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Michèle Boulanger a & Luis A. Escobar b
a Motorola, Wireless Data Group, Schaumburg, IL, 60173
b Department of Experimental Statistics, Louisiana State University, Baton Rouge, LA, 70803
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Experimental Design for a Class of Accelerated Degradation Tests

Michèle BOULANGER
Motorola
Wireless Data Group
Schaumburg, IL  60173

Luis A. ESCOBAR
Department of Experimental Statistics
Louisiana State University
Baton Rouge, LA  70803

Traditionally, reliability assessment of new devices has been based on accelerated life tests. This approach is not practical for highly reliable devices, such as lasers, which are not likely to fail in experiments of reasonable length. An alternative approach is to monitor the devices for a period of time and assess their reliability from the changes in performance (degradation) observed during the experiment. In this article, we propose a methodology for designing experiments for degradation processes in which the amount of degradation over time levels off toward a plateau (maximum degradation) that is a function of stress. We provide (a) the stress levels for the experiment, (b) the proportion of devices to test at each stress level, (c) the times at which to measure the devices, and (d) the total number of devices to test. We apply the proposed methodology to an actual example.

KEY WORDS: D-optimal designs; Mixed nonlinear models; Models for high reliability; Two-step designs.

This article presents an approach for designing a class of reliability experiments called accelerated degradation tests (ADT's). An ADT records changes over time (degradation) in the performance of devices subjected to elevated stress. Data from the ADT are then used to gain insight into the physical mechanisms that underlie the degradation process and to make inferences on the performance of the devices at low stress levels (use conditions) and at operation times far beyond the length of the experiment. These inferences imply extrapolation in two dimensions, stress and time. For example, Figure 1 displays the changes observed in propagation delay of two integrated circuits in an ADT in which the stress was temperature. The objectives of that ADT experiment were (a) to determine if propagation delay was degrading over time and, if so, (b) to estimate the amount of degradation for each device used at a temperature of 303°K (Kelvin) during a period of 25 years (the commercial lifetime of the system). Additional information about that ADT is given in Section 1.

ADT's are different from traditional accelerated life tests (ALT's). An ALT records failure and censoring times of devices subjected to elevated stresses. These data are used to estimate the failure time distribution at test conditions, and extrapolation in stress is then carried out to estimate the failure time distribution at use conditions. For highly reliable devices, ALT's provide little information because few failures, if any, may be observed during the experiment, even at very high levels of stress. On the other hand, ADT's may provide information on the change (degradation) that is occurring on one or more characteristics of each device on test long before a failure actually occurs. These data provide valuable information on the failure mechanisms of the devices, thus explaining the increasing popularity of ADT's in industry today (e.g., see Christou and Unger 1990; Nash 1992; Nelson 1990, chap. 11).

In this article we consider ADT's that exhibit the following characteristics:

1. Each device is subjected to an elevated but constant level of stress over the length of the experiment, but different devices may be subjected to different levels of stress.

2. The degradation of the characteristic of interest levels off to a plateau (maximum degradation) after a period of time. In electronics, this is typical of degradation processes caused by the diffusion of impurities. The same behavior has been observed in material sciences for relaxation mechanisms (see Palmer, Stein, Abraham, and Anderson 1984) and for shrinkage phenomena. In pharmacokinetics, this type of degradation describes cumulative changes caused by drug effects that fade with time.

3. The plateaus are stress dependent, and that dependency can be approximated by a simple function of stress.

When designing an ADT, we need to address the following issues: (a) Select the stress levels to use in
1. A MOTIVATING EXAMPLE

**Background.** The work in this article was motivated by the design of an ADT that we fully solve in Section 6. Here we use the example to illustrate some of the general characteristics of the ADT's considered in the article. The example consists of designing an ADT for a logic integrated circuit (IC), a highly reliable component for a large telecommunication system. No catastrophic failures are expected to occur during the lifetime of the IC nor during an experiment, even under highly elevated stress (temperature). But there is some concern about changes over time in propagation delay, an important characteristic of the IC. If propagation delay degrades too far from its nominal value, the overall system performance will be affected and system failure could occur. Changes in propagation delay are caused by diffusion of impurities in various layers of the IC. Diffusion can be hastened by raising the ambient temperature.

**Objectives.** The objectives are (a) to design an ADT to understand where the diffusion of impurities tends to occur among the many layers of the IC and (b) to estimate how much change in propagation delay the devices will see at the end of their required commercial life of 25 years operating at use conditions of 313°K. The length of the ADT should not exceed 2,000 hours.

**Prior Information.** Carey and Koenig (1991) reported and analyzed the results of a previous ADT on an older generation of IC's. We use some of their results in the design of our new experiment. Table 1 summarizes their ADT. A total of 142 devices were tested at elevated temperatures, and the propagation delay of each device was measured at several points during the experiment. There were no catastrophic failures even at the highest temperature, where a failure was defined as a device that did not function any more. But concern about the stability of the characteristics of the devices was expressed when propagation delay was observed to change as a function of both time and temperature; see Figure 1.

Carey and Koenig (1991) used the following Weibull sigmoidal random-coefficients model to describe the degradation over time \( t \) of a device \( j \) subjected to a constant temperature level \( i \):

\[
y_{ij}(t) = \alpha_j[1 - \exp(-\beta_j t^\gamma)] + \epsilon_j(t).
\]

Here \( y_{ij}(t) \) is the observed change in propagation delay up to time \( t \) of a device \( j \) subjected to a constant temperature level \( i \):
the degradation will level off for device \( j \), \( \gamma \) is fixed and equal to .5, \( \beta_j \) is related to the rate at which the propagation delay for device \( j \) approaches the plateau \( \alpha_{ij} \), and \( \varepsilon(t) \) is the random noise due to the measurement process.

Figure 2 shows a plot of the maximum likelihood estimate (MLE) of the plateaus \( \alpha_j \) versus inverse temperature \( x_i \), where \( x_i = 1/((8.6 \times 10^{-5})S_i) \) and \( S_i \) is the \( i \)th Kelvin temperature level. Carey and Koenig (1991) assumed that the distribution of the log of the plateau across devices was normally distributed with mean \( A + Bx_i \) and variance \( \tau^2 \), where \( A, B \), and \( \tau^2 \) are unknown parameters. MLE’s for \( A, B \), and \( \tau^2 \) are listed on Figure 2. Their predicted value for the median of the plateau distribution at use conditions by the end of the commercial lifetime was acceptable to the designer of the system.

An associate editor pointed to an apparent lack of fit of the model in Figure 2. A likelihood ratio test for a quadratic departure from the simple linear regression model gives a test statistic of \( X^2 = 3.87 \) with a \( p \) value of .05, which is a moderate evidence of lack of fit. For the design of our experiment, it is important to have an estimate of \( \tau^2 \). Under the quadratic model, the MLE of \( \tau^2 \) is \( \hat{\tau}^2 = .011 \) in contrast to an estimate of \( \hat{\tau}^2 = .014 \) under the simple linear regression model. This change in the estimate of \( \tau^2 \) does not have any important effect on the outcome of the planning of the experiment, and in this article, we use the estimate based on the simple linear regression model.

