A FINITE STRAIN BEAM FORMULATION. THE THREE-DIMENSIONAL DYNAMIC PROBLEM. PART I

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0. Introduction

We consider in this paper the continuum basis relevant to the numerical formulation of a finite strain beam theory. The approach discussed here generalizes to the fully 3-dimensional dynamical case the formulation originally developed by Reissner [5] for the plane static problem, and should be regarded as a convenient parametrization of a three-dimensional extension of the classical Kirchhoff–Love rod model [3] due to Antman [9]. This extension, developed in [9] in the context of a director type of approach, includes finite extension and finite shearing of the rod, in contrast with the Kirchhoff–Love model. Here we proceed directly by constraining the 3-dimensional theory with the introduction of the kinematic assumption. A basic step is the formulation of the basic kinematics of the beam in terms of a 3-dimensional orthogonal moving frame defined so that one of its vectors, denoted by \( n \), remains normal to a typical cross-section in any configuration. The resultant force and torque acting on a typical cross-section are then resolved relative to this moving frame. Both for the plane [5], and the three-dimensional problem [9], this leads to a convenient parametrization of the basic equation with theoretical advantages when discussing existence of solutions [9]. We note that this moving frame does not coincide with the convected basis unless shear deformation of the rod is ignored.

Thus, the configurations of the beam are completely defined by specifying the evolution of an orthogonal matrix, the real eigenvalue of which gives the rotation of a typical cross-section, and the position vector of the line of centroids. From a computational standpoint, particularly in the context of the finite element method, such a description allows complete freedom in the best-suited parametrization of the orthogonal transformation. The classical Euler angles, or use of quaternions are two of the possibilities. Furthermore, this kinematic description allows an alternative, computationally much simpler, approach based upon the choice of the incremental vorticity of the moving frame directly as the rotational degree of freedom. This incremental rotation, a skew-symmetric tensor field, is employed to update the total rotation field of the rod described by the evolution of an orthogonal matrix. The simplicity of this updating procedure relies crucially on the simple expression taken by the exponential of a skew-symmetric matrix (not

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necessarily constant) sometimes referred to as the Rodrigues formula. For an extensive discussion of large rotations we refer to Argyris [2].

A basic purpose of this paper is to carefully lay out the basic equations one wishes to numerically treat, parametrized in the present context in terms of a position vector and an orthogonal matrix that depends on the arc length in the reference configuration and time. Note that here, as in the classical Kirchhoff–Love formulation [3, 9], the concepts of rotation and moment have the classical meaning. Thus, rotations are actions of the orthogonal group on the Euclidean space which do not commute. This is in contrast with an alternative approach advocated by Argyris and co-workers (e.g., see [10, 11, 12]) which makes use of the concept of semitangential rotation and semitangential moment. It is also emphasized that in the present context, as in the classical Kirchhoff–Love model, the resulting geometric stiffness, although non-symmetric away from equilibrium, is always symmetric in an equilibrium configuration. These and related issues, together with the numerical implementation of the formulation discussed herein in the context of the finite element method, will be addressed in a separate publication.

The basic contents of this paper can be outlined as follows. After describing the basic kinematics involving the moving frame in Section 1, we summarize the expressions for the linear and angular momentum in Section 2. The former is associated with the acceleration of the centroid and the latter with the time derivative of the vorticity vector of the moving frame. In Section 3, we summarize the basic laws of motion and discuss the spatial and material descriptions of the resultant force and resultant torque acting on a cross-section. This distinction is essential for later developments.

In Section 4 starting with the general 3-dimensional expression for the internal power we develop spatial and material reduced expressions involving the resultant force and torque and their conjugate strain rates. In the material description the reduced expression for the internal power enables one to identify the appropriate strain measures. In particular, the strain measure conjugate to the resultant torque has a simple geometric interpretation as the axial vector of a skew-symmetric tensor associated with the moving frame. In the spatial description, in addition to the appropriate strain measures, the reduced expression for the internal power determines the appropriate (objective) strain rate as the rate of change relative to an observer spinning with the moving frame. We finally show that for the plane problem the present development reduces to the formulation due to Reissner [5], and advance some remarks on the implementation procedure.

1. Basic kinematics

In this section we discuss basic kinematic concepts relevant to the present 3-dimensional nonlinear model.

1.1. Moving basis. Kinematic assumption

Geometrically, the current configuration of a rod is described by defining a family of cross-sections the centroids of which are connected by a curve which we refer to as the line of centroids. Notice that as a result of shearing of the rod, cross-sections are not normal to the line of centroids in the current configuration. Accordingly, to specify the current configuration of the beam we formally introduce the following objects:
(i) A curve defined in an open interval \( \Sigma \subset \mathbb{R} \),

\[
S \in I \rightarrow \varphi_0(S) \in \mathbb{R}^3,
\]

called the line of centroids.

(ii) A family of planes defined by the unit vector field,

\[
S \in I \rightarrow n(S) \in \mathbb{R}^3.
\]

The planes through \( \varphi_0(S) \in \mathbb{R}^3 \) normal to \( n(S), S \in I \), will be referred to as cross-sections of the rod.

