the advantage of circumventing the non-linear nature of the inverse problem without resorting to weak-scattering approximations; rather the linearity is derived through an exact mathematical relationship that exploits singularities in the background Green’s function. Furthermore, in contrast to direct non-linear methods, qualitative approaches do not rely on explicit prior knowledge of the geometric or physical nature of the scatterer(s). Therefore, in addition to fast imaging, qualitative methods can potentially exhibit a level of robustness needed for reliable imaging and feature extraction.

The LSM was first introduced in 1996 [11] and is one of the most established qualitative inverse scattering method. Since the shape of a scatterer is directly reconstructed by the LSM, geometric-based features such as 3D Histogram of Gradients (HoG) [12] or curvature based features [13] can be directly extracted from a qualitative reconstruction. However physics-based features can also be extracted directly and non-directly from a qualitative inversion. For example, the LSM was recently applied for reconstructing interior resonant modes of a particular scatterer in the scene in a completely blind manner. Interior modes of an object can play a key role for deriving salient features since for a particular scatterer they are rotation, translation, and scale invariant [14]. Furthermore, retrieving the quantitative acoustic profile can still be possible with the LSM. For example, the hybrid approach in [15] successfully considered TM electromagnetic inverse scattering where the classical LSM was employed as a pre-processing step. The quantitative values were then reconstructed by using a known relation to estimate the total field in a Lipmann-Schwinger integral. Application of this particular approach to acoustic data can be trivially transposed.

Although there has been some studies reporting the use of the LSM with laboratory experiments [16], [17], the LSM has not received as much consideration throughout the engineering community as it has in the mathematical literature. In our opinion a major factor for this trend perhaps lies in the stability of the method; particularly when insufficient multi-static data is employed. It is known [18], that the LSM relies on a large number of incident directions and a large number of receiver directions to render a physically meaningful image. In many applications, this sensing requirement can be infeasible or too costly, especially for underwater scenes where automated underwater vehicles sensing platforms are employed. In theory, increased frequency bandwidth can compensate for the lack of spatial diversity [5]. However, recent studies concerning multi-frequency LSM [19], [20], [21] have not shown that frequency diversity can effectively mitigate the sparse-aperture problem, i.e., when the observation aperture is only sampled at a few discrete locations. Indeed, simple numerical examples will confirm that current multi-frequency approaches to the LSM fail for sparse aperture data, even when the incident aperture is dense (see [18], [22], [23] for some published examples).

There has been prior work on improving sparse aperture inverse scattering mostly within the context of linear inverse models. For example, Battle [24] sought a maximum-entropy approach to regularize the inversion process. More recently, Tuysuzoglu et al. [25] considered ultrasound image reconstruction also under a linearized inverse model assumption. In that work, a total-variation (TV) penalty was imposed.
on the underlying image and the results showed a notable improvement for linearized aperture reconstruction. The success of a TV approach in linearized inversion can be explained within the context of sparse regularization (compressive sensing). In fact, sparse regularization approaches have been successfully applied in a variety of scientific and engineering applications featuring under-determined systems (see the [26] for a list of examples). To consider a compressive sensing approach, it is necessary that the underlying function one wishes to reconstruct is sparse with respect to some basis. When the underlying signal is not sparse, a common strategy is to find a suitable change of basis in which the projected signal yields a parsimonious representation. In the case of a TV approach, the assumption is that the spatial gradient of the underlying image is sparse, i.e., local regions should exhibit small spatial variation unless traversing a boundary. Investigations on employing compressive sensing approaches for qualitative inverse scattering, and in particular for the LSM, is scarce in the current literature. This may be attributed to the fact that the quantity we need to reconstruct, known as the Herglotz density, is not the actual image. Rather the image and this density are related in a non-linear fashion. In general there is no reason to assume that the Herglotz density should be sparse. In fact, this is most likely not the case. This non-linear relationship between the image and the Herglotz density therefore makes establishing a justifiable sparsity-promoting basis rather difficult. For example, prior work by the author in [27] studied the use of a convex and non-convex TV approach for single frequency LSM. The non-convex approach came as a result of imposing a TV constraint on the image rather than the Herglotz density directly. While the proposed method showed improvement over classical LSM implementations, the approach was quite slow and susceptible to the local min problem. This motivated a second approach where the TV penalty was imposed on the spatial dependence of the Herglotz density resulting in a convex formulation. Although considerably faster than the non-convex formulation, the computational complexity was still much greater than that of the classical LSM approach (see section 4.2 of [5]). Furthermore, the assumption of a sparsity prior in the Herglotz density could not be theoretically justified. Hence, substantial improvement in imaging quality over the classical LSM approach cannot generally be expected. The method proposed in this paper was motivated by a recent study [22] which investigated a partial Frequency Variation (FV) approach for reconstructing the Herglotz density. In that work it was rigorously proved that, at least for the case of a Dirichlet (soft acoustic boundaries) scatterer, any Herglotz density satisfying the so-called far-field inequality should exhibit frequency variation proportional to the frequency step-size. This is the case as long as the frequency step avoided certain interior modes. The conclusion from this rather technical result is that, if the frequency step-size is chosen to be sufficiently small enough, a sparse approximation of the frequency variation of the solution is reasonable. Therefore in contrast to a spatial TV approach, an FV approach offers better guarantees of improved imaging with sparse aperture data. Furthermore, an FV approach is amenable to fast high-resolution imaging since the constraint is not dependent on a particular pixel. Thus all pixels of the resulting image can be processed in an embarrassingly parallel fashion. We show in this paper that a significant improvement in LSM based under-water acoustic image reconstruction quality can be achieved by adopting a FV approach. We also show that by using the principle of reciprocity the FV approach can also be applied to the case of sparse incident aperture, i.e., when the number of interrogation directions is small.