2. APPROACH TO THE DESIGN PROBLEM

In general, ADT’s have several goals—(a) to provide some insight into the physical mechanism that causes the degradation of each device, (b) to estimate the parameters that characterize the degradation for each device under stress, and (c) to provide information on the degradation of a population of devices at use conditions. In this section, we propose a model and an estimation procedure that address these goals and from which we derive our approach to the design of ADT’s.

2.1 General Model

Phenomena that level off after a period of time are often modeled as a sigmoidal growth curve (see Seber and Wild 1989, chap. 7). We use a nonlinear model that incorporates a sigmoidal growth curve, \( F(t, \xi_j) \), to describe the observed degradation \( y_{ij}(t) \) up to time \( t \), for device \( j \), at the \( i \)th stress level.

\[
y_{ij}(t) = \alpha_{ij} F(t, \xi_j) + \varepsilon_{ij}(t), \quad 0 \leq t < \infty.
\]

Here, \( \alpha_{ij} > 0 \) is the plateau for device \( j \) at the \( i \)th stress level. \( F(t, \xi_j) \) is a known, continuous, monotone increasing function of \( t \), with \( F(0, \xi_j) = 0 \) and \( F(\infty, \xi_j) = 1 \). \( F \) and the vector of unknown coefficients \( \xi_j \) determine the rate at which the degradation \( y_{ij}(t) \) levels off toward the plateau \( \alpha_{ij} \). The function \( F \) allows the modeling of numerous sigmoidal growth-curve processes, including the Weibull, logistic, and Gompertz models (see Ratkowsky 1990, chap. 5). In practice, \( F \) should be derived from some understanding of the physical mechanisms underlying the observed degradation. If no such understanding exists, the selection can also be made on empirical basis. But in this case, one must be aware of the difficulties in discriminating between various alternative parametric forms as pointed out by Ratkowsky (1983, p. 63).

Finally, we assume \( \varepsilon_{ij}(t) \sim N(0, \sigma^2) \) with \( \varepsilon_{ij}(t) \) independent over \( i, j, \) and \( t \). Though the measurements are repeated observations on the same device, the assumption of independence is reasonable because in Model (2) each device has its own set of parameters. Because the level of degradation is a function of the stress level \( i \), it is also reasonable to assume that the variance of \( \varepsilon_{ij}(t) \) depends on \( i \) but not on the device \( j \).

To capture the device-to-device variability, we assume that the coefficients \( (\alpha_{ij}, \xi_j) \) are random coefficients over the population of devices. More precisely, we assume that \( \log(\alpha_{ij}) \) are iid with mean \( \eta(x_i) = A + Bx_i \) and variance \( \tau^2 \), where \( x_i \) is a known function of the \( i \)th stress level but \( A \) and \( B \) are unknown. Without loss of generality, we will refer to \( x_i \) as the \( i \)th stress level. In this article, we focus...
primarily on the plateaus and we do not model the remaining random coefficients $\xi_j$ as a function of stress. From a practical point of view, the plateaus are the primary concern of the engineer. If the plateaus, which represent an upper bound for degradation, are acceptable at use conditions, then there is no need to model the whole degradation curve as a function of time. This simplifies greatly both the analysis of the data and the design of the experiment.

The simplest linear relationship proposed for $\eta(x_i)$ includes commonly used acceleration models like the Arrhenius and the inverse power low models (see Nelson 1990, pp. 75–92). The methodology in this article, however, can be extended to situations in which $\eta(x_i)$ is modeled by a general linear function of $x_i$, for example, a polynomial function—but in practice we have found that these models are rarely used.

Nonlinear models with random coefficients similar to Model (2) have been used (see Laird and Ware 1982; Ware 1985) to analyze longitudinal data in pharmacokinetics studies dealing with the behavior of individuals subjected to various doses of a given drug. Lu and Meeker (1993) used these models to derive failure-time distributions in which they defined failure time as the first time a device reaches a prespecified threshold of degradation.

2.2 Estimation

Because the goals set for an ADT at the beginning of Section 2 entail inferences on characteristics of both individual devices and population of devices, the estimation of the parameters of interest is performed in two steps, as follows.

Formerly, let $t_{1j}, \ldots, t_{kj}$ be the measurement times for devices $j$ at $x_i$. In the Appendix, we use Model (2) to obtain the MLE's $\hat{\alpha}_j, \hat{\xi}_j$ for each device. Now, assuming that the bias in the estimation of the log $\hat{\alpha}_j$ is negligible, one can derive the approximate linear regression model

$$\log(\hat{\alpha}_j) = A + B x_i + e_{ij}, \quad (3)$$

where $E(s_0 | i) = 0, V(s_0 | i) = c_i = \tau^2 + v(x_i)$, and $v(x_i) = E_i[V(\log(\hat{\alpha}_j) | i, j)]$.

Carey and Koenig (1991) obtained MLE's of the parameters $A, B$, and $\tau$ in (3), assuming that the $s_0$ are iid normally distributed. In this article, we follow a slightly different method of estimation. We obtain weighted least squares estimates of $A, B$ assuming guess values for $\tau$ and $v(x_i)$. This approach avoids the assumption of normality for $s_0$ and facilitates the development of the experimental plan.

2.3 Approach to the Design of ADT's

To guide the design of our ADT, we set two objectives to ensure that the design meets the three goals set in beginning of Section 2. These two objectives are (1) the ADT should provide sufficiently precise estimates of the parameters $\alpha_j, \xi_j$ for all the devices and (2) the ADT should provide good estimates of population characteristics of the degradation of the devices at use conditions. Formally, the ADT should provide a sufficiently precise estimate of the median of the distribution of the plateaus at use conditions or equivalently of the mean of the distribution of the log of the plateaus.

The first objective is solely concerned with the number and values of the measurement times for each individual device at each stress level. The second objective is primarily concerned with the selection of the stress levels and the proportion of units to allocate at those stresses. The second objective provides for good estimates of population characteristics in opposition to the first one that allows for good estimates of device-specific characteristics. These two objectives jointly affect the overall design of the ADT: To select the number and values of the measurement times, we need to know at which stress levels the ADT will be carried out and, vice versa, to select stress levels and proportion of units we need to have some information on the precision with which the individual characteristics of each device will be estimated.

The two objectives are actually criteria that guide the analysis of the data. In the design phase of the experiment, we propose to study each objective separately, conditionally on having the results of the design for the other objective. This approach allows for a separation of the design into two steps. We propose to start with the second objective because in practice, for cost reasons, engineers are more concerned with the stress levels and the number of devices they need than they are with the actual number of measurement times.