(iii) A fiber within each cross-section defined by the unit vector field,

\[
S \in I \rightarrow t_i(S) \in \mathbb{R}^3.
\]

Thus, at each point of the curve \( S \rightarrow \varphi_0(S) \) we may define an orthonormal frame \( \{t_1(S), t_2(S), n(S)\} \), which we shall refer to as moving or intrinsic frame, such that

\[
\|n(S)\| = 1, \quad \|t_i(S)\| = 1, \quad n(S) \cdot t_i(S) = 0, \quad i = 1, 2, \quad t_1(S) \cdot t_2(S) = 0
\]

and

\[
t_3(S) = n(S) = t_1(S) \times t_2(S), \quad S \in I \subset \mathbb{R}.
\]

For convenience, the notation \( t_3(S) = n(S) \) will often be employed. Our basic kinematic assumption, then, is that the admissible configurations of the rod, denoted by

\[
\varphi : A \times I \rightarrow \mathbb{R}^3,
\]

where \( A \subset \mathbb{R}^2 \) is compact, have the following explicit form:

\[
x = \varphi(\xi_1, \xi_2, S) = \varphi_0(S) + \sum_{r=1}^{2} \xi_r t_r(S).
\]

Here, the parameter \( S \in I \) represents the arc of length of the line of centroids in the reference (unstressed) configuration. The geometric significance of assumption (1.6) is illustrated for the plane case in Fig. 1. For simplicity, in what follows it will be often assumed that the unstressed configuration of the rod, which is taken as the reference configuration, is such that the line of centroids is a straight line so that the moving frame in the reference configuration becomes simply the standard basis in \( \mathbb{R}^3 \), and is denoted by \( \{E_i\} \). As shown in Remark 1.2 below, the consideration of initial curvature is straightforward. For convenience, we shall often use the notation \( \xi = \xi_1E_1 + \xi_2E_2 \).

**Remark 1.1.** We emphasize that the unit vector field \( S \rightarrow n(S) \) is not tangent to the line of centroids \( S \rightarrow \varphi_0(S) \) in the current configuration; but normal to the cross-section passing through \( \varphi_0(S) \).
1.2. Derivatives of the moving basis

Since the moving basis \( \{t_i(S)\} \) is orthonormal for each \( S \in I \), there exists an orthogonal transformation \( S \rightarrow \Lambda(S) \in SO(3) \), where \( SO(3) \) stands for the special orthogonal (Lie) group, such that

\[
t_i(S) = \Lambda(S)E_i \quad \text{or} \quad t_i(S) = \Lambda_{ij}e_j, \quad I = 1, 2, 3,
\]  

(1.7)

where \( \{e_i\} \) denotes the fixed spatial frame, not necessarily coincident with \( \{E_i\} \), and \( \Lambda(S) = \Lambda_{ij}e_i \otimes E_j \) is a two-point orthogonal tensor field. Taking derivative of (1.7) relative to \( S \in I \) we obtain

\[
\frac{d}{dS} t_i(S) = \Omega(S)t_i(S),
\]  

(1.8)

where

\[
\Omega(S) = \left[ \frac{d}{dS} \Lambda(S) \right] \Lambda'(S)
\]  

(1.9)

is a skew-symmetric tensor field; i.e., \( \Omega(S) + \Omega'(S) = 0 \). Since \( \Omega(S) \) is a spatial tensor for each \( S \in I \), its components may be given relative to the moving frame \( \{t_i\} \), and expressed in matrix form as

\[
[\Omega(S)] = \begin{bmatrix}
0 & \kappa_3(S) & -\kappa_2(S) \\
-\kappa_3(S) & 0 & \kappa_1(S) \\
\kappa_2(S) & -\kappa_1(S) & 0
\end{bmatrix}.
\]  

(1.10)

It is convenient to introduce the axial vector field \( S \rightarrow \omega(S) \in \mathbb{R}^3 \) associated with the skew-
symmetric tensor $\Omega(S)$, which is defined by the relation $\Omega(S)\omega(S) = 0$. Thus, relative to the moving frame we have the representation

$$\omega(S) = \kappa_1(S)t_1(S) + \kappa_2(S)t_2(S) + \kappa_3(S)t_3(S).$$  \hfill (1.11)

The derivatives of the moving frame given by (1.8) may then be recast into the alternative expression

$$\frac{d}{dS} t_i(S) = \omega(S) \times t_i(S), \quad i = 1, 2, 3.$$  \hfill (1.12)

This completes the basic kinematic relations needed for subsequent developments. We note the following.

**REMARK 1.2.** If the line of centroids in the reference configuration is an arbitrary curve, not necessarily a straight line, the basis $\{E_i\}$ becomes a function of $S \in I$. Denoting by $\hat{t}_i$ the standard basis in $\mathbb{R}^3$, since $\{E_i\}$ is orthonormal we may write

$$E_i(S) = \Lambda_0(S)\hat{t}_i \Rightarrow \frac{dE_i}{dS} = \Omega_0(S)E_i,$$  \hfill (1.13)

where $\Omega_0 = [d\Lambda_0/dS] \Lambda_0^t$ gives the ‘initial curvature’ of the line of centroids. Hence, the basic equation (1.8) remains valid provided $\Omega$ is replaced by

$$\tilde{\Omega} = \Omega + \Lambda \Omega_0 \Lambda^t.$$  \hfill (1.14)

We may conclude that the consideration of initial curvature in the rod amounts to introducing the second term appearing (1.14) in the definition of $\Omega$ given by (1.9).

