I. INTRODUCTION

Oscillators represent an essential part of radar systems and are commonly used to perform frequency and timing synchronization [1–5]. They are exploited in radio frequency (RF) transmitters and receivers to provide the signal for frequency conversion (upconversion and downconversion), in digital systems to generate the clock signal to synchronize operations, and in analog-to-digital converters to provide the necessary reference phase (clock sampling and synthesis). Unfortunately, the outputs of the oscillators are not perfectly periodic and suffer from many imperfections, making difficult the availability of a precise time reference. Practical clocks [chapter 12 in 4, 6] are affected by phase and frequency instabilities producing the so-called phase noise. It consists of random and unwanted fluctuations that inevitably perturb the time-domain linearity of the phase of an ideal sinusoidal oscillation. Assessing the effect of noisy oscillators on the performance of radar systems may allow the development of algorithms able to mitigate these unwanted distortions, as well as to quantify the resulting performance loss.

Nowadays, most of the modern radar systems are coherent, namely, they detect the phase of the received signal, relative to a well-controlled reference, as well as the time delay and the amplitude. Each slow-time sample [chapter 8, 4] of the received signal is treated as a complex number having an amplitude and a phase angle. The necessity of coherence depends on the specific application of the radar system. However, any system that needs to cancel clutter, to measure the Doppler frequency characteristics, or to generate the electromagnetic image of a target requires phase coherency [chapters 8 and 9 in 1, chapters 2, 4, and 17 in 2, chapter 12 in 4]. Otherwise, stated, to appropriately exploit the phase history of the received signal, the transmitted waveform must exhibit a known phase. To guarantee the previous requirement, the most common technique is to synthesize a transmit signal from a set of very stable continuously operating oscillators, i.e., coherent oscillators (COHO) and stable local oscillators (STALO). The former operates at the intermediate frequency (IF), while the latter works at a frequency on the order of the desired output RF signal.

The integrity of the phase measurement depends on the stability of the oscillators that generate the transmit signal and provide local references for the downconversion process. In fact, the phase measurement is performed comparing the phase of the received signal with that of the reference wave used for generating the transmitted signal. One of the most important consequences of phase noise is that in a clutter cancellation system, such as a moving target indication (MTI) or a pulse-Doppler processor (PDP), some of the clutter signal energy will be spread throughout the passband of the processor, limiting the clutter cancellation and thus the target detectability [chapters 8 and 9 in 1, chapter 12 in 4].

Evidently, the effect of phase noise on the radar performance depends on its statistical characterization,
reveals that phase noise affects I reference to MTI algorithms, the proposed framework addressed in Part II of this two-part manuscript. With algorithms and sidelobe blanking architectures is improvement factor I (operating in the presence of phase noise), in terms of coherent integration techniques when phase noise is to the proposed framework, we assess the performance of the available real phase noise PSD measurements. Owing to the proposed framework, we formalize the radar signal model in the presence of phase noise. Specifically, we develop a fast-time/ slow-time data matrix signal representation [chapters 15 and 17, 4], where the undesired phase fluctuations, affecting the reference signal produced by real radar oscillators, are modeled via multivariate circular distributions [15]. Furthermore, we describe the phase noise PSD through a composite power-law model and show the ability of the proposed parametric model to fit the available real phase noise PSD measurements. Owing to the proposed framework, we assess the performance of coherent integration techniques when phase noise is present, providing an analytic expression to predict the performance degradations experienced by MTI algorithms (operating in the presence of phase noise), in terms of improvement factor I [chapter 17, 4]. The study of PDP algorithms and sidelobe blanking architectures is addressed in Part II of this two-part manuscript. With reference to MTI algorithms, the proposed framework reveals that phase noise affects I directly through its characteristic function (CF). Additionally, I is robust with respect to the actual phase noise multivariate circular distribution, as long as the phase noise PSD model correctly represents the available measurements.

The remainder of Part I is organized as follows. Section II is devoted to the description of the system model. Section III focuses on phase noise modeling, while Section IV provides the performance analysis of coherent integration techniques, with emphasis on MTI algorithms. Section V contains some concluding remarks.

A. Notation

We adopt the notation of using boldface for vectors $\mathbf{a}$ (lowercase), and matrices $\mathbf{A}$ (uppercase). The $(n\text{th}, m\text{th})$ entry of $\mathbf{A}$ and the $n$th element of $\mathbf{a}$ are denoted by $A_{nm}$ and $a_n$, respectively. The transpose, the conjugate, and the conjugate transpose operators are denoted by the symbols $(\cdot)^T$, $(\cdot)^*$, and $(\cdot)^H$, respectively. $\text{tr}(\cdot)$ is the trace of the square matrix argument. $\mathbf{I}$ and $\mathbf{0}$ denote, respectively, the identity matrix and the matrix with zero entries (their size is determined from the context). $\mathbb{R}^N$, $\mathbb{C}^N$, and $\mathbb{H}^N$ are, respectively, the sets of $N$-dimensional vectors of real numbers, of $N$-dimensional vectors of complex numbers, and of $N \times N$ Hermitian matrices. For a given vector $\mathbf{a} \in \mathbb{C}^N$, $\text{diag}(\mathbf{a})$ indicates the $N$-dimensional diagonal matrix whose $i$th diagonal element is $a_i$, for $i = 1, \ldots, N$, whereas its Euclidean norm is denoted by $\|\mathbf{a}\|$. The curled inequality symbol $\succeq$ (and its strict form $\succ$) is used to denote generalized matrix inequality: for any $\mathbf{A} \in \mathbb{H}^N$, $\mathbf{A} \succeq \mathbf{0}$ means that $\mathbf{A}$ is a positive semidefinite matrix ($\mathbf{A} \succ \mathbf{0}$ for positive definiteness). The letter $j$ represents the imaginary unit (i.e., $j = \sqrt{-1}$), while the letter $i$ often serves as an index. For any complex number $x$, $|x|$ and $\arg(x)$ denote the modulus and the argument of $x$, respectively. For any real number $x$, $\lfloor x \rfloor$ represents the nearest integer lower than or equal to $x$. In addition, $\odot$ and $\odot^*$ denote the convolution operator and Hadamard (element wise) product, respectively. Finally, $\mathbb{E}[\cdot]$ denotes statistical expectation.

II. SYSTEM MODEL

We consider a monostatic radar that transmits a burst of $N$ pulses. In Fig. 1, we show the block scheme for the radar transmit and receive chain [chapter 1 in 2, chapter 2 in 16]. Specifically,

$$ s(t) = A \sum_{n=1}^{N} p(t - (n - 1)T), \quad t \in [0, NT) $$


deals with the baseband equivalent of the radar transmitted train, where $A > 0$ is an amplitude factor related to the transmitted power, $p(t)$ is a unit energy rectangular pulse waveform of duration $T_p$, with an effective (peak-to-null) single-side bandwidth $W_p = 1/T_p$, $T$ is the pulse repetition interval (PRI), and $\frac{1}{T}$ is the pulse repetition frequency (PRF);

$$ \exp[j(2\pi f_c t + \phi(t))] \quad \text{and} \quad \exp[-j(2\pi f_c t + \phi(t))] $$

account for the upconversion and downconversion process, respectively, where $f_c$ is the carrier frequency and $\phi(t)$ is the real valued passband random process modeling the oscillator phase noise [chapter 2 in 1, chapter 12 in 4, 10], whose statistical characterization is addressed in Section III;

- the block “channel” models the backscattering from the environment (encompassing the clutter signal $c_{RF}(t)$, as well as a prospective target return $\delta_{RF}(t)$) and the RF white noise $w_{RF}(t)$; its output is the radar received signal $r_{RF}(t) = \delta_{RF}(t) + c_{RF}(t) + w_{RF}(t)$;
the block “matched filtering” represents the matched filter to the radar pulse \( p(t); \) its output is the possibly compressed signal \( y(t) = r(t) \otimes p^*(t); \)

- \( r_{k,n} = y(kT_p + (n-1)T), k = k_{min}, k_{min} + 1, \ldots, \left\lfloor \frac{T}{T_p} \right\rfloor, n = 1, 2, \ldots, N, \) denotes the fast-time/slow-time data matrix, with \( k_{min} \), \( T_p > 0 \) the initial time of the fast-time sampling process, accounting for the radar eclipsing.