The remainder of the paper is organized as follows: We first introduce relevant notation, briefly review the LSM, and discuss its classical implementation in section II. Next, in section III, we discuss the motivation behind the use of a FV approach and offer a few simple examples to illustrate the major concepts regarding the frequency dependence of the Herglotz density. We then proceed to formulate the necessary objective function and discuss a relatively fast numerical reconstruction method based on an ADMM approach. In section IV we establish an initial proof of concept using numerically generated scattering data in three dimensional space. In section V we show, using only two orthogonal incident angles, a 2D reconstruction of cylindrical UXO positioned in an experimental pool facility. Finally we offer concluding remarks in section VI.

II. FORMULATION

Let $\Sigma \subseteq \mathbb{R}^m$, with dimension $m = 2, 3$, represent a region characterized by an isotropic background medium with reference wave speed $c_0 > 0$. Assume that the Green’s function $G_{\Sigma}(r, r'; \omega)$ is computable. Recall that the Green’s function yields the time-harmonic acoustic field at frequency $\omega$ that is observed at position $r$ due to a point source located at point $r'$. We let $\Omega$ denote the (unknown) interior support of the scatterer(s) in the scene which we assume are bounded by a well-defined boundary $\partial \Omega$ separating the exterior region denoted as $\Omega^+$. A multi-static experiment consists of actively probing the scene with a set of acoustic plane-wave sources using different frequencies and different incident directions. Each interrogation in a particular direction $\Gamma_{\text{inc}} \in \Gamma_{\text{inc}}$, where $\Gamma_{\text{inc}}$ is the effective aperture of the probing waves, induces a scattered field which is measured by a set of spatially distributed hydrophones with effective observation aperture $\Gamma_{\text{obs}}$. In the far-field zone the scattered is given as [4]

$$
\hat{u}_{\text{inc}}(\Gamma_{\text{inc}}, \omega) = \frac{e^{i kr}}{r} \left\{ u_{\infty}(\Gamma_{\text{obs}}, \Gamma_{\text{inc}}, \omega) + O \left( r^{-1} \right) \right\}, \tag{1}
$$

for each frequency $\omega \in B$ in the bandwidth of the transmitted pulse. Here $k = \omega / c_0$ is the wavenumber of the background medium, $\Gamma_{\text{obs}} \in \Gamma_{\text{obs}}$ is a unit vector denoting the direction of observation, and $r = |\Gamma_{\text{obs}}|$ is the observation distance relative to the center of the scene. The quantity $u_{\infty}$ is known as the far-field amplitude, which is a complex-valued function depending only on the observation direction, the interrogation direction, and the frequency of the incident plane wave. The inverse problem is to reconstruct the shape of $\Omega$ given partial knowledge of $u_{\infty}$ on the data grid $\Gamma_{\text{obs}} \times \Gamma_{\text{inc}} \times B$. When $u_{\infty}$ is known completely for all multi-static directions and all frequencies on a bandwidth it contains more than enough information to uniquely solve the inverse problem [5].
A. Imaging with the LSM

To render an image of the scene, we first chose a closed and bounded image volume (area for \( m = 2 \)) which we denote as \( V \subset \Omega \). The region \( V \) is chosen such that the image box contains the target(s) we wish to image. For any point \( z \in V \), and frequency \( \omega \in B \), a corresponding integral equation over the incident aperture is constructed

\[
\int_{\Gamma_{inc}} u_\infty(\mathbf{r}_{inc},\omega)g_{\mathbf{r}_{inc}}(\mathbf{r}_{inc},\omega)d\mathbf{s}(\mathbf{r}_{inc}) = G_{\Sigma,\infty}(\mathbf{r}_{inc},z,\omega),
\]

which is known as the far-field equation. The quantity \( G_{\Sigma,\infty} \) appearing on the right-hand side of (2) is the far-field amplitude of the Green’s function \( G_{\Sigma} \) and is assumed to be known. For frequency \( \omega \in B \), recall that \( G_{\Sigma,\infty}(\mathbf{r}_{inc},z,\omega) \) describes the far-field amplitude that will be observed in direction \( \mathbf{r}_{inc} \) due to a radiating point source located at \( z \). The integral kernel \( u_\infty \) is the data, and the quantity we need to solve for \( g_{\mathbf{r}_{inc}} \) is known as the Herglotz density. Upon solving this unstable equation in some reasonable fashion, an image is rendered by computing the intensity at pixel (voxel) \( z \) and frequency \( \omega \) as

\[
I(z,\omega) = \Pi(||g_{\mathbf{r}_{inc}}(\cdot,\omega)||_{L^2(\Gamma_{inc})}).
\]

Here, \( \Pi \) is an arbitrary radial function that depends solely on the angular norm of \( g_{\mathbf{r}_{inc}} \),

\[
||g_{\mathbf{r}_{inc}}(\cdot,\omega)||_{L^2(\Gamma_{inc})}^2 = \int_{\Gamma_{inc}} ||g_{\mathbf{r}_{inc}}(\mathbf{r}_{inc},\omega)||^2d\mathbf{s}(\mathbf{r}_{inc}).
\]

It so happens that the scatterer’s shape will emerge due to certain special properties of the Herglotz density [11]. More specifically, assuming that \( \omega \) does not correspond to an interior mode of \( \Omega \), we can show that \( ||g_{\mathbf{r}_{inc}}||_{L^2(\Gamma_{inc})} \) happens to be small when \( z \in \Omega \) and becomes unbounded (large) as \( z \to \partial\Omega \) from the interior. Furthermore, when \( z \in \Omega^+ \) the Herglotz density remains unbounded. Fig 1 illustrates the procedure for rendering a single-frequency image using the LSM.