**REMARK 1.3.** The moving frame $\{t_i(S)\}$ should not be confused with the convected basis which is often used in the development of rod theories (e.g. [1, 4]), and is defined as follows. Let $F(\xi, S)$ be the deformation gradient and $F_0(S) = F(\xi, S)_{\xi = 0}$. Then, the convected basis $\{\Xi_i\}$ is defined as

$$\Xi_i(S) = F_0(S)E_i, \quad i = 1, 2, 3.$$  \hfill (1.15)

From the basic kinematic assumption (1.6) it easily follows that

$$F_0(S) = \sum_{r=1}^2 t_r(S) \otimes E_r + \frac{d}{dS} \varphi_0(S) \otimes E_3.$$  \hfill (1.16)

The convected base vectors, then, are given by

$$\Xi_1(S) = t_1(S), \quad \Xi_2(S) = t_2(S), \quad \Xi_3(S) = \frac{d}{dS} \varphi_0(S).$$  \hfill (1.17)
Thus, the essential difference between the *convected* basis and the *moving* basis \( \{ t_i(S) \} \) is that \( \Xi_3 \) is *tangent* to the line of centroids whereas \( n(S) \equiv t_3(S) \) is *normal* to the cross-section. Notice also that the *moving* basis \( \{ t_i(S) \} \) is *orthonormal* whereas the *convected* basis is not. If shear deformation is not taken into account the difference between both bases disappears.

**REMARK 1.4.** It is emphasized that the vector field \( S \rightarrow \omega(S) \), although parametrized for convenience by the reference arc length \( S \in I \), takes values on the *current configuration*. Accordingly, its components are given relative to a spatial basis; either \( \{ e_i \} \) or relative to \( \{ t_i \} \) as in (1.11). Alternatively, we may define a *material* vector field by setting

\[
S \rightarrow K(S) = \kappa_t(S)E_i.
\]  

(1.18)

In view of (1.7), \( \omega(S) \) and \( K(S) \) are *spatial* and *material* vector fields related according to

\[
\omega(S) = \Lambda(S)K(S).
\]  

(1.19)

The vector \( K(S) \) appears naturally in the *material* form of the reduced expression for the internal power, as shown in Section 4.

**REMARK 1.5.** The basic kinematic assumption (1.6) precludes changes in the cross-sectional area. More elaborated kinematic assumptions explicitly accounting for this effect may be formulated in the spirit of projections methods; e.g., see [1].

2. Motion. Linear and angular momentum

In this section we extend the kinematic concepts discussed above to account for dynamic effects. The expression for the angular momentum involves the *vorticity* vector associated with the moving frame and has identical structure to the expression found in rigid-body mechanics.

A *motion* of the rod is a curve of configurations parametrized by time; that is,

\[
t \rightarrow \varphi = \varphi_0(S, t) + \sum_{i=1}^{2} \xi_i t_i(S, t),
\]

(2.1)

where \( t \in \mathbb{R}^+ \) is the *time*. Prior to introducing the linear and angular momentum vector fields associated with the motion (2.1), we need the following result.

**Time derivatives of the moving frame.** The moving frame \( \{ t_i(S, t) \} \) is defined by (1.7) where the orthogonal transformation now depends on time; e.g., \( (S, t) \rightarrow A(S, t) \). Denoting by a superposed ‘dot’ the *material* time derivative, we then have

\[
\dot{t}_i(S, t) = [\dot{A}(S, t)A^t(S, t)]t_i(S, t) = W(S, t)t_i(S, t), \quad I = 1, 2, 3,
\]

(2.2)

where \( W(S, t) = -W^t(S, t) \) is a *spatial skew-symmetric* tensor which defines the *spin* of the moving frame. The associated *axial* vector \( w(S, t) \), which satisfies \( W(S, t)w(S, t) = 0 \), gives the *vorticity* of the moving frame. In terms of the vorticity vector, (2.2) may be written as

\[
\dot{t}_i(S, t) = w(S, t) \times t_i(S, t).
\]

(2.3)
Linear and angular momentum. Consider an arbitrary cross-section denoted by $A_t$ and given by $A_t = \varphi|_{s=\text{Fixed}(A)}$, for each $s \in I$. We define the linear momentum per unit of reference arc length, associated with the motion (2.1), by the integral

$$L_t = \int_A \rho_0(\xi, S)\dot{\varphi}_t(\xi, S, t) \, d\xi = A_t \dot{\varphi}_0(S, t),$$

(2.4)

where $\rho_0(\xi, S)$ is the density in the reference configuration, and we have employed the fact that $S \rightarrow \varphi_0(S, t)$ defines the current position of the centroid of the cross-section.