So the first step is based on Equation (3) and optimizes the stress levels and the proportions of devices to allocate to the selected stress levels, given that the variability introduced by the estimation of the $\alpha_j$ is known. The second step is based on Equation (2) and optimizes the times at which to measure the degradation of the devices at a selected stress level $x_i$ and given guess values of the unknown parameters at that stress. After determining the stress levels and the corresponding proportions of units separately from the measurement times, we combine the results of these two steps to determine the total number of devices needed in the ADT as a function of the precision desired for the estimate of the median of the distribution of the plateaus at use conditions.

3. STEP 1: STRESS PLAN

In this section, we consider the selection of the stress levels $x_i$ and the proportion of devices $p_i$ to
allocate at $x_i$ such that they provide a sufficiently precise estimate of the mean of the log plateaus at use conditions $x_U$. By focusing on the plateau parameter $\alpha$ rather than on the whole vector of parameters $(\alpha, \zeta)$, we simplify the design while being conservative because the plateau represents an upper bound for the degradation. In view of the high reliability of the devices under consideration and the limited time for conducting the experiment, we assume that all units in the experiment are observed at stresses higher than use conditions (i.e., $x_U$ does not belong to the experimental region).

The selection of the $x_i$'s is restricted to an experimental region $a \leq x_i \leq b$, which is defined by the engineers. In practice, $a$ and $b$ are the lowest and highest experimental levels of stress allowed by the testing equipment and engineering considerations. For the sake of discussion, we assume that $x_U < a < b$, but all the results hold when $a < b < x_U$, as illustrated in Section 6.

### 3.1 The Minimum Variance Plan

The objective is to minimize the variance of the weighted least squares estimate of the mean of the log plateaus, $\eta(x_U) = A + Bx_U$, at use conditions. Then, the design problem consists of finding the levels of stress $x_i$ and the allocations $p_i$ to $x_i$ that minimize the variance $V[\eta(x_U)]$ where $A, B$ are weighted least squares estimates of $A, B$. The estimates $A, B$ are obtained from Model (3) under the assumption that there is a guess value for $\tau$ and that $\nu(x)$ can be approximated by a positive and continuous function of $x$ over the range of stress levels in the experimental region (see Sec. 6 for an illustration).

Fedorov (1972, theorem 2.13.2) showed that the minimization can be carried out in two steps: (1) Determine the support of the experiment (i.e., the number of design points and their values), and (2) determine the optimal proportion of devices to allocate to the points in the design.

In our case, the optimal design has support on two points in the experimental region $[a, b]$. All of the relevant theory was given by Fedorov (1972, sec. 2.13), and a summary of the procedure of our problem is as follows. Define the function

$$u(x) = \frac{1}{\sqrt{c(x)}} \left[ \delta_0 + \delta_1 x \right],$$

where $c(x) = \tau^2 + \nu(x)$, $x \in [a, b]$, and the coefficients $\delta_0, \delta_1$ are chosen such that $|u(x)| \leq 1$, $x \in [a, b]$, and there are two points, say $x_L$ and $x_H$, such that $a < x_L < x_H < b$, $u(x_L) = 1$, and $u(x_H) = -1$. The points $x_L, x_H$ are the support for the optimal plan. There is always a solution for $\delta_0, \delta_1$ (see Fedorov 1972, pp. 147–151), but, in general, one needs a numerical or a graphical method to find the coefficients. The methodology extends directly to a larger class of functions $\eta(x)$ and in particular it applies when $\eta(x)$ is a regular polynomial in $x$.

The optimal proportions of devices at the stresses $x_L, x_H$—say $p^*_L$ and $p^*_H$—are obtained by minimizing the variance $V[\eta(x_U)]$ under Model (3) and with $p_L$ devices allocated at $x_L$ and $p_H$ devices allocated at $x_H$ (see Fedorov 1972, p. 145). This variance is given by

$$V[\eta(x_U)] = \frac{1}{n} \left[ \frac{c_Lp_L(d_L - d_U)^2 + c_Hp_H(d_H - d_U)^2}{p_Lp_H(d_H - d_L)^2} \right],$$

where $n$ is the total number of devices in the experiment, $p_i$ ($i = L, H$) is the proportion of $n$ allocated at $x_i$, $c_i = \tau^2 + \nu(x_i)$, and

$$d_i = \frac{x_i - a}{b - a}.$$

The optimal allocations $p^*_L$ and $p^*_H$ are

$$p^*_L = \frac{(d_H - d_U)\sqrt{c_L} + (d_L - d_U)\sqrt{c_H}}{(d_H - d_L)\sqrt{c_L}},$$

and $p^*_H = 1 - p^*_L$. The overall minimum variance is obtained by substituting $p^*_L$ and $p^*_H$ in (4) for $p_L$ and $p_H$, which gives

$$V^*_\min = \frac{1}{n} \left[ \frac{(d_H - d_U)\sqrt{c_L} + (d_L - d_U)\sqrt{c_H}}{(d_H - d_L)^2} \right].$$

Observe that the sample size $n$ enters as a simple factor in the formula of $V[\eta(x_U)]$, which implies that the optimum proportions $p^*_L$ and $p^*_H$ do not depend on the value of $n$. The effect of $n$ on the estimator $\exp[\eta(x_U)]$ will be considered in Section 5.

### 3.2 The Compromise Plan: Determination of Allocations

Because the minimum variance plan has support on only two points, it does not allow us to check the validity of the assumed linear relationship between log($\alpha_U$) and $x_i$. To test for a quadratic departure from the linear model, we propose a compromise plan with three levels of stress, $x_L < x_M < x_H$. Meeker (1984) and Meeker and Hahn (1985) used a similar approach to derive compromise plans for ALT's.

The compromise plan assumes that a prespecified proportion of the devices, $p^*_L$, is to be allocated at a chosen middle level of stress, $x_M$. Selection of $p^*_L$ and $x_M$ is discussed in the next section.
In this section, we concentrate on determining the allocations at \( p_L \) and \( p_H \) when everything else is constant. Thus the optimization problem is minimizing, with respect to \( p_L \), the variance \( V(\hat{\eta}(x_U) | p^*_L) \) obtained under Model (3) with \( p_L \) devices at \( x_L \), \( p^*_M \) devices at \( x_M \), and \( p_H \) devices at \( x_H \). The variance is

\[
\begin{align*}
V(\hat{\eta}(x_U) | p^*_L) & = \frac{1}{n} \left\{ c_L c_M (d_L - d_L)^2 + c_L c_H (d_L - d_H)^2 + c_M c_H (d_H - d_M)^2 + p_L c_M (d_M - d_M)^2 + p_H c_H (d_H - d_H)^2 \right\} \\
& = \left( \frac{1}{n} \right) \left\{ c_L c_M (d_L - d_L)^2 + c_L c_H (d_L - d_M)^2 + p_L c_H (d_M - d_H)^2 \right\}
\end{align*}
\]

(7)

where \( p_H = 1 - p^*_M - p_L \), \( c_i = \tau_i^2 + \nu(x_i) (i = L, M, H) \), and \( x_L, x_M, x_H \), and \( p^*_M \) are fixed. This is a constrained minimization in the region \( 0 < p_L < 1 - p^*_M \).