As a first step toward the derivation of the fast-time/slow-time data matrix signal model, let us evaluate the output to the matched filter due to a nonambiguous pointlike moving target with complex backscattering amplitude \( \alpha_1, \) Doppler frequency \( f_d, \) and round-trip delay \( \tau = \frac{2R}{c}, \) where \( R \) is the target range and \( c \) the propagation velocity in the medium. In this case, the downconverted received signal is

\[
r(t) = \tilde{s}_{RF}(t) \exp[-j(2\pi f_c t + \phi(t))]
\]

\[
= A_1 \alpha_1 \sum_{n=1}^{N} A p(t - (n-1)T - \tau) 
\times \exp[j(2\pi f_c(t - \tau) + \phi(t - \tau))] 
\times \exp[j(2\pi f_d(t - \tau))] \exp(-j(2\pi f_c t + \phi(t)))
\]

\[
= A_1 \alpha_1 A \sum_{n=1}^{N} p(t - (n-1)T - \tau) \exp[j(\phi(t - \tau) - \phi(t))] 
\times 2\pi f_d(t - \tau - 2\pi f_c \tau), \quad (1)
\]

with \( A_1 \) accounting for the effects of the transmit-receive antenna gains, the two-way path loss, and other factors involved in the radar equation. Hence, following the steps in [chapter 2 in 16, chapter 2 in 17], we get the signal at the output of the matched filter given by

\[
y(t) = \tilde{\alpha} \sum_{n=1}^{N} \tilde{\chi}_p(t - (n-1)T - \tau, f_d, n, \tau) 
\times \exp[j2\pi(n-1)f_d T], \quad (2)
\]

where \( \tilde{\alpha} = A_1 \alpha_1 A \exp(-j2\pi f_c \tau) \in \mathbb{C} \) and

\[
\tilde{\chi}_p(t_2, f, n, \tau) = \int_{-\infty}^{\infty} p(t_1) p^*(t_1 - t_2) 
\times \exp[j(\delta\phi(t_1 + \tau + (n-1)T))] \exp(j2\pi f_1 t_1) dt_1
\]

indicate a generalized ambiguity function of the pulse waveform \( p(t) \) accounting for the phase noise, with \( \delta\phi(t) = \phi(t - \tau) - \phi(t). \)

Notice that \( \tilde{\chi}_p(t_2, f, n, \tau), n = 1, \ldots, N, \) is a random variable that depends on the random process \( \delta\phi(t). \) Also, the actual integration limits are bounded due to the finite duration of the pulse \( p(t). \) In Appendix A, it is shown that under some mild technical conditions, the phase noise process involved in the \( n \)th PRI, \( n = 1, \ldots, N, \) i.e., \( \delta\phi(t_1 + \tau + (n-1)T), t_1 \in [0, T_p], \) can be approximated as a random variable \( \delta\phi_n = \phi(n - 1)T - \phi(kT_p + (n-1)T), \) where \( k \) is the range bin associated with the round-trip delay \( \tau, \) i.e., \( |kT_p - \tau| \leq \frac{T_p}{2}. \)

Specifically, denoting by \( \delta\phi_n \) the single-side bandwidth where the phase noise PSD is greater than or equal to \( 10^{-5} \frac{W}{Hz}, \) with \( 2W_p \) the single-side radar bandwidth around the carrier frequency, and by \( S_p \) the phase noise power value associated with the frequency interval \([-B_p, B_p], \) then \( \forall t_1 \in [0, T_p] \)

\[
E[|\delta\phi(t_1 + \tau + (n-1)T) - \delta\phi_n|^2] \ll 1, \quad n = 1, \ldots, N,
\]

as long as \( (\pi T_p B_p^2 \delta\phi_n^2) \ll 1. \) Under the previously mentioned assumptions, \( \exp[\delta\phi(t_1 + \tau + (n-1)T) - \delta\phi_n] \approx 1 \) and consequently,

\[
\tilde{\chi}_p(t_2, f, n, \tau) 
\exp(j\delta\phi_n) \left( \int_{-\infty}^{\infty} p(t_1) p^*(t_1 - t_2) \exp(j2\pi f_1 t_1) dt_1 \right) 
\approx \exp(j\delta\phi_n) \left( \int_{-\infty}^{\infty} p(t_1) p^*(t_1 - t_2) dt_1 \right) 
\exp(j\delta\phi_n) \left( \int_{-\infty}^{\infty} p(t_1) p^*(t_1 - t_2) dt_1 \right)
\]

\[
= \exp(j\delta\phi_n) R_p(t_2), n = 1, \ldots, N, \quad (3)
\]

where the last approximation assumes a Doppler-tolerant pulse \( p(t), \) with \( R_p(t_2), t_2 \in \mathbb{R}, \) the autocorrelation function of \( p(t). \) Interestingly, the required conditions hold true for phase noise PSDs of practical interest (please see real phase noise data analysis in Subsection III.B). Based on (3), the signal received from a pointlike target after baseband conversion and filtering can be expressed as

\[
y(t) = \tilde{\alpha} \sum_{n=1}^{N} R_p(t - (n-1)T - \tau) 
\times \exp(j\delta\phi_n) \left( \int_{-\infty}^{\infty} p(t_1) p^*(t_1 - t_2) \exp(j2\pi f_1 t_1) dt_1 \right). \quad (4)
\]

Let us now focus on the output of the matched filter due to the clutter and observe that the clutter signal can be modeled as the superposition of the returns from independent pointlike scatterers [p. 22, 16]. Hence, owing to the linearity of the matched filter, we can compute the clutter output applying the developed approximations (4) to each pointlike source and superimposing the signals belonging to the same range cell.

As to the output of the matched filter due to the RF noise, it is given by

\[
y(t) = \int_{0}^{T_p} p^*(t_1) w(t_1 + t) \exp(-j\phi(t_1 + t)) dt_1, \quad (5)
\]

with \( w(t) \) a complex, zero-mean, circularly symmetric Gaussian random process with constant PSD \( \sigma_w^2 \) within the receiver bandwidth \([-W_p, W_p]. \) Following the same steps as in Appendix A, it is not difficult to show that

\[
E[|\phi(t_1 + t) - \phi(t)|^2] \ll 1, \quad \forall t_1 \in [0, T_p]
\]

\[
\text{1 Notice that the factor } 10^{-4} \text{ can be replaced by any positive real number much lower than } 1 \text{ without affecting the theoretical results obtained in the paper. Nevertheless, from a practical point of view, it is better to specify a value, and to this end, we have chosen } 10^{-4}. \]

---

1 For notational simplicity, we omit the explicit dependence on \( \tau \) of \( \delta\phi(t). \)
\[ y(t) \simeq \left( \int_{0}^{T_p} \exp \left( j \delta \phi(t) \right) \right) \exp(-j \phi(t)). \] (6)

Summarizing, the \( N \)-dimensional vector
\[ r = [r_{k,1}, r_{k,2}, \ldots, r_{k,N}]^T \in \mathbb{C}^N \]
of the received waveform samples obtained after baseband conversion, filtering, and sampling at the range bin of interest \( k \), can be expressed as
\[ r_{k,n} = \alpha p_n \exp(j \delta \phi_n) + c_n \exp(j \delta \phi_n) + w_n, \]
\[ n = 1, \ldots, N, \] (7)

where
- \( p = [p_1, \ldots, p_N]^T \in \mathbb{C}^N \) with \( p_i = \frac{1}{\sqrt{N}} \exp(j2\pi(i-1)\nu_d), i = 1, \ldots, N \), denote the unit norm target steering vector, where \( \nu_d = f_d T \) is the target normalized Doppler frequency, with \( f_d \) in Hz the actual target Doppler frequency;
- \( \alpha \in \mathbb{C} \) accounts for target reflectivity, potential straddle loss, channel propagation effects, and other terms involved into the radar range equation [chapter 1, 2]. The phase of \( \alpha \) is modeled as a uniform random variable over \([0, 2\pi]\), while its amplitude is assumed deterministic and unknown;
- \( y = [\delta \phi_1, \ldots, \delta \phi_N]^T \in \mathbb{R}^N \) represents the oscillator phase noise perturbation, which is statistically independent of \( \alpha, \sigma, \) and \( w \);
- \( c = [c_1, \ldots, c_N]^T \in \mathbb{C}^N \) is the vector of the clutter samples, modeled as a complex, zero-mean, circularly symmetric Gaussian random vector, with covariance matrix \( \mathbb{E}[cc^H] = M \);
- \( w = [w_1, \ldots, w_N]^T \in \mathbb{C}^N \) is the vector of the white noise samples, whose components are independent and identically distributed (iid) complex, zero-mean, circularly symmetric Gaussian random variables, namely, \( \mathbb{E}[ww^H] = \sigma_w^2 I \).