Similarly, the angular momentum per unit of reference arc length, associated with the motion (2.1), and relative to the point $x_0 = \varphi_0(S, t)$, is defined as

$$H_t = \int_A \rho_0(\xi, S)[x - \varphi_0(S, t)] \times \dot{\varphi}_t(\xi, S, t) \, d\xi,$$

(2.5)

where $x = \varphi(\xi, S, t)$. To find a reduced expression for $H_t$ we make use of (2.3) as follows.

Expression for $H_t$. From the kinematic assumption (2.1) and (2.3) we have

$$\dot{\varphi} - \dot{\varphi}_0 = \sum_{r=1}^2 \xi_r \dot{t}_r = \omega \times (\varphi - \varphi_0).$$

(2.6)

Substitution of (2.6) into (2.5) together with the fact that $x_0 = \varphi_0(S, t)$ defines the centroid of the cross-section, yields

$$H = \int_A \rho_0(\varphi - \varphi_0) \times [\omega \times (\varphi - \varphi_0)] \, d\xi$$

$$= \left[ \int_A \rho_0)||\varphi - \varphi_0||^21 - (\varphi - \varphi_0) \otimes (\varphi - \varphi_0) \right] \omega = I_\rho \omega,$$

(2.7)

where $I_\rho$ is the inertia tensor with the following explicit representation relative to the moving frame:

$$I_\rho = \left[ \sum_{A=1}^2 \sum_{B=1}^2 \int_A \rho_0(\xi_A \xi_B) \, d\xi \right]\delta_{AB}1 - t_A \otimes t_B.$$  

(2.8)

Notice that the components of $I_\rho$ relative to the moving frame do not depend on time. Taking the material time derivative of (2.7), noting that $\dot{\omega} = \omega_\dot{t}_t$ and making use of (2.3), we obtain the following expression for $\dot{H}_t$,

$$\dot{H}_t = I_\rho \dot{\omega} \times \omega \times H_t.$$

(2.9)

The analogy between (2.9) and the expression for the angular momentum in rigid-body mechanics is evident.

REMARK 2.1. Let us consider the particular case in which the moving frame $\{t_A(S, t)\}$ is directed along the principal axes of inertia of the cross-section. Introducing the notation

$$I_1(S) = \int_A [\xi_1]^2 \rho_0(\xi, S) \, d\xi, \quad I_2(S) = \int_A [\xi_2]^2 \rho_0(\xi, S) \, d\xi,$$

(2.10)
and denoting by \( J = I_1 + I_2 \) the polar moment of inertia of the cross-section, expression (2.7) for the inertia tensor associated with the cross-section of the rod in its current configuration takes the familiar form

\[
I_p = I_1 t_1 \otimes t_1 + I_2 t_2 \otimes t_2 + J n \otimes n. \tag{2.11}
\]

3. Force and torque. Equations of motion

In this section we summarize the equations of motion for the nonlinear beam model. A comprehensive treatment can be found in e.g. [1, Section 6]. For completeness a simple derivation is included in Appendix A. The component form of these equations in the material description take a particularly simple form involving the orthogonal matrix \([A]\) which is well suited for computational purposes.

Consider a cross-section \( A_r = \phi_r|_{S=\text{Fixed}(A)} \) in the current configuration, and let \( P(\xi, S) \) denote the first Piola–Kirchhoff stress tensor. We may express the two-point tensor \( P(\xi, S) \) as

\[
P(\xi, S) = T_1(\xi, S) \otimes E_1 + T_2(\xi, S) \otimes E_2 + T_3(\xi, S) \otimes E_3. \tag{3.1}
\]

Clearly, \( T_3(\xi, S) = P(\xi, S)E_3 \) is the stress vector (per unit of reference area) acting on the cross-section \( A_r \subset \mathbb{R}^2 \).

The resultant contact force per unit of reference length \( f(S, t) \) over the cross-section \( A_r \), in the current configuration is then given by

\[
f(S, t) = \int_A P(\xi, S)E_3 \, d\xi = \int_A T_3(\xi, S) \, d\xi. \tag{3.2a}
\]

Similarly, the resultant torque per unit of reference arc length \( m(S, t) \) over the cross-section \( A_r \), in the current configuration is given by

\[
m(S, t) = \int_A [x - \phi_0(S, t)] \times T_3(\xi, S) \, d\xi. \tag{3.2b}
\]

The linear and angular momentum balance equations then take the form (see Appendix A)

\[
\frac{\partial}{\partial S} f + \ddot{\xi} = \tilde{A}_t \tilde{\phi}_0, \tag{3.3a}
\]

\[
\frac{\partial}{\partial S} m + \frac{\partial \phi_0}{\partial S} \times f + \dot{\tilde{m}} = \tilde{H}_r \equiv L_r w + w \times H_r, \quad S \in I, \tag{3.3b}
\]

where \( \ddot{\xi} \) and \( \ddot{m} \) are the 'applied' force and torque per unit of reference arc length. In applications, the material form of these equations is often more convenient.