To characterize the compromise plan, let

\[
p^*_L = p^*_L - p^*_M
\]

If \( 0 < p^*_L < 1 - p^*_M \), then the optimal allocations at \( x_L, x_M, \) and \( x_H \) are \( p^*_L, p^*_M, \) and \( 1 - p^*_L - p^*_M \), respectively. We refer to this derived compromise plan as the optimal compromise plan with \((p^*_L, d_M)\). The minimum variance for this compromise plan \( V^*_M \) is obtained by substituting the optimal allocations \( p^*_L, p^*_M, \) and \( p^*_H \) in Equation (7) and is given by

\[
V^*_M = c_M (d_M - d_L)^2 + \left\{ c_L (1 - p^*_M) (d_L - d_L)^2 + p^*_M (\sqrt{c_L} (d_M - d_M) + \sqrt{c_H} (d_H - d_H) - (\sqrt{c_L} + \sqrt{c_H}) (d_M - d_H) \right\}
\]

(9)

On the other hand, if \( p^*_L > 1 - p^*_M \), the optimal compromise plan degenerates to a plan in two stresses. This means that the specified value for \( p^*_M \) is too large. A possible solution to this difficulty is to specify a lower value for \( p^*_M \) and try again.

Figure 3 shows the region \( R_3 \) of feasible choices for \((p^*_M, d_M)\) (or equivalently, of pairs \((p^*_M, x_M)\)) that yield optimal compromise plans with three stress levels. This feasible region is open in the sense that it does not include the boundaries that correspond to compromise degenerate plans in two stress levels.

3.3 The Compromise Plan: Determination of \( p^*_M \) and \( d_M \)

In this section we discuss some criteria for selecting \( p^*_M \) and \( d_M \) (or \( x_M \)).

Relative Efficiency. The optimal compromise plan with \((p^*_M, d_M)\) yields an estimator of \( \eta(x_U) \) with larger variance than the estimator obtained from the minimum variance plan. The ratio of these two variances defines the relative efficiency, \( REST \), of the compromise plan

\[
REST(p^*_M, d_M) = \frac{V^*_M}{V^*_{MIN}} = 1 - p^*_M
\]

For fixed \( p^*_M \), one can use \( REST(p^*_M, d_M) \) to assess the loss in efficiency from allocating a proportion \( p^*_M \) of the devices to the stress level \( x_M \).

Test for Quadratic Model. Another consideration is to choose \((p^*_M, d_M)\) in \( R_3 \), such that the derived optimal compromise plan minimizes the variance of the estimator of \( \eta \), say \( V(\hat{\eta}) \), when fitting the quadratic model \( \eta(x_i) = A + Bx_i + Cx_i^2 \). Experimental plans that give a large \( V(\hat{\eta}) \) provide very little power to detect departures from the assumed simple linear regression model (see Casella and Berger 1990, chap. 8). For fixed \((p^*_M, d_M)\) one derives the optimal compromise plan and computes \( V(\hat{\eta}) \) and \( REST(p^*_M, d_M) \). The ideal situation is when \( V(\hat{\eta}) \) is low.

---

Figure 3. Feasible Regions for \( d_M \) and \( p^*_M \) in Compromise Plan: Area With Single Lines = Region \( R_1 \), \( 0 < p^*_M < 1 - p^*_M \); Area With Crosshatched Lines = Region \( R_2 \), \( 0 < p^*_M < 1 - p^*_M \), REST = 8.

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TECHNOMETRICS, AUGUST 1994, VOL. 36, NO. 3
and \( \text{REST} (\rho_{d_M}^*, d_M) \) is high. Unfortunately, these two criteria often conflict in the sense that an experimental plan that provides a high relative efficiency may have a large \( V(C) \) and vice versa. For this reason, we propose to combine the two criteria as follows. Specify a level \( s \), \( 0 < s < 1 \), as a maximum tolerable limit for \( \text{REST} (\rho_{d_M}^*, d_M) \) and define as \( B_2 \) the subset region of \( B_1 \) such that \( \text{REST} (\rho_{d_M}^*, d_M) \geq s \) (see Fig. 3). Then one can choose the optimal compromise plan in \( B_2 \) that minimizes \( V(C) \). The actual derivation of this plan has to be done numerically, and we discuss an example in Section 6.1.

**Other Criteria.** There are other alternative compromise plans. An obvious choice is to use the optimal plan for extrapolation at \( x_U \) under an assumed quadratic model \( n(x) = A + Bx + Cx^2 \). The procedure to obtain this best for quadratic extrapolation stress plan is similar to the development in Section 3.1, and all the needed theory was given by Fedorov (1972, theorem 2.13.2). In the example in Section 6, we illustrate that this plan may have very low efficiency with respect to the optimal plan for extrapolation derived in Section 3.1. This is to be expected because the relative efficiency of this plan is calculated under the assumption that the simple linear model is correct.

4. **STEP 2: MEASUREMENT–TIME PLAN**

In this section, we discuss the problem of selecting appropriate times to measure the devices to obtain sufficiently precise estimates of the \((a_{ij}, \xi_j)\) in Model (2). Though a primary interest is to obtain sufficiently precise estimates of the plateau parameter \( \alpha \), the measurement-time plans should also provide precise estimates of the parameters \( \xi \) to meet the goals stated at the beginning of Section 2.3. Theoretically, one can entertain a measurement-time plan for every single device, but this is not practical due to the many devices to be tested. Therefore, we assume that all the devices at a given level of stress \( x_i \) are measured at the same time but that we may have different time plans at different levels of stress. For this reason, and to simplify the presentation, in this section we drop the subindices \( i, j \) and simply write \( x_i = x, (\alpha, \xi) = (a_{ij}, \xi_j) \), and so forth.

The measurement-time plans proposed in this article are derived under the following constraints: (a) The measurements must be made in a prespecified interval \([t_{min}, t_{max}]\), where \( t_{max} \) is the predetermined duration of the experiment and \( t_{min} > 0 \) is the earliest time at which the devices can be measured, and (b) the total number of measurement times, \( k \), is fixed and known. In the following section, we show how to assess the effect of the number of measurement times on the precision of the estimates.

Next, we describe a statistical optimal plan and two heuristic plans for the selection of the measurement times.

4.1 The D-Optimal Plan

Here we determine the measurement times based on the D-optimal (DO) criterion. This criterion, when applied to Model (2), selects the measurement times such that they minimize the determinant of the asymptotic variance–covariance matrix of the MLE's \((\hat{\alpha}, \hat{\xi})\), as summarized in the following. See Seber and Wild (1989, chap. 5) for details.