It is important to note that mathematically speaking there is a technical gap in this procedure. We point out that an exact solution to the far-field equation (2) only exists in very limited circumstances (such as when \( \Omega \) is a sphere). However, as was rigorously shown by Cakoni [28], there always exists an appropriate approximation to the far-field equation up to an arbitrary level of tolerance. In other words, for any given \( \epsilon > 0 \), there exists some \( g_{\mathbf{r}_{inc}} \) satisfying the inequality

\[
||Fg_{\mathbf{r}_{inc}}(\cdot,\omega) - G_{\infty}(\cdot,z;\omega)||_{L^2(\mathbb{R}^{m-1})} < \epsilon,
\]

where \( F \) is the integral operator specified in equation (2), that will retain the characteristic dissection property we mentioned above. In practice an unconstrained minimum residual approach to finding the desired Herglotz density rarely results in a physically meaningful image. This is due to the fact that the far-field operator is extremely smoothing (blurring) and the Eigen-spectrum of the \( F^HF \) operator will contain no lower bound. Even a small amount of noise in the multi-static data will cause wild variations in the reconstructed Herglotz density. Instead, the predominant approach in the literature is to apply a Tikhonov regularized approach

\[
g_{\mathbf{r}_{inc}} = \arg \min_{g \in L^2(\mathbb{R}^{m-1})} \|Fg(\cdot,\omega) - G_{\infty}(\cdot,z;\omega)\|^2_{L^2(\mathbb{R}^{m-1})} + \alpha \|g\|^2_{L^2(\mathbb{R}^{m-1})}.
\]

In [19] a multi-frequency version of the LSM was studied. The procedure follows what was described above, i.e., solving for the Herglotz density for each frequency in isolation, but with a recommendation that a parallel indicator function be used to reconstruct each spectral component of the image

\[
I(z,\omega) = 1/\|g_{\mathbf{r}_{inc}}(\cdot,\omega)\|_{L^2(\Gamma_{inc})}.
\]

This specific image reconstruction function was chosen in order to mitigate effects due to traversing a possible interior mode of \( \Omega \). An issue we discuss more in depth in the following section. The final image is then simply a sum of each spectral component:

\[
I(z) = \int_B I(z,\omega)d\omega.
\]

B. Fully Discrete Formulation

We consider a fully discrete formulation of the multi-frequency LSM procedure discussed above. This is accomplished by first discretizing the far-field equations (2) and then applying a regularization scheme. To simplify the discussion, we use the term pixel to describe an element of a 2D or 3D image grid. It is assumed that the data will be obtained on a sampled version of the data grid \( \Gamma_{obs} \times \Gamma_{inc} \times B \). Let us denote these grid points as \( \{\mathbf{r}_{inc}^{(j)}\}_{j=1}^{n_j} \), \( \{\mathbf{r}_{inc}^{(j)}\}_{j=1}^{n_j} \), and \( \{\omega^{(k)}\}_{k=1}^{n_k} \), for the incident aperture samples, the observation aperture samples, and the frequency samples respectively. We do not assume that spatial sample points are given in a uniform fashion on their respective spheres, however we do assume a uniform sampling of the frequency bandwidth with step-size \( \Delta \omega > 0 \). Now, for each fixed pixel \( z \in V \), where \( V \) is assumed to contain \( \Omega \), we formulate the block diagonal system

\[
\begin{bmatrix}
F^{(1)} & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & F^{(n_\omega)} & \vdots \\
\end{bmatrix}
\begin{bmatrix}
g^{(1)}_z \\
\vdots \\
g^{(n_\omega)}_z \\
\end{bmatrix} =
\begin{bmatrix}
\Phi^{(1)}_z \\
\vdots \\
\Phi^{(n_\omega)}_z \\
\end{bmatrix}.
\]

Each left-hand block \( F^{(k)}g^{(k)}_z \) is a discretized approximation of the far-field operator applied to the Herglotz density for a single frequency, i.e.,

\[
F^{(k)}g^{(k)}_z = \sum_{j=1}^{n_j} w_j u_\infty(\mathbf{r}_{inc}^{(j)}\mathbf{r}_{inc},\omega^{(k)})g_{\mathbf{r}_{inc}}(\mathbf{r}_{inc},\omega^{(k)}),
\]

where \( w_j \) are the quadrature weights to approximate the integral over the unit sphere in equation (10). For specific details on how to compute the quadrature weights \( w_j \) in the case of 3D imaging we refer the reader to section 3.1 of [22]. Similarly, each right-hand block is given as

\[
\Phi^{(k)}_z = G_{\Sigma,\infty}(\cdot,z,\omega^{(k)}).
\]
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For the sake of notational brevity we write (9) as

$$Fg = \Phi,$$  \hspace{1cm} (12)