Exploting the previously mentioned definitions, the received signal (7) can be expressed in a vectorial compact form as
\[ r = \alpha \tilde{y} \odot p + \tilde{y} \odot c + w, \] (8)

with \( \tilde{y} = [\exp(j \delta \phi_1), \ldots, \exp(j \delta \phi_N)]^T \). Before concluding this section, it is worth pointing out that the vector \( y \) accounts for both the phase noise perturbation introduced at the transmitter side (upconversion), as well as the phase noise contribution produced at the receiver side (downconversion). The next section is devoted to the statistical characterization of the overall phase noise contribution.

III. PHASE NOISE MODELING

According to the developed signal model, phase noise observations \( \phi(t) \) can be regarded as circular samples, i.e., points on the unit circle. Otherwise stated, the process \( \nu(t) = \exp(j \phi(t)) \) belongs to the class of “Circular Data” processes, characterized by circular distributions [15]. In Subsection III.A, we introduce some statistical models for multivariate circular variables (wrapped distribution), and in Subsection III.B, we describe a phase noise spectral model.

A. Multivariate Circular Models

In this subsection, we describe the important family of wrapped distributions. Given a random vector of \( \mathbb{R}^N \) with an assigned probability distribution, we can wrap each component around the circumference of the unit radius circle so as to produce a wrapped distribution. Otherwise stated, if \( x_i, i = 1, \ldots, N \), denote random variables on the line, i.e., \( x = [x_1, x_2, \ldots, x_N]^T \) represents a random vector of the Euclidean space \( \mathbb{R}^N \), the corresponding (wrapped) random variables \( x_i^w, i = 1, \ldots, N \), are given by
\[ x_i^w = x_i (\text{mod} 2\pi), \quad i = 1, \ldots, N. \] (9)

As a consequence, if \( x_1, x_2, \ldots, x_N \) have joint pdf \( f(\cdot) \), then the joint pdf \( f_w(\cdot) \) of \( x_1^w, x_2^w, \ldots, x_N^w \) is [chapter 3, 15]
\begin{align}
\bar{\Phi}(p_1, p_2, \ldots, p_N) &= \Psi_x(p_1, p_2, \ldots, p_N), \\
p_1, p_2, \ldots, p_N &= 0, \pm 1, \pm 2, \ldots,
\end{align}

i.e., the samples of the standard CF.

Interestingly, it can be proved that any multivariate circular distribution is determined by its CF, a property referred to as “uniqueness property” [chapter 4, 15]; otherwise stated, there exists a one-to-one mapping between the pdfs and their joint trigonometric moments, in contrast with classic distributions on the Euclidian space and their standard moments [chapter 30, 18]. Based on the previously mentioned observation, a meaningful approach to obtain statistical inference on circular data is to estimate their joint trigonometric moments. Also, this important property has been exploited in [19] to approximate the von Mises density via a suitable mixture of zero-mean one-dimensional wrapped Gaussian distributions.

We now present some special cases of multidimensional wrapped distribution.

Wrapped Gaussian distribution: The wrapped Gaussian distribution \( \mathcal{WN}(\mu, \Sigma) \) is obtained wrapping the \( N \)-dimensional Gaussian distribution \( \mathcal{N}(\mu, \Sigma) \), where \( \Sigma > 0, \mu = [\mu_1, \mu_2, \ldots, \mu_N]^T \in \mathbb{R}^N \) [20, 21]. From (10), the pdf corresponding to \( \mathcal{WN}(\mu, \Sigma) \) is

\[ f_w(\theta; \mu, \Sigma) = \frac{1}{\sqrt{2\pi \det(\Sigma)}} \sum_{k_1, k_2, \ldots, k_N = -\infty}^{\infty} h(\theta, k_1, k_2, \ldots, k_N, \mu, \Sigma) \]

with

\[ h(\theta, k_1, k_2, \mu, \Sigma) = \exp \left[ -\frac{1}{2} \Sigma^{-\frac{1}{2}} \begin{bmatrix} \theta_1 - \mu_1 + k_1 2\pi \\
\theta_2 - \mu_2 + k_2 2\pi \\
\vdots \\
\theta_N - \mu_N + k_N 2\pi \end{bmatrix}^2 \right]. \]

Because the CF of \( x \sim \mathcal{N}(\mu, \Sigma) \) is given by \( \Psi_x(t) = \exp(j \mu^T t - \frac{1}{2} t^T \Sigma t) \), with \( t \in \mathbb{R}^N \), from (12), we have

\[ \bar{\Phi}(p_1, p_2, \ldots, p_N) = \exp \left[ j \left( \sum_{i=1}^{N} \mu_i p_i \right) - \frac{1}{2} \left( \sum_{i=1}^{N} \sum_{i' \geq i} \Sigma_{i,i'} p_i p_{i'} \right) \right]. \]

(13)

Notice that the first-order distributions of \( x_i^w, i = 1, \ldots, \bar{N} \) are wrapped Gaussian distributions; specifically, \( x_i^w \sim \mathcal{WN}(\mu_i, \Sigma_{i,i}), i = 1, 2, \ldots, \bar{N} \), [chapter 3, 15].

Wrapped generalized asymmetric Laplace distribution: The wrapped generalized asymmetric Laplace (WGAL) distribution \( \mathcal{WGAL}(\mu, \Sigma, s) \) is obtained wrapping the \( N \)-dimensional generalized asymmetric Laplace (GAL) distribution \( \mathcal{GAL}(\mu, \Sigma, s) \), where \( s \geq 1, \Sigma > 0, \mu = [\mu_1, \mu_2, \ldots, \mu_N]^T \in \mathbb{R}^N \). From [22] and (10), the pdf connected to \( \mathcal{WGAL}(\mu, \Sigma, s) \) is

\[ f_w(\theta; \mu, \Sigma, s) = c_1 \sum_{k_1, k_2, \ldots, k_N = -\infty}^{\infty} K_{s-1}(Q(\theta, k_1, k_2, \ldots, k_N, \Sigma)C(\Sigma, \mu)) \times \left( \frac{Q(\theta, k_1, k_2, \ldots, k_N, \Sigma)}{C(\Sigma, \mu)} \right)^{(s-1)} \times \exp(G(\theta, k_1, k_2, \ldots, k_N, \Sigma, \mu)), \]

(14)

where \( K_{s}(\mu) \) is the modified Bessel function of the third kind with index \( \lambda > 0 \) [22] and

\[ c_1 = \frac{2}{\Gamma(s) \sqrt{\det(\Sigma) 2\pi}}, \]

\[ G(\theta, k_1, k_2, \ldots, k_N, \Sigma, \mu) = \begin{bmatrix} \mu_1 \\
\mu_2 \\
\vdots \\
\mu_N \end{bmatrix}^T \Sigma^{-1} \begin{bmatrix} \theta_1 + k_1 2\pi \\
\theta_2 + k_2 2\pi \\
\vdots \\
\theta_N + k_N 2\pi \end{bmatrix}. \]

(15)

Notice that the CF of \( x \sim \mathcal{GAL}(\mu, \Sigma, s) \) is given by \( \Psi_x(t) = (1 + \sum_{i=1}^{ \bar{N} } \frac{k_{i}^s}{\lambda_{i}^s})^{-1}; \) as a consequence, from (12),

\[ \bar{\Phi}(p_1, p_2, \ldots, p_N) = \exp \left[ \frac{1}{2} \left( \sum_{i=1}^{\bar{N}} \frac{\bar{N} - \sum_{i=1}^{N} \mu_i p_i}{p_i p_{i'}} \right) \right]. \]

Interestingly, a random vector \( x \sim \mathcal{GAL}(\mu, \Sigma, s) \) has mean vector \( s\mu \) and covariance matrix \( s\Sigma \). Additionally, it can be generated via [22]

\[ x = \mu + \sqrt{\zeta} y, \]

(16)

where \( y \sim \mathcal{N}(0, \Sigma) \) and \( z \) is a standard gamma distribution with shape parameter \( s \). Consequently, the first-order distributions of \( x_i^w, i = 1, \ldots, \bar{N} \) are \( \mathcal{WGAL}(\mu_i, \Sigma_{i,i}, s), i = 1, 2, \ldots, \bar{N} \).