Let \( g(t) \) be the \((q + 1) \times 1 \) vector of first derivatives of \( g(\xi) \) with respect to \((\alpha, \xi)\) evaluated at a set of guess or prior values \((\alpha^*, \xi^*)\), where \( q \) is the dimension of \( \xi \). Consider a \( k \times 1 \) vector of measurements in the experimental region, say \( t = (t_1, \ldots, t_k) \), \( (t_{min} = t_1 \leq \cdots \leq t_k \leq t_{max}) \), and define the \( k \times (q + 1) \) gradient matrix \( G(t) = \{g(t_1), \ldots, g(t_k)\} \). A vector of measurement times \( t_{DO} = (t_{DO}^1, \ldots, t_{DO}^k) \) constitutes a DO plan if \( t_{DO} \) maximizes the determinant

\[
\Delta(t) = |G(t)'G(t)| = (\alpha^*)^2 |M(t)'M(t)|, 
\]

where \(|G(t)'G(t)|^{-1}\) is an asymptotic estimate of the covariance matrix of \((\hat{\alpha}, \hat{\xi})\) (see Seber and Wild 1989, chap. 12),

\[
M(t) = \begin{bmatrix} F(t_1, \xi^*) & \ldots & F(t_k, \xi^*) \\ \frac{\partial F(t_1, \xi)}{\partial \xi} & \ldots & \frac{\partial F(t_k, \xi)}{\partial \xi} \end{bmatrix}',
\]

and all the derivatives are evaluated at \( \xi^* \).

Clearly, a DO plan is invariant to choices of the input values \( \alpha^* \), but it depends on the guess values \( \xi^* \). In this sense a DO plan is local optimal (see Chernoff 1952). In most cases, there is no close-form formula to compute \( t_{DO} \) and numerical methods are used to maximize (10). Two other important characteristics of DO plans are (see also Atkinson and Donev 1992, p. 116):

1. The plan has support on \( q + 1 \) distinct points, and each of these points is replicated \( k/(q + 1) \) times. Though we do not have an analytic proof, we suspect that \( t_k = t_{max} \), we have observed this in all our numerical computations. It is interesting to notice that, when \( \xi \) is known, Model (2) is a simple linear regression through the origin and a DO plan allocates the \( k \) measurement to \( t_{max} \). The effect of not knowing \( \xi \) is to move \( kq/(q + 1) \) measurements away from \( t_{max} \) to \( q \) distinct points in the interval \([t_{min}, t_{max}]\) for estimating \( \xi \) according with the shape of the function \( F \).

2. A DO plan may be very sensitive to changes in input values \( \xi^* \), and it is not robust to departure from Model (2).
DO plans are rarely suitable for real applications because if the data collected according to a DO plan do not support the assumed Model (2), one can only fit alternative models that have at most the same number of parameters as the assumed model—that is, \( q + 1 \). DO plans, however, provide a useful baseline for comparison with alternative or "heuristic" plans like the ones we discuss in the following section.

4.2 Heuristic Plans

In practice, scientists tend to use alternative experimental plans that are easier to implement than DO plans. In this section, we discuss two plans used often in practice and which, as shown in the example in Section 6 later, can be highly efficient with respect to DO plans.

Equal Degradation (ED) Plan. This plan is motivated by the need to keep track of the changes in degradation of the device in the interval \([t_{\text{min}}, t_{\text{max}}]\). The idea is to choose the measure times, say \( t_{\text{ED}}^i = (t_{\text{ED}}^{i,1}, ..., t_{\text{ED}}^{i,q})\), with \( t_{\text{ED}}^{i,1} = t_{\text{min}}, t_{\text{ED}}^{i,q} = t_{\text{max}}\) such that the expected amount of observed degradation between two consecutive times is constant. Thus

\[
F(t_{\text{ED}}^i, \xi^0) = F(t_{\text{ED}}^i, \xi^0) = \frac{F(t_{\text{max}}, \xi^0) - F(t_{\text{min}}, \xi^0)}{k - 1}, \quad i = 1, \ldots, k.
\]

It is easy to show that

\[
t_{\text{ED}}^i = F^{-1} \left[ \frac{(l - 1)F(t_{\text{max}}) + (k - l)F(t_{\text{min}})}{k - 1} \right], \quad l = 1, \ldots, k. (12)
\]

The ED plan makes a single measurement at each of the measurement times. As for DO plans, the ED plan has the following properties: (a) The plan depends on \( t_{\text{max}}, t_{\text{min}}\) and \( \xi^0\) but it does not depend on guess values for \( \alpha \); (b) the plan can be very sensitive to model departures. In contrast with a DO plan, the ED plan has the following advantages: (a) It is easy to compute, (b) it is not as sensitive to small changes in guess values \( \xi^0\), and (c) when \( k > q + 1\), the plan allows the fitting of models with more than \( q + 1\) parameters.

Equal Log-Spacing (EL) Plan. These measurement times are obtained such that they are equidistant in the log-time scale. This plan makes more time measurements in the lower end of the interval \([t_{\text{min}}, t_{\text{max}}]\) than in the upper end of it. The rationale for the EL plan is based on the observation that many degradation processes show a great deal of degradation early into the observation period before stabilizing. Thus it makes sense to get more observations early when the degradation rate is faster to get precise estimates of the parameters \( \xi \). Let \( t_{\text{EL}}^l = (t_{\text{EL}}^{l,1}, ..., t_{\text{EL}}^{l,q})\) be the measurement times for the EL plan, then

\[
t_{\text{EL}}^l = e^{\exp \left[ \frac{(l - 1)\log(t_{\text{max}}) + (k - l)\log(t_{\text{min}})}{k - 1} \right]}, \quad l = 1, \ldots, k. (13)
\]

This plan does not depend on guess values, and the same measurement-time plan is used for all of the devices at all levels of stress. One should be cautious, however, in using this kind of plan because it ignores the functional form of \( F \) and any previous knowledge on the degradation process.

4.3 Relative Efficiencies

In this section, we compare the heuristic plans against the DO plans, when each plan has the same number of time measurements, \( k \). The comparison is made in terms of relative efficiency, \( \text{RETI} \), of each heuristic plan with respect to the corresponding DO plan. The \( \text{RETI} \) are (see Atkinson and Donev 1992, p. 16)

\[
\text{RETI}(\text{ED}) = \left( \frac{\Delta(t_{\text{DO}})}{\Delta(t_{\text{ED}})} \right)^{1/(q + 1)},
\]

\[
\text{RETI}(\text{EL}) = \left( \frac{\Delta(t_{\text{EL}})}{\Delta(t_{\text{DO}})} \right)^{1/(q + 1)},
\]

where \( \Delta(\cdot) \) is the determinant function defined by (10). The \( 1/(q + 1) \) power is to keep the comparison in terms of number of measurements \( k \).

It is difficult to characterize fully the behavior of these relative efficiencies, because they are a function of (a) the function \( F \); (b) the prior guesses \( \xi^0 \); (c) \( t_{\text{min}}, t_{\text{max}}\); and (d) the number of measurements \( k \). We, however, have observed the following for a given function \( F \):

1. The relative efficiencies are more sensitive to the parameter \( k \) than to \( t_{\text{min}}, t_{\text{max}} \) or prior guess values \( \xi^0 \); see Table 4, Section 6.