where the matrix $F \in \mathbb{C}^{m \times n}$ has the dimensions $m = n_{\text{obs}}n_\omega$, and $n = n_{\text{inc}}n_\omega$. Once an appropriate Herglotz solution $\hat{g}$ is computed, the image is rendered with a discrete version of (8),

$$I(z) = \sum_{\omega_k \in B} \frac{1}{\|g^{(k)}\|_{2,m=1}},$$  \hspace{1cm} (13)

where

$$\|g^{(k)}\|_{2,m=1} = \left( \sum_{j=1}^{n_r} |w_j| g_{k} \Phi_{j}^{(k)}, \omega_k \right)^{1/2},$$  \hspace{1cm} (14)

is an approximation of the Herglotz energy on the sphere for a fixed frequency. Although we stated that equation (12) is only for one pixel, in practice we solve for all pixels simultaneously by including multiple right-hand side vectors, i.e., $\Phi$ will be an $m \times n_z$ matrix, where $n_z$ is the number of pixels in the image grid. This approach first suggested in [29] obviously speeds up the imaging procedure considerably, and from our own numerical experience, we have not observed major improvements to imaging quality by considering a regularization scheme (i.e., different parameters) for each pixel in isolation.

The Tikhonov solution to equation (12) is given as

$$\hat{g} = \arg \min_{g \in \mathbb{C}^{n_z}} \|Fg - \Phi\|^2 + \alpha\|g\|^2,$$  \hspace{1cm} (15)

where again $\alpha > 0$ is a regularization parameter used to impose stability. The Tikhonov approach has many advantages; firstly it is a well defined problem, i.e., it has a unique solution for any $\alpha > 0$. Secondly, when $u_{\infty}$ is densely known on the data grid $\Gamma_{\text{obs}} \times \Gamma_{\text{inc}} \times B$ the reconstructions are stable and accurate. Thirdly, the solution (15) can be computed quite efficiently using a singular value decomposition (SVD) factorization scheme. The factorization can be performed rather rapidly since each block $F^{(j)}$ is typically a small scale matrix. This is advantageous especially when one wishes to sweep through different regularization parameters or spatial grid samplings.

III. COMPRESSION MULTI-FREQUENCY FORMULATION

It is well known that the LSM experiences difficulties when one or both apertures $\Gamma_{\text{inc}}$ and $\Gamma_{\text{obs}}$ is under-sampled. Indeed, from equation (10) one can observe that failure to adequately sample $\Gamma_{\text{inc}}$ will result in a poor approximation of the far-field integral operator. In the case that $\Gamma_{\text{inc}}$ is sampled sufficiently enough but $\Gamma_{\text{obs}}$ is sparsely sampled, we are left with an under-determined and ill-conditioned system of equations. Here we will address the latter problem, however we do wish to make a note about the incident aperture problem. If, on the other hand, one had a dense sampling of the observation aperture for only a few incident directions, one can employ a reciprocity condition to integrate over the observation aperture instead. In a sense the roles of the observation and incident directions will be reversed. We provide an example of how to do this in section IV below. The point being here is that our approach should suffice as long as one of the apertures is dense enough. We leave the dual sparse aperture problem for future work.

A. Frequency Dependence of the Herglotz Density

Although an in-depth discussion on the frequency dependence of the Herglotz density is well beyond the scope of this article, we wish to give the reader a brief overview into this important behavior. The first concept to understand is how an approximation to the far-field equation behaves when the frequency of the probing plane wave coincides with (or is near enough to) what is known as an interior mode. Interior modes are those frequencies that align with the natural internal vibrations of a scatterer. These modes can be shown to uniquely correspond to the shape of the scatterer. As shown in [19], if $\omega$ happens to be close to an interior mode, then the Herglotz density becomes large for almost every pixel $z$. An enlightening example of this phenomenon can be illustrated through a simple example; the case of a soft acoustic disk of radius $a > 0$ in $\mathbb{R}^2$. This is one of the cases where an exact solution to the far-field equation exists. For any pixel $z = (\rho, \phi)^T$ represented in polar coordinates, the Herglotz density satisfying the far-field equation is given analytically as

$$g_{\Phi}(\theta; k) = \sum_{n=-\infty}^{\infty} a_n(z,k)e^{in\theta},$$  \hspace{1cm} (16)
where

\[ a_n(z, k) = \frac{i^{-n}H_n^{(1)}(ka)J_n(k\rho)}{2\pi J_n(ka)}e^{-in\phi}. \]  

(17)

We see here that, with the exception of the case \( \rho = \alpha \), the coefficients \( a_n \) become singular whenever \( ka \) coincides with a zero of the Bessel function \( J_n \). In Fig 2 we show an example reconstruction with dense data of a unit disk for two frequencies approaching the first known interior mode. One important observation here is that the transition is quite sharp (on a wavenumber scale). The work in [22] generalized these observations to an arbitrarily shaped scatterer. The key result in that paper can be loosely stated as follows: in the case that \( B \) is a Dirichlet (acoustically soft) scatterer, we have for any \( \epsilon > 0 \) and frequencies \( \omega_1 < \omega_2 \) contained in a band free of any interior Dirichlet eigenvalues of \( \Omega \) that

\[ \| g(z, \omega_1) - g(z, \omega_2) \|_{L_2(\mathbb{R}^{2m-1})} \leq C|\omega_2 - \omega_1| + \epsilon, \]  

(18)

for all pixels \( z \in \Omega \). Since on a given frequency band \( B \subset \mathbb{R}^2 \), a certain scatterer’s interior modes should only occur at discrete locations, result (18) implies that we can choose a frequency step-size small enough so that variations in the Herglotz density with respect to frequency is approximately zero across the band. Fig 3 shows an example of this concept for the simple case of a spherical scatterer. It is important to note that the positive constant \( C \) in (18) is not dependent on \( \omega_1 \) or \( \omega_2 \), but only on the distance of the frequency band to an interior mode and the distance of \( z \) to the boundary \( \partial \Omega \). Therefore, the optimal choice of the frequency step-size needed to justify a sparsity prior would only depend on the size of the target and center frequency of the bandwidth.