B. Phase Noise Spectral Models

Empirical models based on measurements suggest that the phase noise PSD \( S_\phi(f) \) can be described as a power-law function [10, 23], i.e.,

\[ S_\phi(f) = K_\alpha f^{-\alpha}, \quad \alpha \in [0, 1, 2, 3, 4], \quad K_\alpha \in \mathbb{R}, \]

(17)

or through their suitable combinations, where \( f \) denotes the offset frequency from the actual carrier frequency [chapter
TABLE I
Phase Noise Models

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Phase noise</th>
<th>Color of $\Omega(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f^{-\alpha}$</td>
<td>White phase noise</td>
<td>Purple</td>
</tr>
<tr>
<td>$f^{-1}$</td>
<td>Flicker phase noise</td>
<td>Blue</td>
</tr>
<tr>
<td>$f^{-2}$</td>
<td>White FM phase noise</td>
<td>White</td>
</tr>
<tr>
<td>$f^{-3}$</td>
<td>Flicker FM phase noise</td>
<td>Pink</td>
</tr>
<tr>
<td>$f^{-4}$</td>
<td>Random walk FM phase noise</td>
<td>Brown</td>
</tr>
</tbody>
</table>

12, 4]. The characterization of the oscillator output PSD based on the knowledge of the phase noise spectral characteristics is a long-standing issue and has been addressed in many papers. Among them, the Leeson model [24] remains probably the most widely used. Moreover, in recent work [10], the authors applied correlation theory methods to obtain a model for the near-carrier oscillator PSD. Further interesting results can be found in [25].

The phase noise PSD is related to frequency fluctuations PSD $S_\Omega(f)$ via the spectral transformation

$$S_\phi(f) = \frac{S_\Omega(f)}{f^2}. \quad (18)$$

Notice that the different noise sources are usually referred to as “colored” [10]. Specifically, the correspondence between the terminology for frequency noise sources and the resulting phase noise is summarized in Table I. It is worth pointing out that this representation does not correspond to the actual phase noise generation mechanisms, but it is useful as a mathematical abstraction.

In the following, we focus on a simple mathematical model often used in practice [chapter 12 in 4, 10], which approximates the phase noise PSD as

$$S_\phi(f) = K_0 + \frac{K_1}{|f|} + \frac{K_2}{|f|^2} + \frac{K_3}{|f|^3} + \frac{K_4}{|f|^4}, \quad (19)$$

where the weights $K_i$, $i = 0, 1, 2, 3, 4$ allow to suitably combine the power-law processes. Notice that $K_0$ defines the minimum phase noise level at high offset frequencies.

It is worth highlighting that both the combined model (19) and the power-law spectra (17) with $\alpha \geq 1$ produce an infinite value at 0 Hz, so the use of these models is usually subject to some maximum phase noise PSD, which generally occurs around the 5- to 10-Hz offset frequency [chapter 12, 4]. As already pointed out, the question of near-carrier oscillator PSD has been successfully addressed in open literature for the power-law phase noise processes described in Table I [10]. As a matter of fact, the effect of slowly varying frequency fluctuations in oscillators is characterized through their long-term stability and can manifest itself through non-negligible drifts in the oscillation frequency. Nevertheless, it is not of primary importance in a radar application, as long as, over the coherent processing interval (CPI), the induced frequency drift remains constant or can be compensated.

Before concluding this subsection, we present some real phase noise PSD measurements and their adherence with properly tuned composite power-law models. The data have been generated from a stable oven-controlled crystal oscillator (OCXO), operating at 100 MHz, and a direct digital synthesizer (DDS). The available measurements refer to the single-side PSD in dBc/Hz [chapter 12 in 4], for different values of the oscillator operating carrier frequency and carrier power. Table II summarizes the characteristics of the available data set.

**TABLE II**
Phase Noise Data Set Characteristics

<table>
<thead>
<tr>
<th>File Number</th>
<th>Carrier Frequency (MHz)</th>
<th>Carrier Power (dBm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>file1</td>
<td>1000</td>
<td>9.6</td>
</tr>
<tr>
<td>file2</td>
<td>1610</td>
<td>12</td>
</tr>
<tr>
<td>file3</td>
<td>2000</td>
<td>17.3</td>
</tr>
<tr>
<td>file4</td>
<td>2140</td>
<td>10.4</td>
</tr>
<tr>
<td>file5</td>
<td>2390</td>
<td>7.6</td>
</tr>
<tr>
<td>file6</td>
<td>390</td>
<td>-1.9</td>
</tr>
<tr>
<td>file7</td>
<td>500</td>
<td>17.6</td>
</tr>
<tr>
<td>file8</td>
<td>500</td>
<td>17.8</td>
</tr>
</tbody>
</table>
where \( \mathbf{h} = [h_1, \ldots, h_N]^T \in \mathbb{C}^N \) are known deterministic weights. Equation (20) encompasses a wide class of signal processing algorithms, such as coherent detection (for instance, PDP), clutter cancellation (e.g., MTI), linear filtering, and spectral analysis (discrete Fourier transform computation). Interestingly, the statistic \( z \) can be expressed as the sum of two contributions \( z_T \) and \( z_d \), with

\[
\begin{align*}
\mathbf{z}_T &= \alpha \sum_{n=1}^{N} h_n^* p_n \exp(j\delta \phi_n) = \alpha \mathbf{h}^\dagger \text{diag}(\hat{\mathbf{y}}) \mathbf{p}, \\
\mathbf{z}_d &= \sum_{n=1}^{N} h_n^* (c_n \exp(j\delta \phi_n) + w_n) = \mathbf{h}^\dagger \text{diag}(\hat{\mathbf{y}}) \mathbf{c} + \mathbf{h}^\dagger \mathbf{w},
\end{align*}
\]

denoting the useful term;

\[
\begin{align*}
\mathbf{z}_T &= \alpha \sum_{n=1}^{N} h_n^* p_n \exp(j\delta \phi_n) = \alpha \mathbf{h}^\dagger \text{diag}(\hat{\mathbf{y}}) \mathbf{p}, \\
\mathbf{z}_d &= \sum_{n=1}^{N} h_n^* (c_n \exp(j\delta \phi_n) + w_n) = \mathbf{h}^\dagger \text{diag}(\hat{\mathbf{y}}) \mathbf{c} + \mathbf{h}^\dagger \mathbf{w},
\end{align*}
\]

Conditioned on the random phase vector

\[
\mathbf{\varphi} = \left[ \phi(0), \phi(\bar{k}T_p), \phi(T), \phi(T + \bar{k}T_p), \ldots, \right. \\
\left. \phi((N-1)T), \phi((N-1)T + \bar{k}T_p) \right]^T,
\]

\( z_T \) is a complex random variable with uniform phase and deterministic amplitude \( |\alpha| |\mathbf{h}|^\dagger \text{diag}(\hat{\mathbf{y}}) \mathbf{p} | \) (nonfluctuating target assumption), and \( z_d \) is a complex, zero-mean, circularly symmetric Gaussian random variable with variance \( (\|\mathbf{M}^\dagger \text{diag}(\hat{\mathbf{y}}) \mathbf{h}\|^2 + \sigma^2_w \| \mathbf{h} \|^2) \). As a consequence, in the presence of phase noise, the useful and the interference signals are no longer statistically independent (i.e., only conditional independence holds true).

Let us now analyze the second-order statistical characterization of \( z_T \) and \( z_d \) that will be useful for subsequent mathematical derivations. Due to the linearity of the mean operator and the statistical independence of \( \alpha, \varphi, \mathbf{w}, \) and \( \mathbf{c} \), the mean value of \( z_T \) and \( z_d \) are, respectively, given by

\[
\begin{align*}
\mathbb{E}[z_T] &= \mathbb{E}[\alpha] \sum_{n=1}^{N} h_n^* p_n \mathbb{E}\left[ \exp(j\delta \phi_n) \right] = 0; \\
\mathbb{E}[z_d] &= \sum_{n=1}^{N} h_n^* (\mathbb{E}[c_n] \mathbb{E}[\exp(j\delta \phi_n)] + \mathbb{E}[w_n]) = 0.
\end{align*}
\]