3.1. Material description

The vector fields \( f(S, t) \) and \( m(S, t) \), although parametrized for convenience by the reference arc length \( S \in I \), take values on the current configuration; i.e., their components are
given relative to a spatial basis, either \{e_i\} or \{t_i\}. Alternatively, we define material vector fields

\[ S \rightarrow N = N_iE_i, \quad S \rightarrow M = M_iE_i, \quad S \in I, \]  

(3.4)

by pulling back\(^1\) the vector fields \( f(S, t) \) and \( m(S, t) \) to the reference configuration \( A \times I \subset \mathbb{R}^3 \) with the orthogonal transformation \( S \rightarrow A(S, t) \). Accordingly, we have the relations:

\[ f = \Lambda N \quad \text{and} \quad m = \Lambda M. \]  

(3.5)

The geometric meaning of \( N(S, t) \) and \( M(S, t) \) follows from the observation that

\[ f = N_i \Lambda E_i = N_i t_i, \quad \text{and} \quad m = M_i t_i. \]  

(3.6)

Thus, the components of the force and moment vectors \( f \) and \( m \) relative to the moving frame \( \{t_i\} \) equal those of \( N \) and \( M \) relative to the reference frame \( \{E_i\} \).

The component form of the equations in the material description are obtained by substitution of (3.5) into (3.3a), (3.3b).

**REMARK 3.1.** The classical equations of thin rods of Kirchhoff–Love [3, pp. 387–388] may be now recovered from (3.3a) and (3.3b) as follows. First, we introduce the current arc length defined by the map

\[ S \rightarrow s(S) = \int_0^S \| \partial \varphi_0(\mu, t) / \partial \mu \| d\mu, \]  

(3.7)

which may be regarded as a smooth reparametrization. Next, we note that if no shearing effect is considered, we must have

\[ \frac{\partial}{\partial S} \varphi_0(S, t) = \frac{ds}{dS} n(s), \quad \| n(s) \| = 1, \]  

(3.8)

which is simply the first Frenet formula. From (3.6)\(_2\) we have

\[ \frac{\partial m}{\partial S} = \frac{ds}{dS} \left[ \frac{\partial M_i}{\partial S} - \Omega_{ij}M_j \right] t_i. \]  

(3.9)

Making use of (3.8) and (3.6)\(_1\), since \( n \times t_3 = t_2 \) and \( n \times t_2 = -t_1 \), we also have

\[ \frac{\partial \varphi_0}{\partial S} \times f = \frac{ds}{dS} [N_1 t_2 - N_2 t_1]. \]  

(3.10)

Substitution of (3.9) and (3.10) into (3.3b) leads, for the static case and with the assumption that \( \ddot{n} = 0 \), to the Kirchhoff–Love moment equilibrium equations\(^2\) [3; p. 388, Equation (11)]. The force equilibrium equation follows at once from (3.3a) and (3.6)\(_1\).

\(^1\)For a formal definition of the pull-back operation see e.g. [8].

\(^2\)Notice that \( \Omega \) defined by (1.10) would have opposite sign with the convention in [3: p. 384, Equation (5)].
4. Internal power and strain measures. Constitutive equations

Our purpose in this section is to formulate properly invariant reduced constitutive equations in terms of global kinetical and kinematical objects. Our first step is to obtain a reduced expression for the internal power from the general expression of 3-dimensional theory, by introducing the basic kinematic assumption (2.1). This reduced expression yields the appropriate definition of strain measures conjugate to the resultant force and moment in the spatial as well as in the fully material descriptions.

4.1. Internal power. Strain measures

We first consider the reduced expression for the internal power in terms of the spatial force \( f(S, t) \) and torque \( m(S, t) \) defined by (3.2a) and (3.2b), respectively. The basic result is summarized in the following

**PROPOSITION 4.1.** With the kinematic assumption (2.1) in force, the internal power \( II \) may be expressed as

\[
II = \int_{A \times I} P : \dot{F} \, d\xi \, dS - \int_I (f \cdot \gamma + m \cdot \nu) \, dS, \tag{4.1a}
\]

where \( \omega(S, t) \) is the spatial vector with components given by (1.11), \( \gamma(S, t) \) is a spatial vector defined by

\[
\gamma(S, t) = \frac{\partial \varphi_0}{\partial S}(S, t) - n(S, t), \tag{4.1b}
\]

and \( (\cdot) \) stands for the following objective rate:

\[
(\cdot)^\nu = \frac{\partial}{\partial t}(\cdot) - \omega \times (\cdot). \tag{4.1c}
\]

**PROOF.** To prove (4.1a), (4.1b) we first compute the deformation gradient. From (2.1) and using (1.12) we obtain

\[
F = \sum_{l=1}^2 t_r \otimes E_r + \left[ \frac{\partial \varphi_0}{\partial S} + \omega \times (x - \varphi_0) \right] \otimes E_3. \tag{4.2}
\]

Taking the material time derivative and making use of (2.3), we have

\[
\dot{F} = \sum_{l=1}^2 (w \times t_r) \otimes E_r + \left[ \frac{\partial \varphi_0}{\partial S} + \omega \times T_3 \right] \otimes E_3 + \left[ \omega \times \{w \times (x - \varphi_0)\} \right] \otimes E_3. \tag{4.3}
\]