2. It seems that for large \( k \) the efficiencies of the heuristic plans decrease monotonically. This is not too surprising because when \( k \) increases the "heuristic" plans keep increasing the number of measurement times rather than replicating the number of measurements at some given times.

3. For a fixed \( k \), as \( t_{\text{max}} \) increases, the relative efficiency of the heuristic plans tends to increase. This can be explained by the fact that the DO plan put all of its support in only \( q + 1 \) points in a very large interval.

Finally, because the DO plans seem to be more sensitive to the input guess values of the parameters \( \xi^0 \) than the heuristic plans, we expect that the actual
relative efficiencies are larger than the relative efficiencies computed under the assumption that the guess values are correct. We provide some additional insights into the relative performance of these measurement-time plans in the example discussed in Section 6.

5. SAMPLE SIZE AND PRECISION

In this section we relate the total number of devices on test, \( n \), to the precision of the estimate of the median of the distribution of the plateaus, \( \exp[\bar{\eta}(x_U)] \), at use conditions.

Let \( 0 < r < 1 \) be the maximum relative percentage of error that can be tolerated in the estimation of the median of the plateaus at use conditions. Then, the precision and the sample size \( n \) are related through

\[
\Pr \left( \left| \frac{\exp[\bar{\eta}(x_U)] - \exp[\bar{\eta}(x_U)]}{\exp[\bar{\eta}(x_U)]} \right| < r \right) \geq 1 - \rho,
\]

where \( \bar{\eta}(x_U) = \hat{A} + \hat{B}x_U \) and \( (1 - \rho) \) is a prespecified confidence level. In large samples (which is the usual case in these kinds of experiments), \( \bar{\eta}(x_U) \) is approximately normally distributed, and straightforward manipulation of Equation (14) leads to the sample size

\[
n = \frac{4z^2_{\rho/2}}{\left[ \log \left( \frac{1 + r}{1 - r} \right) \right]^2} \left( n V_{\min}^{**} \right),
\]

where \( n V_{\min}^{**} \) is independent of \( n \) and given by Equation (9) and \( z_{\rho/2} \) is the \( \rho/2 \) quantile from a standard normal variable. Then one can compute the precision of \( \bar{\eta}(x_U) \) as a function of \( n \) or choose \( n \) to achieve a desired precision. Observe that \( n \) is a function of (a) hard-engineering constraint \( a, b \); (b) the stress levels selected in Step 1, (c) the variance component \( \sigma_i \), which depends on \( t_{\min} \) and the measurement times (the number \( k \) and their values) selected in Step 2, and (d) the requirements on \( \rho \) and \( r \). Then one can vary the nonhard-engineering constraints to measure the effect of these factors on sample size. In particular, one can study the effect of \( t_{\min} \) and \( k \) on the precision or sample size if \( \sigma_i \) can be modeled as a function of those variables, but this might be difficult. A limiting case is when \( \sigma_i(x_U) \) is equal to 0 for all the stress levels. This corresponds to having no error in the estimation of the plateaus \( \alpha_i \), which can approximately happen if \( t_{\min} \) is large.

6. MOTIVATING EXAMPLE CONTINUED

In this section, we design an ADT to assess the degradation in propagation delay of a new logic IC. The objective of the ADT and prior information available on the IC have been described in Section 1. We start with Step 1 by determining the temperature levels and proportions of devices at each temperature. Then we continue with Step 2 by determining the measurement times for some a priori fixed number of measurements. Finally, we compute the number of devices needed to meet a prespecified requirement on the precision of the estimator of interest at use conditions. Some sensitivity analysis of the results of the design as a function of the length of the experiment \( t_{\min} \) and of the number of measurements \( k \) is also provided.

6.1 Step 1: Selecting Temperature Levels

**Physical Constraints.** The experimental region for the Kelvin temperature, \( S \), is the interval \([373^\circ K, 448^\circ K]\). This region is determined by physical constraints: \( 373^\circ K \) is the minimum temperature at which the measuring equipment will be able to detect the small amount of change expected in propagation delay by the end of the experiment, and \( 448^\circ K \) is the temperature beyond which the plastic package of the IC may be damaged. The use temperature is \( 313^\circ K \). Therefore, in the inverse temperature scale \( x = 1/(8.6 \times 10^{-5} S) \), the experimental region is the interval \([a, b]\), with \( a = 25.96, b = 31.17 \). The use condition corresponds to \( x_U = 37.15 \).

**Prior Information.** From Carey and Koenig (1991) we get a guess value for \( \tau^2 \) of .014. We also derive a guess functional form for the variance \( v(x) \) as \( v(x) = -2.565 + .1x \) by fitting a simple regression line to the unpublished \( \hat{\sigma}(x) = E\{V[\log(\hat{\alpha}_i)]/i, j]\} \) values.

**Minimum Variance Plan.** The minimum variance plan has support on the two extreme temperatures of the experimental region. To show this, define \( \mu(x) = (5.01778 - .18511x)/m \), where \( \mu(x) = \tau^2 + v(x) \). The coefficients for \( \mu(x) \) were obtained graphically using the methodology give by Fedorov (1972, p. 151). It is easy to show that \( \mu(x) \leq 1 \) for all \( x \) in the interval \([a, b]\) and \( \mu(a) = 1, \mu(b) = -1 \). Thus, according to Section 3.1, it follows that \( x_L = a \) and \( x_H = b \) are the support points for the optimal plan that correspond to the extreme temperature \( 373^\circ K \) and \( 448^\circ K \). The optimal allocations \( p_L^* = .13, p_H^* = .87 \), and the minimum variance \( V_{\min} \) are obtained by substituting \( d_H = 1, d_L = 0, d_U = 2.15, c_L = .014 + v(x_L) = .045, \) and \( c_H = .014 + v(x_U) = .566 \) in Equations (5) and (6), respectively.