Also, \( z_T \) and \( z_d \) are uncorrelated random variables; in fact,

\[
\begin{align*}
\mathbb{E}[z_T z_d^*] &= \mathbb{E}[\alpha] \sum_{n=1}^{N} \sum_{m=1}^{N} h_n^* p_n h_m^* \\
&\times (\mathbb{E}[c_m] \mathbb{E}[\exp(j\delta \phi_n) \exp(j\delta \phi_m)]) \\
&+ \mathbb{E}[\exp(j\delta \phi_n)] \mathbb{E}[w_n] = 0; \\
\mathbb{E}[z_T z_d^*] &= \mathbb{E}[\alpha] \sum_{n=1}^{N} \sum_{m=1}^{N} h_n^* p_n h_m^* \\
&\times (\mathbb{E}[c_m^*] \mathbb{E}[\exp(j\delta \phi_n) \exp(-j\delta \phi_m)]) \\
&+ \mathbb{E}[\exp(j\delta \phi_n)] \mathbb{E}[w_m^*] = 0.
\end{align*}
\]
Finally, the mean power of $z_T$ and $z_d$ are, respectively,

$$
\mathbb{E}[|z_T|^2] = \mathbb{E}[|\alpha|^2] \mathbb{E}\left[ \left( \sum_{n=1}^{N} h_n^* p_n \exp(j \delta \phi_n) \right)^2 \right] 
$$

and

$$
\mathbb{E}[|z_d|^2] = \mathbb{E}[|\alpha|^2] \mathbb{E}\left[ \left( \sum_{n=1}^{N} h_n^* c_n \exp(j \delta \phi_n) + w_n \right)^2 \right] 
$$

as long as $\mathbb{E}[\delta \phi_n^2] \ll 1$, $n \in \{1, 2, \ldots, N\}$, i.e., in this regimen $A$ only depends on the second-order moments of $\phi$.

Interestingly, $\mathbb{E}[|z_T|^2]$ and $\mathbb{E}[|z_d|^2]$ are functionally dependent on the phase noise only through the matrix $A$. Furthermore, indicating the CF of the phase vector $\phi$ (see Subsection III.A) as $\ddot{\phi}(i) = [i_1, i_2, \ldots, i_{2N}]^T \in \{0, \pm 1, \pm 2, \ldots\}^{2N}$, for all $(m, n) \in \{1, 2, \ldots, N\}^2$, it holds

$$
A_{n,m} = \begin{cases} 
1 & n = m \\
\mathbb{E}\left[ \exp(j (\delta \phi_n - \delta \phi_m)) \right] = \ddot{\phi}(i^{(m,n)}) & n \neq m 
\end{cases} 
$$

with $i^{(m,n)} \in \{0, \pm 1, \pm 2, \ldots\}^{2N}$ given by

$$
i_k^{(m,n)} = \begin{cases} 
1, & k = 2(n - 1) + 1 \\
-1, & k = 2(n - 1) + 2 \\
1, & k = 2(m - 1) + 2, \ k = 1, 2, \ldots, 2N \\
-1, & k = 2(m - 1) + 1 \\
0, & \text{otherwise}
\end{cases} 
$$

The previous equations imply that the phase noise influences the second-order moments of $z_T$ and $z_d$ directly through its CF. It is worth highlighting that the conducted analysis fully agrees with the study addressed in [7], where the case of iid Gaussian random phase noise samples is considered.

A. MTI Processing

In this subsection, we exploit the previously derived general results to characterize the performance of MTI systems. In this context, the figures of merit are given by [chapter 17, 4]:

- clutter attenuation (CA) that measures the reduction in clutter power at the output of the MTI filter compared with the clutter power at the input;
- improvement factor ($I$) that quantifies the increase in signal-to-clutter power ratio (SCR) due to MTI filtering; as a consequence, it accounts for the effect of the filter both on the target and the clutter.

**Clutter attenuation**: CA directly evaluates the effectiveness of the MTI filter $h$ to suppress the clutter energy. It is simply the ratio between the clutter power $C_{in}$ at the input and the clutter power $C_{out}$ at the output of the filter:

$$
CA = \frac{C_{in}}{C_{out}} = \frac{\frac{1}{N} tr(M)}{h^*(A \odot M) h}. 
$$

**Improvement factor**: $I$ is formally defined as the SCR at the output of the MTI filter divided by the SCR at the input, averaged over all target radial velocities of interest. As a first step toward the evaluation of $I$, let us consider a specific normalized target Doppler shift $v_d$, hence, exploiting (22) and (23), the improvement factor can be
factored into the form
\[
I(v_d) = \frac{SCR_{out}}{SCR_{in}} = \frac{S_{out} C_{in}}{S_{in} C_{out}} = \frac{N|\alpha|^2 h^1 \text{diag}(p) \text{Adiag}(p^*) h}{|\alpha|^2} \frac{1}{h^1 (A \otimes \tilde{M}) h}.
\]
(29)

Finally, averaging over the normalized target Doppler interval \([v_1, v_2](-\frac{1}{2} \leq v_1 \leq v_2 \leq \frac{1}{2})\), we obtain
\[
I_{\nu_1} = \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} I(v_d) dv_d = NH^1 (A \otimes E [pp^*]) h
\times \frac{1}{h^1 (A \otimes \tilde{M}) h} = \frac{h^1 (A \otimes \Phi^\nu_1) h}{h^1 (A \otimes \tilde{M}) h},
\]
(30)

with \(\tilde{M} = \frac{M}{\pi \text{tr}(M)}\) the normalized clutter covariance matrix and
\[
\Phi^\nu_1 = \begin{cases}
1 & \text{if } l = m
\end{cases}
\]
with \(l, m \neq m\) (31)

As \(v_1 = -\frac{1}{2}\) and \(v_2 = \frac{1}{2}\), (30) reduces to the conventional average improvement factor
\[
I = \int_{-\frac{1}{2}}^{\frac{1}{2}} I(v_d) dv_d = \frac{\|h\|^2}{h^1 (A \otimes \tilde{M}) h}.
\]
(32)

Interestingly, when the phase noise is absent, \(A_{n,m} = 1, \forall (n, m) \in \{1, 2, \ldots, N\}^2\), leading to the ideal improvement factor \(I_{\text{ideal}} = \frac{\|h\|^2}{h^1 \tilde{M} h}\). As a consequence, the improvement factor loss \(L\), due to the phase noise, is given by
\[
L = \frac{I_{\text{ideal}}}{I} = \frac{h^1 (A \otimes \tilde{M}) h}{h^1 \tilde{M} h},
\]
(33)

which allows to quantify the impairment effect on the performance of the MTI algorithms. The previous equation highlights that the loss factor \(L\) depends on the employed MTI filter, the CPI, the clutter correlation, and the phase noise CF, but it is functionally independent of the clutter power.

B. Performance Analysis of MTI Systems in the Presence of Phase Noise

In this subsection, we analyze the performance of the single canceller, double canceller, and third-order canceller in the presence of phase noise, clutter, and thermal noise, for a radar employing a rectangular pulse of duration \(T_p = 10^{-1}\) μs. This is tantamount to considering as MTI filters the following vectors [chapter 2 in 2, chapter 17 in 4], belonging to the class of binomial filters,

1) \(h_{sc} = [1, -1, 0, 0, \ldots, 0]^T \in \mathbb{C}^N\), for the single canceller;
2) \(h_{dc} = [1, -2, 1, 0, \ldots, 0]^T \in \mathbb{C}^N\), for the double canceller;
3) \(h_{tc} = [1, -2, 2, -1, 0, \ldots, 0]^T \in \mathbb{C}^N\), for the third-order canceller.

In all the numerical examples, we consider a Gaussian-shaped clutter autocorrelation function, namely \([11]\),
\[
r_c(m) = E \left[ c(n)c(n + m) \right] = P_c \exp \left[ -\frac{(mT)^2}{2\sigma^2_f} \right], m \in \mathbb{N},
\]
(34)

where \(c(n), n \in \mathbb{N}\) denotes the clutter random process; the clutter vector \(e\) in (7) is consequently given by \(e = [c(1), c(2), \ldots, c(N)]^T\). \(P_c\) represents the clutter power, and \(\sigma_f = 1/(2\pi\sigma_f)\) with \(\sigma_f\) the spectral width of the clutter PSD. Otherwise stated, the \((i, k)\)th element of the clutter covariance matrix \(C_m\) is modeled as
\[
M_{i,k} = P_c \rho_T^{(i-k)^2},
\]
(35)

with \(\rho_T = \exp(-\frac{T^2}{2\sigma_f^2})\), the one-lag correlation coefficient.