Since \( P = \sum_{l=1}^3 T_l \otimes E_l \), it follows that

\[
P : \dot{F} = T_3 \cdot \frac{\partial \varphi_0}{\partial S} + [(x - \varphi_0) \times T_3] \cdot \omega + T_3 \cdot [\omega \times \{w \times (x - \varphi_0)\}] + \sum_{l=1}^2 w \cdot (t_r \times T_r). \tag{4.4}
\]
We now make use of the angular momentum balance condition (see Appendix A) \( \partial \varphi / \partial \xi_t \times T_t = 0 \), and (1.12) to express the last term in (4.4) as

\[
\sum_{r=1}^{3} w \cdot (t_r \times T_r) = w \cdot \sum_{r=1}^{3} \frac{\partial}{\partial S} (x - \varphi_0) \times T_r
\]

\[
= -w \cdot \left[ \frac{\partial \varphi}{\partial S} \times T_3 \right] = -T_3 \cdot \left[ w \times \frac{\partial \varphi_0}{\partial S} + w \times \{ \omega \times (x - \varphi_0) \} \right].
\]

(4.5)

Substitution of (4.5) into (4.4), use of definitions (3.2a) and (3.2b), together with the identity

\[
\{ w \times (x - \varphi_0) \} = \{ w \times (x - \varphi_0) \} - \{ w \times \omega \} (x - \varphi_0) = \{ w \times \omega \} \times (x - \varphi_0),
\]

leads to the following reduced expression for the internal power:

\[
\Pi = \int \left\{ f \cdot \left[ \frac{\partial}{\partial t} \left( \frac{\partial \varphi_0}{\partial S} \right) - w \times \frac{\partial \varphi_0}{\partial S} \right] + m \cdot [ \omega - w \times \omega ] \right\} dS,
\]

(4.6)

which proves the proposition.

REMARK 4.2. The physical significance of the rate \( \cdot \) should be clear. It gives the rate of change of \( (\cdot) \) relative to an observer which moves with the spatial frame \( \{ t_t \} \), since the effect of the spin of the moving frame \( \{ t_t \} \) given by \( w \) is subtracted from the material time derivative. Thus, one often speaks of a corrotated rate. This interpretation follows at once from (2.3).

Alternatively, we may recast the reduced expression (4.1a) for the internal power in terms of material fields \( N \) and \( M \) as follows.

PROPOSITION 4.3. With the kinematic assumption (2.1) in force, the reduced expression for the internal power may be expressed as

\[
\Pi = \int_{I} P : \mathbf{F} \, d\xi \, dS = \int_{I} [N \cdot \mathbf{F}^t + \mathbf{M} \cdot \mathbf{K}] \, dS,
\]

(4.7)

where \( N(S, t) \) and \( M(S, t) \), are material vector fields defined by (3.5), and \( I'(S, t) \), \( K(S, t) \) are material vector fields given by

\[
\Gamma = \Lambda \frac{\partial \varphi_0}{\partial S} - \mathbf{E} = \Lambda \left[ \frac{\partial \varphi_0}{\partial S} - \mathbf{n} \right], \quad K = \Lambda \omega.
\]

(4.8a)

PROOF. The result follows at once from (2.2)–(2.3) by noting that for any spatial vector \( h = h_t t_t \) we have

\[
\frac{v}{h} = \frac{\partial}{\partial t} h - w \times h = \frac{\partial}{\partial t} h - Wh = \Lambda \frac{\partial}{\partial t} [\Lambda' h].
\]

(4.9)
Since \( m, \mathbf{M}, \mathbf{f} \) and \( \mathbf{N} \) are related according to (3.5), use of Proposition 4.1 and (4.8b) supplies the result.

**REMARK 4.4.** It is interesting to examine the limiting case of the Kirchhoff–Love situation considered in Remark 3.1. As a result of assumption (3.8) the strain measures \( \gamma \) and \( \Gamma \) reduce to

\[
\gamma = \left[ \frac{ds}{dS} - 1 \right] \mathbf{n} \quad \text{and} \quad \Gamma = \left[ \frac{ds}{dS} - 1 \right] \mathbf{E}_3. \tag{4.10}
\]

since \( \mathbf{n} = \mathbf{t}_3 = \mathbf{A} \mathbf{E}_3 \). Thus, shear deformation of the rod vanishes identically. Actually, the situation discussed in [3, Sections 255–256, pp. 388–393] corresponds to that of a superposed infinitesimal deformation, and may be obtained as a particular case of Proposition 4.3 by consistent linearization procedures.

**REMARK 4.5.** The strain measures (4.8b) could also be obtained by starting with the material form of the balance equations and making use of a one-dimensional virtual-work type of argument as in [5]. Our approach, however, proceeds directly from the 3-dimensional theory.

Next, we formulate global constitutive equations.