### B. A Frequency Variation Approach

Motivated by the results discussed above, we propose to augment (15) with an additional penalty term:

\[ \hat{g} = \arg \min_{g \in \mathbb{C}^n} \| Fg - \Phi \|_2^2 + \alpha \| g \|_2^2 + \beta \| D_\omega g \|_1. \]  

(19)

The matrix \( D_\omega : \mathbb{C}^{n} \rightarrow \mathbb{C}^{n'} \), \( n' = n_{\text{inc}}(n_{\omega} - 1) \), is a linear frequency differencing operator, and the parameter \( \beta > 0 \) controls how much variation across frequency will be permitted. It is natural to question if the second term in (19) became relatively large. It is given as

\[ \min_{g \in \mathbb{C}^n} \| Ag - b \|_2^2 + \beta \| D_\omega g \|_1, \]  

(20)

where,

\[ A = \begin{bmatrix} F & \sqrt{\alpha} \Phi \\ \Phi & 0 \end{bmatrix}, \quad b = \begin{bmatrix} \Phi \\ 0 \end{bmatrix}. \]  

(21)

We then introduce an auxiliary variable \( w \) and penalty parameter \( \rho > 0 \) (see [30]) to form the constrained equivalent problem

\[ \min_{(g, w)} \| Ag - b \|_2^2 + \beta \| w \|_1 + \frac{\rho}{2} \| D_\omega g - w \|_2^2 \]  

subject to \( D_\omega g = w \). \]  

(22)

Introducing a Lagrange multiplier, rearranging terms using a completion of the square, and employing the method of ascent [31], the constrained problem (22) can be solved by the following iterative procedure:

\[ \begin{align*}
\left( g^{(i+1)}, w^{(i+1)} \right) &= \arg \min_{(g, w)} L_\rho((g, w), u^{(i)}), \\
u^{(i+1)} &= u^{(i)} + D_\omega g^{(i+1)} - w^{(i+1)},
\end{align*} \]  

(23)

(24)

where the augmented Lagrangian \( L_\rho \) is given as

\[ L_\rho((g, w), u) := \| Ag - b \|_2^2 + \beta \| w \|_1 + \frac{\rho}{2} \| D_\omega g - w + u \|_2^2. \]  

(25)

Problem (23) can be solved by alternating between minimizing with respect to \( g \) while holding \( w \) fixed, and then minimizing with respect to \( w \) while holding \( g \) fixed. The advantage here is that under the assumption that \( N(A) \cap N(D_\omega) = \{0\} \), where \( N(.) \) denotes the null-space, both sub-problems have closed form expressions,

\[ g^{(i+1)} = K^{-1} \left( A^H b + \rho D^H (w^{(i)} - u^{(i)}) \right), \]  

(26)

\[ w^{(i+1)} = \text{Shrink}(D_\omega g^{(i+1)} + u^{(i)}, \beta/\rho), \]  

(27)

where

\[ K = A^H A + \rho D^H D_\omega \]  

(28)

C. An ADMM Approach for Reconstruction

There are a variety of numerical approaches for minimizing the convex objective function in (19). The operators involved are typically large scale and very sparse. Therefore it is highly recommended that first-order type approaches are considered, i.e., methods that avoid approximating a dense Hessian matrix. In [22] an Fast Iterative Shrinkage Thresholding Algorithm (FISTA) was employed which relied on an inner iterative process to compute the associated proximity or “de-noising” operator. In addition to increasing the computational time per global iteration, the inner iterative process had a tendency to become unstable, particularly when \( \rho \) became relatively large. In [27] an Alternating Direction Lagrangian Method (ADLM) was employed for the proposed single-frequency TV approach, however this approach required the design of effective preconditioners to deal with the slow rate of convergence. Introducing an augmented Lagrangian formulation however leads to better stability per iteration and thus faster convergence times. We briefly discuss our approach: the first step is to write problem (19) into a standardized form,

\[ \min_{g \in \mathbb{C}^n} \| Ag - b \|_2^2 + \beta \| D_\omega g \|_1, \]  

(20)

We then introduce an auxiliary variable \( w \) and penalty parameter \( \rho > 0 \) (see [30]) to form the constrained equivalent problem

\[ \min_{(g, w)} \| Ag - b \|_2^2 + \beta \| w \|_1 + \frac{\rho}{2} \| D_\omega g - w \|_2^2 \]  

subject to \( D_\omega g = w \). \]  

(22)

Introducing a Lagrange multiplier, rearranging terms using a completion of the square, and employing the method of ascent [31], the constrained problem (22) can be solved by the following iterative procedure:

\[ \begin{align*}
\left( g^{(i+1)}, w^{(i+1)} \right) &= \arg \min_{(g, w)} L_\rho((g, w), u^{(i)}), \\
u^{(i+1)} &= u^{(i)} + D_\omega g^{(i+1)} - w^{(i+1)},
\end{align*} \]  