As to the phase noise contribution \(\varphi\) in (21), we model it as a zero-mean random vector, with covariance matrix \(R\). For comparison purposes, we analyze the MTI performance as \(\varphi\) is drawn from both a wrapped Gaussian distribution and a WGAL distribution (see Subsection III.A). To comply with the empirical evidence, we model the one-side phase noise PSD via a composite power law, namely,
\[
S_{\phi}(f) = K_0 + \frac{K_1}{f} + \frac{K_2}{f^2} + \frac{K_3}{f^3} + \frac{K_4}{f^4}, f \in [f_0, W_p],
\]
(36)

where \(f > 0\) denotes the one-side offset frequency (the PSD is a symmetric function) and \(f_0\) is the lowest offset frequency of interest. Based on Wiener-Khintchine theorem, the autocorrelation function of the process \(\varphi(t)\) is given by
\[
r_{\phi}(\tau) = E [\varphi(t)\varphi(t - \tau)] = 2 \int_{f_0}^{W_p} S_{\phi}(f) \cos (2\pi f \tau_1) df.
\]
(37)

In Appendix C, it is shown that the autocorrelation function of the composite power-law model (36) can be expressed as
\[
r_{\phi}(\tau_1) = 2K_0 \sin(2\pi W_p \tau_1) - \sin(2\pi f_0 \tau_1)
+ 2K_1[\cos(2\pi f_1 W_p) - \cos(2\pi f_1 f_0)]
+ 4K_2\pi \tau_1 \left[ \frac{\cos(2\pi f_1 f_0)}{2\pi f_1 f_0} - \frac{\cos(2\pi f_1 W_p)}{2\pi f_1 W_p} \right]
- \text{Si}(2\pi f_1 W_p) + \text{Si}(2\pi f_1 f_0)
+ 4K_3\pi^2 \tau_1^2 \left[ \frac{\cos(2\pi f_1 f_0)}{(2\pi f_1 f_0)^2} - \frac{\cos(2\pi f_1 W_p)}{(2\pi f_1 W_p)^2} \right]
+ \text{ci}(2\pi f_1 f_0) - \text{ci}(2\pi f_1 W_p)
+ \frac{\sin(2\pi f_1 W_p)}{2\pi f_1 W_p} - \frac{\sin(2\pi f_1 f_0)}{2\pi f_1 f_0}
+ \frac{16}{3} K_4\pi^3 \tau_1^3 \left[ \frac{\cos(2\pi f_0 f_0)}{(2\pi f_0 f_0)^3} - \frac{\cos(2\pi W_p f_1)}{(2\pi W_p f_1)^3} \right]
\]
\[\begin{align*}
+ & \frac{1}{2} \sin(2\pi W_\rho \tau_i) - \frac{1}{2} \sin(2\pi f_0 \tau_i) \\
- & \left[ \frac{1}{2} \cos(2\pi f_0 \tau_i) - \cos(2\pi \tau_i W_\rho) \right] \\
- & \sin(2\pi \tau_i W_\rho) + \sin(2\pi f_0 \tau) \right].
\end{align*}\] (38)

where

\[ci(x) = -\int_{x}^{\infty} \frac{\cos(t)}{t} dt,\]

is the cosine integral function [chapter 5, p. 231, 30] and

\[Si(x) = \int_{0}^{x} \frac{\sin(t)}{t} dt,\]

is the sine integral function [chapter 5, p. 231, 30].

Note that model (36) does not specify the value \(P_\phi\) of the phase noise PSD over the bandwidth \([0, f_0]\). However, as shown in Appendix C, under some mild technical conditions, the value of \(P_\phi\) does not affect the radar performance, as long as \(\frac{f_0}{f_0} \gg NT\), where \(NT\) represents the radar CPI. As an example, the performance of a radar employing \(N = 16\) pulses with a PRI \(T = 1\) ms, mainly depends on the phase noise spectral components greater than or equal to 3 Hz; otherwise stated, the phase noise PSD value \(P_\phi\) over \([0, f_0]\), with \(f_0 = 3\) Hz, is “irrelevant.”

As to the parameter values of the composite power model, we consider \(K_0 = 10^{-11}, K_1 = 0.0000025 \cdot 10^{-4}, K_2 = 0.00021 \cdot 10^{-4}, K_3 = 0.04 \cdot 10^{-4}, \) and \(K_4 = 10^{-5}\) (approximatively corresponding to the parameters involved in the optimized fitting of file2; see Subsection III.B), unless otherwise stated.

In the following analysis, we study the improvement factor loss \(L\), due to the phase noise, as a function of the PRF, phase noise pdf, and clutter spectral variance, for different binomial filters, assuming a clutter power \(P_{cl} db = 20\) dB and \(\sigma_{cl}^2 = 0\) dB. Moreover, the performance of the radar in terms of \(I\) and \(L\) does not depend on the number of pulses \(N\), as long as \(N\) is greater than or equal to the number of MTI filter taps \(K_T\). In fact, if \(h_i = 0, i = K_T + 1, \ldots, N,\)

\[h^i (\mathbf{A} \circ \mathbf{M}) = \tilde{h}^i (\tilde{\mathbf{A}} \circ \tilde{\mathbf{M}}) \tilde{h},\]

where \(\tilde{h} = [h_1, \ldots, h_{K_2}]^T \in \mathbb{C}^{K_2}\), while \(\tilde{A}_{i,j} = A_{i,j}\), and \(\tilde{M}_{i,j} = M_{i,j}; (i, j) \in \{1, 2, \ldots, K_2\}^2\).

In Fig. 6, we provide the ideal improvement factor \(I_{ideal}\) (Fig. 6a) and the improvement factor loss \(L\) (Figs. 6b–6d) versus PRF, assuming a target range \(R = 30\) km, which is tantamount to fixing the range bin of interest \(y\) influencing \(x\), and \(\sigma_f = 50\) Hz, for \(h_{sc}, h_{dc}\), and \(h_{c}\).\(^3\) Specifically, Fig. 6b accounts for wrapped Gaussian phase noise, Fig. 6c addresses WGAL phase noise with \(s = 1\), and Fig. 6d refers to WGAL phase noise with \(s = 10\), to deal with distributions sharing different spreading levels. Also, in all the plots, the red curve refers to the single canceller, the green curve refers to the double canceller, and the blue curve refers to the third-order canceller. Moreover, the dashed lines refer to the improvement factor loss predicted via the small angle approximation method (25), which is usually applied in the radar field. Finally, dashed-dotted lines in Fig. 6b account for the predation of \(L\) according to [chapter 17 in 4, 7], which is based on the assumption of stationary clutter and white Gaussian phase noise.

Inspection of Fig. 6a reveals that in the ideal case, increasing PRF is tantamount to improving the MTI filter effectiveness. This is a reasonable behavior because higher values of PRF correspond to highly correlated clutter samples that can be better cancelled. Also, Figs. 6b–6d show that the higher PRF, the higher \(L\), highlighting that the phase noise impairments grow as the clutter correlation increases. A possible justification of the previously stated results is that the phase noise spectral spreading becomes more and more pronounced as the clutter samples PRI

\(^3\) Notice that the one-lag correlation coefficient involved in our analysis is a function of \(T\).
becomes narrower: as a matter of fact, if the clutter PSD is flat,\(^4\) the phase noise does not impair MTI performance, whereas if the clutter PSD is a Dirac delta function,\(^5\) the phase noise significantly spreads the original clutter PSD, and as a consequence, it affects MTI performance.

Interestingly, the curves in Fig. 6b highlight that the simpler and approximated method [chapter 17 in 4, 7] may lead to a poor prediction of the phase noise effects. Moreover, the results in Figs. 6b–6d suggest that the small angle approximation holds true for the available phase noise measurements. In fact, they show that the actual loss factor \(L\) coincides with that predicted by the small angle approximation method, regardless of the phase noise pdf. This behavior reflects the fact that the MTI performance degradation only depends on the second-order phase noise statistical characterization, as long as the phase noise power is small (see equation (25)). To further assess the phase noise impairments, in Fig. 7, we consider the same scenario as in Fig. 6, but with a higher phase noise power, assuming \(K_0 = 10^{-8}, K_1 = 0.00025 \cdot 10^{-4}, K_2 = 0.021 \cdot 10^{-4}, K_3 = 4 \cdot 10^{-4},\) and \(K_4 = 10^{-4}\). Wrapped Gaussian phase noise (black dashed-dotted lines), WGAL phase noise with \(s = 1\) (red dashed lines), WGAL phase noise with \(s = 10\) (cyan-dotted lines), and small angle approximation predictions (green solid lines). Single canceller \((\circ\text{-marked curves})\); double canceller \((\cdot\text{-marked curves})\); third-order canceller \((\Box\text{-marked curves})\).