### 4.2. Constitutive equations

In what follows, attention is restricted to the elastic case and the pure mechanical theory. More general situations including heat conduction may be considered by the methods in e.g. [4] or [1]. For present purposes we simply note that as a result of Proposition 4.1 for elastic behavior we may define a stored energy function \( \psi(S, \gamma, \omega) \) such that

\[
f = \frac{\partial \psi(S, \gamma, \omega)}{\partial \gamma} \quad \text{and} \quad m = \frac{\partial \psi(S, \gamma, \omega)}{\partial \omega}, \quad S \in I. \tag{4.11}
\]

Similarly, in the material description, as a result of Proposition 4.3 we may define a stored energy function \( \Psi(S, \Gamma, K) \) such that

\[
\mathbf{N} = \frac{\partial \Psi(S, \Gamma, K)}{\partial \Gamma} \quad \text{and} \quad \mathbf{M} = \frac{\partial \Psi(S, \Gamma, K)}{\partial K}, \quad S \in I. \tag{4.12}
\]

The function \( \Psi \) is subjected to the usual invariance requirements under superposed rigid-body motions [1, 4]. For computational purposes, particularly for inelasticity, the rate form of constitutive equations (4.11) and (4.12) is often needed. In the material description taking the material time derivative of (4.12) we simply have

\[
\begin{bmatrix}
\dot{\mathbf{N}} \\
\dot{\mathbf{M}}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial \Psi}{\partial \Gamma} & \frac{\partial \Psi}{\partial K} \\
\frac{\partial \Psi}{\partial K} & \frac{\partial \Psi}{\partial K}
\end{bmatrix}
\begin{bmatrix}
\dot{\Gamma} \\
\dot{K}
\end{bmatrix}
= \mathbf{C}(S, \Gamma, K)
\begin{bmatrix}
\dot{\Gamma} \\
\dot{K}
\end{bmatrix}. \tag{4.13}
\]
Making use of (4.9) and the chain rule, (4.13) may be expressed in the spatial description as

\[
\begin{bmatrix}
\mathbf{f} \\
\mathbf{m}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \psi}{\partial \gamma} & \frac{\partial \psi}{\partial \omega} \\
\frac{\partial \phi}{\partial \omega} & -\frac{\partial \phi}{\partial \omega}
\end{bmatrix} \begin{bmatrix}
\mathbf{v} \\
\omega
\end{bmatrix} = \mathbf{c}(S, \gamma, \omega) \begin{bmatrix}
\mathbf{v} \\
\omega
\end{bmatrix}.
\]

(4.14)

We refer to \( \mathbf{C} \) and \( \mathbf{c} \) as the material and spatial elasticity tensors, respectively. In particular, one often assumes in applications that the material elasticity tensor \( \mathbf{C} \) in the rate constitutive equations (4.13) is diagonal with constant coefficients. This is equivalent to assuming a quadratic (uncoupled) expression for the material stored energy function \( \Psi(S, F, K) \). Of this particular type are the constitutive equations of the classical Kirchhoff–Love rod theory. This completes our discussion of constitutive equations.

**Remark 4.6.** With the assumption that the material elasticity tensor \( \mathbf{C} \) in (4.13) is diagonal with constant coefficients, one can formulate simple inelastic constitutive models which are properly invariant and account for viscoplastic response. Such models are particularly useful in computational applications. See [7].

5. Concluding remarks: The plane case

It is first shown that for the plane problem the formulation heretofore presented reduces to that proposed by Reissner [5]. We conclude this section with a remark on the significance of the parametrization employed in this paper in a finite element solution procedure.

(i) Assume that the motion of the beam takes place in the coordinate plane normal to \( E_2 = e_2 = t_2 \), illustrated in Fig. 1. The orthogonal tensor \( A(S) \) then admits, for all \( S \in I \), the matrix representation

\[
A(S) = \begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}.
\]

(5.1)

The axial vector \( \mathbf{\omega}(S) \) given by (1.11), now coincident with \( K(S) \) defined by (1.18), and the vorticity vector \( \mathbf{w} \) have the expressions

\[
\mathbf{\omega} = \frac{\partial \phi}{\partial S} E_2, \quad \mathbf{w} = \frac{\partial \phi}{\partial t} E_2.
\]

(5.2)

The strain measures \( \Gamma \) defined by (4.8b) take the form

\[
\Gamma_3 = (1 + u') \cos \theta + v' \sin \theta - 1, \quad \Gamma_1 = -(1 + u') \sin \theta + v' \cos \theta,
\]

(5.3)

where \( \partial \phi / \partial S = (1 + u')e_3 + v'e_1 \). Introducing the notation
where \( s(S) \) is the current arc length defined by (3.7) so that

\[
\frac{ds}{dS} = \sqrt{(1 + \mu')^2 + (v')^2},
\]

(5.5)

the strains \( \Gamma_3 \) and \( \Gamma_1 \) given by (5.3) may be expressed as

\[
\Gamma_3 = (1 + \varepsilon) \cos(\alpha - \vartheta) - 1, \quad \Gamma_1 = (1 + \varepsilon) \sin(\alpha - \vartheta),
\]

(5.6)

which coincide with the expressions given in \([5, 6]\). Notice that \( \alpha - \vartheta \) defines the shear angle in the natural way. Explicit component expressions for the 2-dimensional equilibrium equations in terms of \( \mathbf{N} \) and \( \mathbf{M} \) follow at once by substitution of (3.5) into (3.3a) and (3.3b).