**Optimal Compromise Plan.** Table 2 displays the values of \( x_M^* \) and \( p_M^* \) for compromise plans that minimize the variance of the quadratic coefficient, \( V(\hat{C}) \), while keeping the relative efficiency \( REST \) above a
Table 2. Optimal Compromise Plans

<table>
<thead>
<tr>
<th>Min REST required %</th>
<th>Effective REST %</th>
<th>nV(\hat{C})</th>
<th>p_L^*</th>
<th>p_M^*</th>
<th>d_M</th>
<th>x_M</th>
<th>T_M °K</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>90</td>
<td>41.51</td>
<td>.130</td>
<td>.127</td>
<td>.481</td>
<td>28.47</td>
<td>408</td>
</tr>
<tr>
<td>80</td>
<td>80</td>
<td>23.45</td>
<td>.127</td>
<td>.247</td>
<td>.461</td>
<td>28.36</td>
<td>410</td>
</tr>
<tr>
<td>70</td>
<td>70</td>
<td>17.91</td>
<td>.124</td>
<td>.358</td>
<td>.437</td>
<td>28.24</td>
<td>412</td>
</tr>
<tr>
<td>60</td>
<td>60</td>
<td>15.69</td>
<td>.119</td>
<td>.461</td>
<td>.412</td>
<td>28.11</td>
<td>414</td>
</tr>
<tr>
<td>50</td>
<td>50</td>
<td>15.06</td>
<td>.113</td>
<td>.557</td>
<td>.384</td>
<td>27.96</td>
<td>416</td>
</tr>
<tr>
<td>40</td>
<td>50</td>
<td>15.06</td>
<td>.113</td>
<td>.557</td>
<td>.384</td>
<td>27.96</td>
<td>416</td>
</tr>
<tr>
<td>30</td>
<td>50</td>
<td>15.06</td>
<td>.113</td>
<td>.557</td>
<td>.384</td>
<td>27.96</td>
<td>416</td>
</tr>
<tr>
<td>01</td>
<td>50</td>
<td>15.06</td>
<td>.113</td>
<td>.557</td>
<td>.384</td>
<td>27.96</td>
<td>416</td>
</tr>
</tbody>
</table>

NOTE: \(d_L = 0, d_H = 1, \gamma_p' = \gamma_p - p_T\).

certain level. These values were obtained numerically by minimizing \(V(\hat{C})\) among all the possible optimal compromise plans that meet the minimum required REST value. Figure 3 displays the region \(R_1\) corresponding to all possible choices of \((p_M^*, d_M)\) for the example and the region \(R_2\) of choices of \((p_M^*, d_M)\) leading to a REST value of at least 80%. Observe that the optimal compromise plan that provides a global minimum for \(V(\hat{C})\) has a REST value of only 50%.

Table 2 shows that only small decreases in \(V(\hat{C})\) are obtained by lowering the REST below 80%. Thus a sensible compromise plan is obtained by requiring a minimum REST value of 80%. From Table 2, we extract (entries were rounded off to two decimal digits) \(p_L^* = .13\) at \(x_L = 25.96\) (or \(S_L = 448°K\)), \(p_M^* = .25\) at \(x_M = 28.36\) (or \(S_M = 410°K\)), and \(p_H^* = .62\) at \(x_H = 31.17\) (or \(S_L = 373°K\)). Then using Equation (9) one obtains \(V_{min}^* = 4.34/n\).

The Best Plan for Quadratic Extrapolation. The optimal plan for extrapolation at \(x_U\) under an assumed quadratic model \(\eta(x) = A + Bx + Cx^2\) can be obtained from Fedorov (1972, theorem 2.13.2). This plan has the following structure: \(p_L = .11\) at \(x_L = 25.96\) (or \(S_H = 448°K\)), \(p_H = .47\) at \(x_M = 27.83\) (or \(S_M = 418°K\)), \(p_H = .42\) at \(x_H = 31.17\) (or \(S_L = 373°K\)). For the simple linear regression model, this plan yields a variance of \(V[\hat{\eta}(x_U)]_{\{p_M^*\} = 6.15/n\}; see Equation (7). Then the REST efficiency of the plan, with respect to the minimum variance plan, is only about 56%; see Figure 3.

6.2 Step 2: Selection of Measurement Times

In this section, we derive the measurement times for the three plans (DO, ED, and EL) under the compromise plan for the stress levels and associated allocations described in Section 6.1.

Physical Constraints. The results of the experiment must be available to decide whether the new IC is satisfactory and can be sent to be mass manufactured by \(t_{max} = 2,000\) hours; \(t_{min}\) is set to 24 hours, which corresponds to one day after the devices are first put in the temperature oven. A fixed total number of measurements \(k = 9\) seemed reasonable because it corresponds to about one measurement a week per device.

Prior Information. The guess values \(\zeta^0\) needed for the planning are parameter estimates from the previous experiment of Carey and Koenig (1991).

DO, ED, and EL Plans. Table 3 shows the measurement times derived for the three plans. The ED and EL plans were obtained from Equations (12) and (13), respectively. The DO plans were obtained using the Simplex algorithm of Nelder and Mead (1965) to carry out the constrained maximization.

Table 3. Measurement-Time Plans for \(k = 9, t_{max} = 2,000\) Hours and \(t_{min} = 24\) Hours

<table>
<thead>
<tr>
<th>Temperature and guess values</th>
<th>DO</th>
<th>CD</th>
<th>EL</th>
</tr>
</thead>
<tbody>
<tr>
<td>(373°K)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\beta = 8.6 \times 10^{-4})</td>
<td>31, 31, 31, 24, 64, 129, 126, 219, 381, 24, 42, 73,</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\gamma = .5)</td>
<td>573, 573, 573, 228, 372, 581, 662, 1,324, 2,000, 662, 1,150, 2,000</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2,000, 2,000, 2,000</td>
<td>2,000, 2,000, 2,000</td>
<td></td>
</tr>
<tr>
<td>(410°K)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\beta = 1.1 \times 10^{-4})</td>
<td>41, 41, 41, 24, 62, 180, 24, 42, 73,</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\gamma = .5)</td>
<td>694, 694, 694, 323, 517, 772, 126, 219, 381,</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2,000, 2,000, 2,000</td>
<td>2,000, 2,000, 2,000</td>
<td></td>
</tr>
<tr>
<td>(448°K)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\beta = 7.3 \times 10^{-5})</td>
<td>42, 42, 42, 24, 85, 187, 24, 42, 73,</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\gamma = .5)</td>
<td>707, 707, 707, 335, 535, 793, 126, 219, 381,</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2,000, 2,000, 2,000</td>
<td>2,000, 2,000, 2,000</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1,117, 1,516, 2,000</td>
<td>662, 1,150, 2,000</td>
<td></td>
</tr>
</tbody>
</table>

NOTE: The DO plans are replicated at three distinct points and depend on \(\zeta^0 = (\beta, \gamma)\). The EL plans are invariant to choices of \(\zeta^0\). The ED plans depend on \(\zeta^0\).
Relative Efficiency. To illustrate the sensitivity of the relative efficiency, RETI, to the experiment length $t_{\text{max}}$ and to the number of measurement $k$, Table 4 shows RETI values for a prespecified set of values of $t_{\text{max}}$ and $k$. These efficiencies are fairly constant. Although we do not show the computations, the efficiencies are also fairly constant in relation to various choices of the prior values for the parameters. In this case the EL plan is the preferred heuristic plan because it has efficiencies of the same order as the ED plan, is prior-information independent, and is simple to calculate. Interestingly, the measurement times of the previous experiment were selected roughly according to the EL plan.