(23)

(24)

where the augmented Lagrangian \( L_\rho \) is given as

\[ L_\rho((g, w), u) := \| Ag - b \|_2^2 + \beta \| w \|_1 + \frac{\rho}{2} \| D_\omega g - w + u \|_2^2. \]  

(25)

Problem (23) can be solved by alternating between minimizing with respect to \( g \) while holding \( w \) fixed, and then minimizing with respect to \( w \) while holding \( g \) fixed. The advantage here is that under the assumption that \( N(A) \cap N(D_\omega) = \{0\} \), where \( N(.) \) denotes the null-space, both sub-problems have closed form expressions,

\[ g^{(i+1)} = K^{-1} \left( A^H b + \rho D^H (w^{(i)} - u^{(i)}) \right), \]  

(26)

\[ w^{(i+1)} = \text{Shrink}(D_\omega g^{(i+1)} + u^{(i)}, \beta/\rho), \]  

(27)

where

\[ K = A^H A + \rho D^H D_\omega \]  

(28)
(a) $k = 0.9682k_1^*$

(b) $k = 0.9997k_1^*$

Fig. 2. Illustrating a Tikhonov solution with dense multi-static data for two wavenumbers (frequency) near the first interior mode at $k_1^*$ for a unit disk.

A Single Interior Pixel Intensity as a Function of Frequency

(a)

A Single Interior Pixel Intensity as a Function of Frequency

(b)

Fig. 3. Reconstruction of a Dirichlet sphere using dense aperture data for a single voxel $z = [0, 0, 0]^T$. a) Spatial norm (see equation (14)) of the Herglotz density as a function of frequency. The peaks in the plot correspond to the interior Dirichlet eigenvalues. b) Spatial norm of the frequency derivative of the Herglotz density, which clearly can be approximated by a sparse function.

and Shrink($\ldots$) is the well-known shrinkage operator defined component-wise as

$$(\text{Shrink}(z, \mu))_i := \text{sgn}(z_i) \max\{|z_i| - \mu, 0\}. \quad (29)$$

This entire procedure from variable splitting, augmenting the Lagrangian, and introducing a Lagrange multiplier is known as an ADMM approach. The $K$ matrix in (26) is symmetric, moderately sized, and sparse which suggests an iterative approach such as the Conjugate Gradient (CG) algorithm. However as mentioned before, we want to solve for multiple right-hand sides $b$ simultaneously. For this work we employ a sparse LU factorization for solving equation (26). For convenience we summarize our overall numerical approach in algorithm 1. Note that the input data matrix $A$ in algorithm 1 implicitly contains the $\alpha$ parameter defined in the original approach (19).

Algorithm 1 FV-LSM via an ADMM procedure

function ADMMFV(A, b, $\beta$, $\rho$)

$g^{(0)} \leftarrow 0$
$w^{(0)} \leftarrow 0$
$K \leftarrow (A^H A + \rho D_\omega^H D_\omega)$

for $i \leftarrow 0$ to $i_{\text{max}}$ do

$g^{(i+1)} \leftarrow K^{-1} (A^H b + \rho D_\omega^H (w^{(i)} - u^{(i)}))$

$w^{(i+1)} \leftarrow \text{Shrink}(D_\omega g^{(i+1)} + u^{(i)}, \beta/\rho)$

$u^{(i+1)} \leftarrow u^{(i)} + D_\omega g^{(i+1)} - w^{(i+1)}$

if $\max\{|g^{(i+1)} - g^{(i)}|_2 \} < \text{tol}$ then

break

end if

end for

return $g^{(i+1)}$

end function
IV. Numerical Experiments

In this section we present some simple numerical experiments using simulated data to illustrate a basic proof of concept for the proposed compressive approach. We consider 3D imaging of a U-shaped obstacle (shown in Fig 4) which was embedded in a constant infinite background. The target was interrogated by acoustic plane waves from 64 uniformly distributed directions on a bandwidth corresponding to wavenumbers \( k_0 \in [1, 6] \) using \( n_\omega = 100 \) uniformly stepped frequencies. The scattered far-field amplitude was observed along a limited aperture using 8 observation directions whose positions are shown in Fig 5. The scattered wave in the far-field was computed by means of a first-order boundary element code and was corrupted with 2% white additive noise. We note that here the far-field amplitude is computed directly using an exact formula (see section 3 of [4]), thus there is no need to specify the stand-off distance \( r_{\text{obs}} \) for this experiment.

The objective for this experiment is to assess whether the proposed frequency variation approach provides increased stability, particularly with sparse aperture observation aperture. We first apply a block Tikhonov approach, i.e., \( \beta \) stability, particularly with sparse aperture observation aperture. The proposed frequency variation approach provides increased stability, particularly with sparse aperture observation aperture.

Fig. 4. A U-shaped obstacle with dimensions relative to the center wavelength \( \lambda_c \) of the corresponding bandwidth. For the experiments \( \lambda_c \approx 1.57 \), which corresponded to a wavenumber of \( k = 4 \).

Finally, for the sake completeness, we show a 3D iso-surface reconstruction of the target in Fig 9.

V. Multi-Static Imaging of a Submerged UXO

We consider application of FV-LSM imaging to the case of a UXO positioned proud of a sediment layer. We refer to Fig 10, which shows a photograph of the UXO in question and an illustration of the experimental configuration. A more detailed description and analysis of the experiment was described by Bucaro [32], however we summarize the experimental configuration for convenience to the reader.