(ii) As already mentioned in the Introduction, the parametrization employed through this paper is particularly convenient in a numerical treatment of the problem by the finite element method. To illustrate the point, we consider briefly the basic procedure for the configuration update which typically arises in an iterative solution strategy based on Newton's method. In the present context, the configurations of the rod at step \( t_n \) are characterized by the pair \((\varphi_n, \Lambda_n)\). Through the solution process, one obtains an incremental displacement vector \( \Delta \mathbf{u}_{n+1} \), and an (infinitesimal) incremental rotation vector \( \Delta \mathbf{\theta}_{n+1} \). Let \( \Delta \mathbf{\theta}_{n+1} \) be the skew-symmetric tensor associated with the (infinitesimal) rotation vector \( \Delta \mathbf{\theta}_{n+1} \). The configuration update procedure, then, is given simply by

\[
\varphi_{n+1} = \varphi_n + \Delta \mathbf{u}_{n+1}, \quad \Lambda_{n+1} = e^{\Delta \mathbf{\theta}_{n+1}} \Lambda_n.
\]

(5.7)

A crucial point to realize is that the exponential of the skew-symmetric matrix \( \Delta \mathbf{\theta}_{n+1} \) appearing (5.7) can be computed very efficiently without any approximation. This can be achieved with the aid of formulae pertaining to classical rigid-body mechanics and recently discussed in detail by Argyris \([2]\). By making use of (5.7) in a variational setting one obtains a formulation analogous to the one employed by Simo et al. \([7]\) for the treatment of the plane case. This and related aspects will be discussed in detail in a forthcoming paper.

Appendix A

To develop equations of motion expressed in terms of the resultant force \( \mathbf{f}(S, t) \) and the resultant moment \( \mathbf{m}(S, t) \), we proceed from the material form of the balance of linear and angular momentum principles of the 3-dimensional theory, which may be expressed as

\[
\text{DIV} \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \dot{\mathbf{\varphi}}_t, \quad \mathbf{F} \mathbf{P}' = \mathbf{P} \mathbf{F}',
\]

(A.1)

where \((\xi, S) \rightarrow \mathbf{B}(\xi, S, t)\) is the body force field, and \(\text{DIV} \mathbf{P} = \partial \mathbf{T}_t / \partial \xi_t\).

Balance of linear momentum. From (3.1) and (3.2), making use of (A.1)_1, we have
\[ \frac{\partial}{\partial S} f(S, t) = \int_{\Omega} \frac{\partial}{\partial S} T_{3} \, d\xi \]
\[ = - \int_{\Omega} \left[ \sum_{r=1}^{2} \frac{\partial T_{r}}{\partial \xi_{r}} + \rho_{0}B \right] \, d\xi + \int_{\Omega} \rho_{0}\ddot{\varphi}_{r} \, d\xi. \]  
(A.2)

Applying the divergence theorem, and defining the applied load as
\[ \ddot{q}(S, t) = \sum_{r=1}^{2} \int_{\partial \Omega} [T_{r}\nu_{r}] \, d\Gamma + \int_{\Omega} \rho_{0}B \, d\xi, \]  
(A.3)

where \( \nu = \nu_{1}E_{1} + \nu_{2}E_{2} \) is the vector field normal to the ‘lateral’ contour \( \partial \Omega \) of the beam, we obtain the balance equation
\[ \frac{\partial}{\partial S} f(S, t) + \ddot{q}(S, t) = L_{t} = A_{r}\ddot{\varphi}_{o}(S, t), \quad S \in I. \]  
(A.4)

Balance of angular momentum. From (3.2a) and (3.2b) we have
\[ \frac{\partial}{\partial S} m(S, t) = \int_{\Omega} \frac{\partial}{\partial S} \times T_{3} \, d\xi - \frac{\partial}{\partial S} \times f + \int_{\Omega} [\varphi - \varphi_{o}] \times \sum_{r=1}^{2} \frac{\partial T_{r}}{\partial \xi_{r}} \, d\xi + \int_{\Omega} \rho_{0}\ddot{\varphi} \, d\xi \]
\[ = \ddot{H}_{t} + \int_{\Omega} \frac{\partial}{\partial \xi_{r}} \times T_{r} \, d\xi - \frac{\partial}{\partial S} \times f - \ddot{m}(S), \]  
(A.5)

where use has been made of (2.11), the divergence theorem, and the following notation for the applied moment field:
\[ \ddot{m}(S, t) = \sum_{r=1}^{2} \int_{\partial \Omega} [x - \varphi_{o}] \times [T_{r}\nu_{r}] \, d\Gamma + \int_{\Omega} \rho_{0}[x - \varphi_{o}] \times B \, d\xi. \]  
(A.6)

From the balance of angular momentum condition (A.1)_2 it follows that \( \partial \varphi/\partial \xi_{r} \times T_{r} = 0 \). Thus, (A.5) reduces to
\[ \frac{\partial}{\partial S} m(S, t) + \frac{\partial}{\partial S} \times f + \ddot{m}(S, t) - \ddot{H}_{t} = I_{p}\dot{\omega} + \omega \times H_{t}, \quad S \in I, \]  
(A.7)

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References