In Fig. 8, we consider the same analysis as in Fig. 6, but for \(\sigma_f = 20\) Hz. As for Fig. 6, the higher PRF, the higher \(L\), highlighting that the MTI performance degradation increases as the clutter spectral width reduces. Interestingly, both \(l\) and \(L\) are greater than the counterparts in Fig. 6. This is a reasonable behavior because now the clutter PSD is narrower than Fig. 6; this implies that in the ideal condition, the cancellers can be more effective, but at the same time, the phase noise spectral spreading for the scenario of Fig. 8 can be more deleterious. Again, the loss factor \(L\) is the same, regardless of the phase noise pdf, confirming that the small angle approximation holds true.

Table III reports the improvement factor loss for a typical study case with PRF = 1250 Hz, as in a AN/SPS-48E radar [31], when the oscillator phase noise is compatible with the data in file2, the clutter PSD is Gaussian shaped, and the target range is \(R = 30\) km.

---

\(^4\)In this case, the cancellers are unable to separate the target from clutter also in the ideal condition of phase noise absence.

\(^5\)In the ideal condition of phase noise absence, the clutter can be perfectly filtered out by cancellers.
V. CONCLUSIONS

In this paper, we have formalized the radar signal model when the phase noise affects the reference signal produced by real radar oscillators. Specifically, we have developed a fast-time/slow-time data matrix signal representation, modeling the undesired phase fluctuations via a multivariate circular distribution and describing the phase noise PSD through a composite power-law model. Interestingly, the proposed parametric spectral model well fits the available real phase noise PSD measurements.

We have assessed the performance of coherent integration techniques operating in the presence of phase noise, providing an analytic expression to predict the performance degradations experienced by MTI algorithms. The proposed framework has highlighted that phase noise affects $I$ directly through its CF. Additionally, $I$ is robust with respect to the actual phase noise circular multivariate distribution, as long as the phase noise PSD correctly represents with the available measurements.

In the second part of this two-part paper, we continue the same paper on the phase noise effects on radar signal processors focusing on the performance of PDP algorithms and sidelobe blanker.

ACKNOWLEDGMENT

We thank Ing. G. Tonelli (Selex ES) who has kindly provided the real data used in this study. The authors thank the associate editor and the referees for their interesting comments and careful revision of the paper.

APPENDIX A. AMBIGUITY FUNCTION APPROXIMATION

Notice that for all $t_1 \in [0, T_p]$,

$$\mathbb{E}[|\delta \phi(t_1 + \tau + (n-1)T) - \delta \phi_n|^2] \leq 2 \left( \mathbb{E}[|\phi(t_1 + (n-1)T) - \phi((n-1)T)|^2] \right)$$

$$+ \mathbb{E}[|\phi(t_1 + \tau + (n-1)T) - \phi(\hat{k}T_p + (n-1)T)|^2],$$

where the inequality stems from $x^2 + y^2 \geq 2xy$. Let us analyze $\mathbb{E}[|\phi(t_1 + (n-1)T) - \phi((n-1)T)|^2]$; resorting to the Wiener-Khintchine theorem,

$$\mathbb{E}[|\phi(t_1 + (n-1)T) - \phi((n-1)T)|^2] = 4 \int_{-\infty}^{\infty} \sin^2(\pi f t_1) S_\phi(f) df$$

$$\leq 8 \int_0^{B_\phi} f^2 \pi^2 t_1^2 S_\phi(f) df + 8 \int_{B_\phi}^{W_p} S_\phi(f) df$$

$$\leq 4 S_\phi B_\phi^2 \pi^2 t_1^2 + 8 \cdot 10^{-4} \forall t_1 \in [0, T_p],$$

where $S_\phi(f)$ is the phase noise PSD, $W_p$ is the radar baseband single-side bandwidth, $B_\phi$ is the single-side bandwidth, the phase noise PSD is greater than or equal to $\frac{4 S_\phi B_\phi^2 \pi^2 t_1^2}{W_p}$, and $S_\phi$ is the value of phase noise power level over the frequency interval $[-B_\phi, B_\phi]$. Finally, in (41), we use $\sin^2(x) \leq x^2$ and $\sin^2(x) \leq 1$, and in (42), we exploit $f^2 \leq B_\phi^2, \ f \in [0, B_\phi]$.

To study $\mathbb{E}[|\phi(t_1 + \tau + (n-1)T) - \phi(\hat{k}T_p + (n-1)T)|^2]$, we can apply the previous steps to

$$\mathbb{E}[|\phi(t_1 + (n-1)T) - \phi((n-1)T)|^2],$$

with $t_1 = t_1 + \tau - \hat{k}T_p$, where $|\tau| \leq \frac{1}{2} T_p$ because $\hat{k}$ is the range bin and consequently, $|\tau - \hat{k}T_p| \leq \frac{1}{2} T_p$.

APPENDIX B. COMPOSITE POWER-LAW MODEL PARAMETER ESTIMATION

As a first step toward the estimation of the model parameters, let us denote by

- $f_i, i = 1, 2, \ldots, N$, the discrete set of offset frequencies, where the phase noise PSD has been measured;
- $S_{m}(f_i), i = 1, 2, \ldots, N$, is the phase noise PSD measurement in dBc/Hz obtained in correspondence of the offset frequency from the carrier $f_i$, $i = 1, 2, \ldots, N$.

Hence, the estimation of the parameters $K_0, K_1, K_2, K_3, K_4$, is obtained via the least-square (LS) fitting of the model $S_{mB}(f) = 10 \log_{10} (S_\phi(f))$ with the PSD measurements $S_{m}(f_i), i = 1, 2, \ldots, N$, namely, they are defined as the optimal solution to

$$\min_{K_{0,1,2,3,4}} \sum_{i=1}^{N} |S_{mB}(f_i) - S_{m}(f_i)|^2.$$  (43)

Problem (43) is a non-convex optimization problem, and we can obtain suboptimal solution performing a search over a discrete grid of points or using a search grid algorithm. In both cases, we need a good initial point to compute an optimized solution with a low computational complexity. To this end, we fix a set of four different frequencies, $f_1, f_2, f_3, f_4$ and evaluate the mentioned estimate according to the following procedure:

- $K_0 = 10^{N_{floor}/10}$, with $N_{floor} = \frac{1}{M} \sum_{i=n_1}^{n_1+M} S(f_i)$, the average value of the phase noise PSD measurements, over the last $M$ offset frequencies;
- $K_1, K_2, K_3, K_4$, the solution to the following system of linear equations

$$K_1 \frac{1}{f_1} + K_2 \frac{1}{f_2} + K_3 \frac{1}{f_3} + K_4 \frac{1}{f_4} = 10 S_{m}(f_1)^{1/10} - K_0,$$  (44)

$$K_1 \frac{1}{f_2} + K_2 \frac{1}{f_3} + K_3 \frac{1}{f_4} + K_4 \frac{1}{f_4} = 10 S_{m}(f_2)^{1/10} - K_0,$$  (45)

$$K_1 \frac{1}{f_3} + K_2 \frac{1}{f_4} + K_3 \frac{1}{f_4} + K_4 \frac{1}{f_4} = 10 S_{m}(f_3)^{1/10} - K_0,$$  (46)

AUBRY ET AL.: PHASE NOISE MODELING AND ITS EFFECTS ON MTI FILTERS 709
\[ K_1 \frac{1}{f_4} + K_2 \frac{1}{f_4^2} + K_3 \frac{1}{f_4^3} + K_4 \frac{1}{f_4^4} = 10^{\log(f_4)/10} - K_0. \]  
(47)

This is tantamount to predicting \( k_0 \) as the phase noise PSD floor, evaluated as the average value of the last \( M \) PSD values and estimate the remaining parameters to match the phase noise PSD measurements in correspondence of the frequencies \( f_1, f_2, f_3, f_4 \).