### 6.3 Determination of Sample Size and Precision

In Table 5 we give the total number of devices for several choices of $r$ and $p$. The results in this table were calculated for the compromise plan recommended in Section 6.2. The first part of the table shows the sample sizes under the assumption $v(x) = -2.565 + .1x$ given in Section 6.2, and the second part shows the sample sizes for the limiting case $v(x) = 0$. The limiting case leads to sample sizes about 20 times smaller than the other situation. This is explained by the small value of $\tau^2 = .014$, which indicates that the plateaus at a given level of stress are not very different from each other. Thus, if we can improve the estimation procedure of the plateaus by using a larger $t_{\text{max}}$ or by increasing $k$, we would need fewer devices in the ADT to reach a given precision level. In other words, if the hard constraints of $t_{\text{max}} = 2,000$ hours or $k = 9$ can be relaxed, we can reduce the number of devices in the ADT.

### 7. CONCLUSION

In this article, we propose a two-step approach for the design of reliability experiments in which devices are monitored under stress for a period of time. The approach is very intuitive, can be easily explained to engineers and physicists, and parallels the two-step analysis of the degradation given by Carey and Koenig (1991).

The proposal consists of treating the selection of stress levels and of measurement times separately but integrating the outcomes of these two steps when making a confidence statement about the precision of the results. We allow for optimization in the selection of the stress levels, in the selection of the times to measure the devices, and in the determination of the sample size. We do not, however, claim overall optimality for our approach because the objective is to provide practical guidelines in designing a useful and informative degradation experiment. Our methodology allows engineers to consider how all of the important variables in the problem interrelate in the design and evaluate the trade-off of some alternative plans.

An obvious generalization of this work is to extend it to other classes of ADT’s, which, for instance, would not have a degradation mechanism expected to level off after a period of time.

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APPENDIX: MAXIMUM LIKELIHOOD ESTIMATION FOR MODEL (2)

We consider Model (2), $y(t) = aF(t, \xi) + \epsilon(t)$, for the degradation of unit $j$ at stress $i$ at time $t$, where, to simplify the notation, we dropped the sub-indices $i, j$. Let $\mathcal{L}$ be the log-likelihood based on $t_1, \ldots, t_k$ independent observations. Then

$$\mathcal{L} = \frac{k}{2} \log(2\pi) - k \log(\sigma) - \frac{1}{2\sigma^2} \sum_{i=1}^k [y_i - aF(t_i, \xi)]^2.$$  

Straightforward computations give

$$\frac{\partial \mathcal{L}}{\partial \sigma} = \frac{1}{\sigma^2} \sum_{i=1}^k [y_i - aF(t_i, \xi)] F(t_i, \xi)$$

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{1}{\sigma^2} \sum_{i=1}^k [y_i - aF(t_i, \xi)] F(t_i, \xi)$$

$$\frac{\partial \mathcal{L}}{\partial \sigma^2} = -k + \frac{1}{\sigma^2} \sum_{i=1}^k [y_i - aF(t_i, \xi)]^2.$$  

To obtain the MLE for the parameters, we equate the derivatives in (A.1) to 0 and solve for the unknowns. It follows that the MLE $(\hat{a}, \hat{\xi})$ for $(a, \xi)$ are the simultaneous solution to

$$\sum_{i=1}^k [y_i - aF(t_i, \xi)] \frac{\partial F(t_i, \xi)}{\partial \xi} = 0$$

$$\alpha = \frac{\sum_{i=1}^k y_i F(t_i, \xi)}{\sum_{i=1}^k F(t_i, \xi)}.$$  

In general, these equations are solved numerically. After obtaining $\hat{a}, \hat{\xi}$, the MLE $\hat{\sigma}$ of $\sigma$ is obtained from the closed-form solution

$$\hat{\sigma} = \left(\frac{1}{k} \sum_{i=1}^k [y_i - aF(t_i, \xi)]^2\right)^{1/2}.$$  

An estimate of the covariance matrix of $(\hat{a}, \hat{\xi}, \hat{\sigma})$ is obtained from the Hessian matrix, say $H$, of $\mathcal{L}$ as follows. Let $\theta = (\alpha, \xi, \sigma)$, then

$$H = \begin{bmatrix}
\frac{\partial^2 \mathcal{L}}{\partial \alpha^2} & \frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \sigma} & \frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \xi} \\
\frac{\partial^2 \mathcal{L}}{\partial \sigma \partial \alpha} & \frac{\partial^2 \mathcal{L}}{\partial \sigma^2} & \frac{\partial^2 \mathcal{L}}{\partial \sigma \partial \xi} \\
\frac{\partial^2 \mathcal{L}}{\partial \xi \partial \alpha} & \frac{\partial^2 \mathcal{L}}{\partial \xi \partial \sigma} & \frac{\partial^2 \mathcal{L}}{\partial \xi^2}
\end{bmatrix}.$$  

where all the derivatives are evaluated at the MLE $\theta = (\hat{a}, \hat{\xi}, \hat{\sigma})$. It is easy to show that

$$\frac{\partial^2 \mathcal{L}}{\partial \alpha^2} = \frac{1}{\sigma^2} \sum_{i=1}^k \frac{\partial F(t_i, \xi)}{\partial \alpha} \frac{\partial r_i}{\partial \alpha} + \frac{1}{\sigma^2} \sum_{i=1}^k \frac{\partial F(t_i, \xi)}{\partial \sigma} \frac{\partial r_i}{\partial \sigma}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \sigma \partial \alpha} = 0$$

$$\frac{\partial^2 \mathcal{L}}{\partial \xi \partial \sigma} = \frac{2k}{\sigma^2} \sigma^2.$$  

where $O$ is a null matrix and $r_i = y_i - aF(t_i, \xi)$. Observe that

$$\frac{\partial r_i}{\partial \alpha} = -\frac{\partial}{\partial \alpha} [aF(t_i, \xi)]$$

$$\frac{\partial^2 r_i}{\partial \alpha^2} = -\frac{\partial^2}{\partial \alpha^2} [aF(t_i, \xi)].$$  

Then,

$$\mathcal{H}(\hat{\theta}, \hat{\sigma}) = H^{-1} = \hat{\sigma}^2 \begin{bmatrix} Y & O \end{bmatrix} \begin{bmatrix} O & 1 \\ 1 & 2k \end{bmatrix} (A.2)$$

with

$$Y = \begin{bmatrix} \sum_{i=1}^k \frac{\partial r_i}{\partial \alpha} \frac{\partial r_i}{\partial \alpha} + \sum_{i=1}^k \frac{\partial^2 r_i}{\partial \alpha^2} \end{bmatrix}^{-1} - \begin{bmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{bmatrix}.$$  

Thus $v_{11} = \text{var}(\hat{\alpha})$ and

$$\text{var}[\log(\hat{\alpha})] = \frac{\text{var}(\hat{\alpha})}{\hat{\sigma}^2} = \frac{v_{11}}{\hat{\sigma}^2}.$$  

Finally, observe that here var[log(\hat{\alpha})] refers to the conditional variance of log(\hat{\alpha}) for a stress $x_i$ and device $j$.

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