The UXO has a length of approximately \( l = 0.4486 \) m and a maximum diameter of approximately \( a = 0.127 \) m. The target was positioned \( \delta = 0.1 \) m above the sediment layer of the pool. Acoustic interrogation of the UXO was performed from a distance \( r_{\text{inc}} = 2.7 \) m using angles of \( \theta_{\text{inc}} = 0^\circ \) and \( \theta_{\text{inc}} = 90^\circ \). The interrogation distance \( r_{\text{inc}} \) is measured from the target center and angular direction \( \theta_{\text{inc}} \) is measured relative to the normal vector of the front side of the UXO. The pseudo plane wave sources were driven by a short-time pulse of approximately 20 \( \mu \)s resulting in a frequency band in excess of 25 kHz. The scattered response to each incident direction was sensed using a controlled robotic arm positioned \( r_{\text{obs}} = 2.0 \) m relative to the target center in \( \Delta \theta_{\text{obs}} = 1^\circ \) angular increments along the entire \( 360^\circ \) aperture. The receiver data was transformed to the frequency domain using a frequency spacing of roughly 122.13 Hz. For the estimated background wave speed of \( c^v = 1482 \) m/s, this frequency spacing corresponded to a wavenumber spacing of \( \Delta k = 1.03 \) m\(^{-1} \).

A. Pre-Processing Procedures

In order to generate the right-hand side of (12) we require the Green’s function of the medium. Wall reflections due to the finite size of the pool were eliminated by first exciting the scene with no UXO present and then subtracting the response from the data collected with the UXO present [32]. To account for reflections due to the sediment layer and the water-air interface a stratified Green’s function should be used [33]. However in this particular case, where we lack any depth diversity in the measurements, we can assume that reflections from \( \Omega \) are toroidal. In which case reflections from the lower and upper boundaries should not significantly contribute to the observed response. Therefore a reasonable (and computationally efficient) approach is to use a 2D free-space Green’s function

\[
G_{\Sigma}(r_{\text{obs}}, r_{\text{src}}) = \frac{i}{2\pi} H_0^1(k|r_{\text{obs}} - r_{\text{src}}|),
\]

\( \alpha = 0 \) fixed. Fig 7 shows the 2D slice results using for some representative values of \( \beta \). We can immediately observe a clear improvement in the reconstruction quality. In particular now the opening of the target is clearly visible and much of the blur on the rear side of the object has been mitigated. We observed that in this particular example, the energy term, i.e., the \( \alpha \) parameter, had little effect on the overall reconstruction quality for the FV-LSM approach. This behavior can be observed in Fig 8, where we compare two values of \( \alpha \) while holding \( \beta \) fixed.

Fig. 5. Positions of the 8 observation directions used for reconstruction of the U-shaped target.
whose far-field amplitude is given as [4]

\[ G_{\Sigma,\infty}(\hat{r}_{obs}, \hat{r}_{inc}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{r}_{obs} \cdot \hat{r}_{inc}}. \]  

(31)

Next we estimate the far-field amplitude from the raw multi-static data. In the far-field zone, i.e., \( r_{obs} \gg kl \), the far-field amplitude can simply be estimated as

\[ u_{\infty}(\hat{r}_{obs}, \hat{r}_{inc}, \omega) \approx r_{obs} e^{-ikr_{obs}} u_s(\hat{r}_{obs}, \hat{r}_{inc}, \omega). \]  

(32)

A visualization of the estimated far-field data for both incident angles are shown in Fig 11. Interestingly, from Fig 11 we observe that the far-field response for the 0° incident case seems to be corrupted with a higher level of noise than the 90° case. We note that we did not apply any low-pass filtering to smooth out the pattern.

As discussed above, the FV approach is applicable to the case of sparse observation aperture with dense incident aperture. However in this case we have a sparse incident aperture (only two angles), and a dense observation aperture. In fact, generally in these laboratory experiments it is less costly to observe the field at multiple angles than it is to interrogate the scene with a large number of angles. Fortunately, we can adapt to this case by employing the reciprocity principle [4]. More specifically, we use as input data the following:

\[ u_{input}^{\infty}(\hat{r}_{obs}, \hat{r}_{inc}, \omega) = u_{\infty}(\hat{r}_{inc}, \hat{r}_{obs}, \omega). \]  

(33)

With this simple transformation we can now approximate the integral over the unit-circle using 360 quadrature points, but leaving only two observation angles at 180° and 270°.
Fig. 9. Isosurface of an FV reconstruction for the U-shaped target. An iso-value of 0.6 was used for this particular case.

(a) Picture of UXO target

(b) Multi-static geometry

Fig. 10. Showing the UXO target as well as the multi-static configuration used in this experiment.