**APPENDIX C. AUTOCORRELATION FUNCTION OF POWER-LAW COMPOSITE PSD**

Due to the linearity of the integral operator involved in (37), we can separately analyze the different addends in (36). Specifically, we have

- **White phase noise**

  \[ 2 \int_{f_0}^{W_p} K_0 \cos(2\pi f_1) df = 2K_0 \sin(2\pi W_p f_1) - \sin(2\pi f_0 f_1), \]

  \( (48) \)

- **Flicker phase noise**

  \[ 2 \int_{f_0}^{W_p} K_1 \frac{1}{f} \cos(2\pi f f_1) df = 2 \int_{2\pi f_1 f_0}^{2\pi f_1 W_p} K_1 \frac{1}{v} \cos(v) dv \]

  \[ = 2K_1 \left[ 2\pi f_1 f_0 \cos(2\pi f_1 W_p) - \cos(2\pi f_1 f_0), \right. \]

  \( (49) \)

- **White FM phase noise**

  \[ 2 \int_{f_0}^{W_p} K_2 \frac{1}{f^3} \cos(2\pi f f_1) df \]

  \[ = 4\pi f_1 \int_{2\pi f_1 f_0}^{2\pi f_1 W_p} K_2 \frac{1}{v^3} \cos(v) dv \]

  \[ = 4K_2 \pi f_1 \left[ -\cos(v) \int_{2\pi f_1 f_0}^{2\pi f_1 W_p} \frac{1}{v^3} \sin(v) dv \right. \]

  \[ - \sin(2\pi f_1 W_p) + \sin(2\pi f_1 f_0), \]

  \( (50) \)

- **Flicker FM phase noise**

  \[ 2 \int_{f_0}^{W_p} K_3 \frac{1}{f^3} \cos(2\pi f f_1) df \]

  \[ = 8\pi^2 f_1^2 \int_{2\pi f_1 f_0}^{2\pi f_1 W_p} K_3 \frac{1}{v^3} \cos(v) dv \]

  \[ = 4K_3 \pi^2 f_1^2 \left[ -\cos(v) \int_{2\pi f_1 f_0}^{2\pi f_1 W_p} \frac{1}{v^3} \sin(v) dv \right. \]

  \[ + \sin(2\pi f_1 W_p) - \sin(2\pi f_1 f_0), \]

  \( (51) \)

\[ \text{• Random Walk FM phase noise} \]

\[ 2 \int_{f_0}^{W_p} K_4 \frac{1}{f^4} \cos(2\pi f f_1) df \]

\[ = 16\pi^3 f_1^3 \int_{2\pi f_1 f_0}^{2\pi f_1 W_p} K_4 \frac{1}{v^4} \cos(v) dv \]

\[ = \frac{16}{3} K_4 \pi^3 f_1^3 \left[ -\cos(v) \int_{2\pi f_1 f_0}^{2\pi f_1 W_p} \frac{1}{v^3} \sin(v) dv \right. \]

\[ + \sin(2\pi f_1 W_p) - \sin(2\pi f_1 f_0) \]

\[ \left. + \frac{1}{2} \sin(2\pi f_1 W_p) + \frac{1}{4} \sin(2\pi f_1 f_0), \right. \]

\( (52) \)

Let us now analyze the effect of the value \( P_\phi \) of the phase noise PSD in the bandwidth \([0, f_0]\) on the system performance. To this end, we assume that \( \phi_1(t) = \phi(t) \otimes h_1(t) \) and \( \phi_2(t) = \phi(t) \otimes h_2(t) \) are statistically independent random processes,\(^6\) where \( h_1(t) \) and \( h_2(t) \) denote, respectively, the ideal low-pass filter with single-sideband bandwidth \( f_0 \) and the ideal bandpass filter over \([f_0, W_p] \). Hence, we have that \( \phi(t) \) can be expressed as the sum of the two statistically independent processes \( \phi_1(t) \) and \( \phi_2(t) \), where \( \phi_2(t) \) shares the autocorrelation function (38), while \( \phi_1(t) \) is characterized by the autocorrelation function

\[ r_\phi(t) = P_\phi \frac{\sin(2\pi f_0 t)}{(2\pi f_0 t)^3} \approx P_\phi, \quad |t| \leq \text{CPI}. \]

(53)

Equation (53) means that \( \phi_1(t) \) can be modeled as a random variable. Because \( \exp(\phi(t)) \) produces a fixed rotation on the received signal, it does not affect the system performance.

\(^6\) By doing so, the realization of \( \phi(t) \) does not influence the pdfs describing the random process \( \phi_2(t) \).
REFERENCES

Coherent Radar Performance Estimation. Dedham, MA: 

[2] Skolnik, M. 

House, 2005.

Principles of Modern Radar: Basic Principles. Raleigh NC: 

On the statistical models for slow varying phase noise in 
communication systems, Tech. Rep., 
http://www.mehrpouyan.info/Report_Phase_Noise_V1.2_ 

Low-noise CMOS LC oscillator with dual-ring structure. 

[7] Richards, M. 
Coherent integration loss due to white Gaussian phase noise. 

Upper bound of coherent integration loss for symmetrically 
distributed phase noise. 

[9] Richards, M. 
A slight extension of coherent integration loss due to white 
Gaussian phase noise, 
http://users.ece.gatech.edu/mrichard/Coherent%20 
Integration%20with%20Phase%20Noise,%20Extended.pdf, 
last access Feb. 11, 2016.

[10] Chorti, A., and Brookes, M. 
A spectral model for RF oscillators with power-law 
phase noise. 

Performance comparison of optimum and conventional MTI 
and Doppler processors. 
IEEE Transactions on Aerospace and Electronic Systems, 
AES-20, 6 (Nov. 1984), 707–715.

Time and phase synchronisation via direct-path signal 
for bistatic synthetic aperture radar systems. 

Echo-domain phase synchronization algorithm for bistatic SAR 
in alternating bistatic/ping-pong mode. 
IEEE Geoscience and Remote Sensing Letters, 9, 4 (July 
2012), 604–608.

[14] Lopez-Dekker, P., Mallorqui, J. J., Serra-Morales, P., 
and Sanz-Marcos, J. 
Phase synchronization and Doppler centroid estimation in 
fixed receiver bistatic SAR systems. 
IEEE Transactions on Geoscience and Remote Sensing, 46, 11 
(Nov. 2008), 3459–3471.


[16] Ward, J. 
Space-time adaptive processing for airborne radar. 
Massachusetts Institute of Technology, Lincoln laboratory, 

[17] Bandiera, F., Orlando, D., and Ricci, G. 
Advanced Radar Detection Schemes Under Mismatched 

[18] Billingsley, P. 

[19] de Abreu, G. T. F. 
On the generation of Tikhonov variates. 
IEEE Transactions on Communications, 56, 7 (July 2008), 
1157–1168.

Wrapped Gaussian mixture models for modeling and high-rate 
quantization of phase data of speech. 
IEEE Transactions on Audio, Speech, and Language 
Processing, 17, 4 (May 2009), 775–786.

[21] Bahlmann, C. 
Directional features in online handwriting recognition. 
Pattern Recognition, 39, 1 (Jan. 2006), 115–125.

Multivariate generalized Laplace distributions and related 
random fields. 
Matematiska vetenskaper, Göteborg 2010, 
http://www.math.chalmers.se/Math/Research/Preprints/2010/47.pdf., 
last access Feb. 11, 2016.

[23] Allan, D., Hellwig, H., Kartaschoff, P., Vanier, J., Vig, J., 
Winkler, G., and Yannoni, N. 
Standard terminology for fundamental frequency and time 
metrology. 
In Proceedings of the 42nd Annual Frequency Control 

[24] Leeson, D. B. 
A simple model of feedback oscillator noise spectrum. 

Phase noise in oscillators: A unifying theory and numerical 
methods for characterization. 
IEEE Transactions on Circuits and Systems I, 47, 5 (May 

McGuigal, T., Mullen, J. W. S., Jr., Sydnor, R., Vessot, R., 
and Winkler, G. 
Characterization of frequency stability. 
IEEE Transactions on Instrumentation and Measurement, 20, 
2 (May 1971), 105–120.

[27] Levanon, N. 

[28] Hsiao, J. K. 
On the optimization of MTI clutter rejection. 
IEEE Transactions on Aerospace and Electronic Systems, 

[29] Guerci, J. R. 
Theory and application of covariance matrix tapers for robust 
adaptive beamforming. 
IEEE Transactions on Signal Processing, 47, 4 (Apr. 1999), 
977–985.

Handbook of Mathematical Functions with Formulas, Graphs, 
and Mathematical Tables. New York: Dover Publications, 
1964.

[31] AN/SPS-48E radar specifics, 
http://www.radartutorial.eu/19.kartei/karte510.en.html, last 
access Feb. 11, 2016.
学霸图书馆

www.xuebalib.com

本文献由“学霸图书馆-文献云下载”收集自网络，仅供学习交流使用。

学霸图书馆（www.xuebalib.com）是一个“整合众多图书馆数据库资源，提供一站式文献检索和下载服务”的24小时在线不限IP图书馆。

图书馆致力于便利、促进学习与科研，提供最强文献下载服务。

图书馆导航：

图书馆首页 文献云下载 图书馆入口 外文数据库大全 疑难文献辅助工具