B. Imaging Results

The image plane shown as the turquoise plane in Fig 10-b was chosen to have dimensions $V = [-1, 1] \times [-1, 1]$ with a spatial resolution of $20 \times 20$. For visual purposes, we post-processed the LSM reconstructed images to have pixel intensities lie in the range $[0, 1]$, and to have a spatial resolution of $100 \times 100$. Using the discretization procedure we described in section II, the far-field block-diagonal matrix $F$ consisted of 318 rows and 57240 columns. Our first experiment goes towards validating that the FV-LSM approach provides adequate stability for sparse aperture imaging. Fig 12 shows various reconstructions comparing the Tikhonov and the FV approaches. The first two reconstructions shown in Figs 12-a and 12-b, are the resulting images for a pure Tikhonov approach. We can clearly see that a Tikhonov approach fails to provide a physically meaningful image. Fig 12-c corresponded to the case of a pure FV approach (i.e., $\alpha = 0$), and also failed to detect the presence of the UXO. We note that for this case the matrix $K$ was rather ill-conditioned, regardless of the choice of parameter $\rho$. This could have been an indication that the null-spaces of the $A$ and $D_\omega$ had a non-trivial intersection. On the other hand, the results in Figs 12-d through f, where we varied the value of $\alpha > 0$ while keeping a fixed value of $\beta = 3e-4$, show a remarkable improvement in reconstruction quality. Although in these examples we can observe some distortion appearing on the top of the UXO (the 90° viewing angle), most likely due to the lack of any observation data from that direction, we can see that choosing $\alpha$ appropriately helps to mitigate this particular image artifact.

We also conducted experiments to observe the effect of frequency step-size. We note that using only half of the frequency samples did not result in any significant degradation in the reconstruction. However subsequent attempts using a frequency step-size larger than $\Delta k \approx 2 \text{ m}^{-1}$ resulted in unwanted artifacts appearing in the image. An example of which is shown in Fig 13.

We now turn our focus onto illustrating the effect of $\beta$ on reconstruction quality; specifically we wish to see how sensitive the method is to small changes in the $\beta$ parameter. It is usually the case with $\ell_1$ type penalties, that changes with respect to the regularization parameter are piece-wise constant. This is in contrast to $\ell_2$ type penalties where reconstruction variations are smooth with respect to parameter variations. To quantitatively assess this hypothesis, we ran the proposed FV-LSM imaging procedure for some values of $\beta$ while holding $\alpha = 1e-7$ fixed. For each reconstruction in this set, we thresholded the resulting image using a constant cut-off value of $c = 0.55$, which yields a binary image that estimates the UXO support. We then proceeded to estimate the dimensions (length and diameter) of the labeled target area by computing the maximum and minimum values in both the X and Y directions. These estimated parameters were then compared to the actual dimensions of the UXO. What we observed for this particular case was that, once $\beta$ was chosen to be sufficiently large enough, the procedure exhibited a robust dependence with respect variations in $\beta$. The results displayed in table I support this conclusion. We note that a similar observation was made in the previous numerical experiment with the U-shaped obstacle.

VI. Summary

In this paper we presented a regularization approach for multi-frequency acoustic LSM imaging suitable for sparse aperture configurations. Based on the argument that any Herglotz density satisfying the far-field inequality should exhibit frequency variation proportional to the frequency step size, a pixel-independent compressive approach can be applied when using a small enough frequency step-size. This was accomplished here by augmenting the classical Tikhonov approach with an $\ell_1$ penalty on the solution’s frequency differences. An iterative imaging procedure based on the ADMM was then derived to minimize the proposed objective function. Proof of concept of the approach was demonstrated on a
Fig. 11. Polar plots of the estimated far-field scattering data shown here for 3 particular frequencies.

(a) $\theta = 0^\circ$

(b) $\theta = 90^\circ$

Fig. 12. UXO imaging results using both Tikhonov and FV-LSM approaches. A large improvement in imaging quality can be observed when considering the FV-LSM approach.

Table I
A tabulation of the estimated dimensions of the reconstructed UXO for varying values of $\beta$ and fixed $\alpha = 1e^{-7}$.

<table>
<thead>
<tr>
<th>$\beta$-Value</th>
<th>Estimated Length (m)</th>
<th>Length Error (%)</th>
<th>Estimated Diameter (m)</th>
<th>Diameter Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1e - 6$</td>
<td>0.400</td>
<td>9.9</td>
<td>0.323</td>
<td>154.5</td>
</tr>
<tr>
<td>$1e - 5$</td>
<td>0.444</td>
<td>0.9</td>
<td>0.363</td>
<td>186.3</td>
</tr>
<tr>
<td>$1e - 4$</td>
<td>0.424</td>
<td>5.4</td>
<td>0.161</td>
<td>272.2</td>
</tr>
<tr>
<td>$1e - 3$</td>
<td>0.424</td>
<td>5.4</td>
<td>0.161</td>
<td>272.2</td>
</tr>
<tr>
<td>$1e - 2$</td>
<td>0.424</td>
<td>5.4</td>
<td>0.161</td>
<td>272.2</td>
</tr>
</tbody>
</table>

numerically generated data set as well as physical underwater scattering measurements. In both cases, more stable and accurate reconstructions were achievable by the proposed FV approach as compared to the traditional Tikhonov approach.

This was especially the case in the physical experiment, where only two orthogonal views were used in the reconstruction. We also presented some results showing how both parameters can influence the reconstruction quality. We also note that,
although the procedure described in this manuscript applied to the case of sparse observation aperture only, we showed in the physical experiment that the algorithm can be used with a sparse incident aperture if the observation aperture is sufficiently dense.

Future directions of this research will focus on employing sparse aperture qualitative methods for the purpose of feature extraction. This will entail investigation into the feasibility of using detected interior modes (which can be estimated using no prior information), and hybrid approaches with non-linear inversion methods to extract other types of salient features. Future work will also consider sparse-sparse multi-static configurations for LSM based imaging.

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