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Small-slope approximation for electromagnetic wave scattering at a rough interface of two dielectric half-spaces

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Abstract. The small-slope approximation (SSA) for wave scattering at the rough interface of two homogeneous half-spaces is developed. This method bridges the gap between two classical approaches to the problem: the method of small perturbations and the Kirchhoff (or quasi-classical) approximation. In contrast to these theories, the SSA is applicable irrespective of the wavelength of radiation, provided that the slopes of roughness are small compared with the angles of incidence and scattering.

The resulting expressions for the SSA are given for the entries of an S-matrix that represents the scattering amplitudes of plane waves of different polarizations interacting with the rough boundary. These formulae are quite general and are valid, in fact, for waves of different origins. Apart from the shape of the boundary, some functions in these formulae are coefficients of the expansion of the S-matrix into a power series in terms of elevations. These roughness-independent functions are determined by a specific scattering problem. In this paper they are calculated for the case of electromagnetic scattering at the interface of two dielectric half-spaces. In contrast to an earlier paper by the author, where only the formulae for the reflected field were presented, in this paper both reflected and transmitted fields are considered in detail.

The a priori symmetry relations that this scattering problem should obey (reciprocity and energy conservation) are formulated in terms of the S-matrix.

The statistical moments of scattering amplitudes are directly related to the mean-reflection coefficient and scattering cross sections, which are usually determined experimentally. The corresponding formulae are given here for the case of Gaussian space-homogeneous statistics of roughness.

1. Introduction

The problem of electromagnetic (EM) waves scattering at rough boundaries is of practical interest and has been addressed many times in different papers and monographs [1, 2]. Solutions to this problem usually used the method of small perturbations (MSP) [3, 4] or the Kirchhoff (or tangent plane or semiclassical) approximation (KA) [1, 2]. However, these two methods cannot cover the entire range of practical problems. Their combination, i.e. the two-scale model [5, 6], extends the potential of these calculations but there is a drawback: a more or less arbitrary parameter arises that divides roughness into two classes, large scale and small scale. For the continuous spectrum of roughness, this parameter cannot be uniquely determined and its choice can affect the results.

Therefore a method free of this disadvantage is desired. Some methods have been proposed to attain this. In particular, some methods that are more powerful than the MSP, e.g. phase perturbation-type methods, have been used [7, 8]. Bahar [9] considered this problem with the help of the full-wave approach (which could also be considered as a specific version of the reference waveguide method).
Recently, a new general approach to the problem of wave scattering at rough surfaces that is applicable to waves of an arbitrary nature was suggested; it is called the SSA [10]. This method bridges the gap between the two classical approaches: MSP and KA. It should be valid for arbitrary roughness having small slopes, irrespective of the wavelength of the radiation (a more accurate validity statement is given in section 7). The formulae for the SSA can be readily obtained with the help of some general procedures if the solution of the appropriate scattering problem is known in the framework of the conventional MSP.

EM scattering appears to be a relatively cumbersome problem and some care should be taken to present the final results in a clear form convenient for practical applications. A good possibility is to use an S-matrix that describes mutual transformations of plane waves of different polarizations interacting with the boundary. The statistical moments of the scattering amplitudes (SA) that constitute the S-matrix represent quantities that, in fact, are usually measured in experiments. Moreover, some a priori restrictions, such as the reciprocity relations or energy conservation (unitarity condition), can be easily formulated in terms of the S-matrix.

The final expressions for the SSA are quite universal and can be applied to waves of an arbitrary nature. They present explicit formulae for scattering amplitudes in integral form. Besides the shape of the roughness, there are some functions in these formulae which can be found within the limits of the conventional MSP method. These functions are, in fact, coefficients of the expansion of the S-matrix into the integral power series in terms of elevations.

The main purpose of this paper is to provide corresponding general expressions for the SA that arise in the SSA, and to determine the coefficient functions for the EM case. As a result, the theory presented describes the scattering of EM waves at rough boundaries that have small slopes. This situation is often encountered in practice and is of obvious interest. Both deterministic and statistical cases are considered.

The paper is organized as follows. Section 2 is devoted to the formulation of the problem and introduction of the S-matrix. In section 3 the reciprocity relations and unitarity condition are formulated and in section 4 the power expansion of the S-matrix up to the second order in terms of elevations is given. Section 5 presents a detailed description of the procedure for transforming formulae of the MSP into expressions of the SSA. The results of that section are, in fact, quite general and can be applied to waves of arbitrary origin. In section 6 the mean-reflection coefficients (first statistical moments of the SA) and scattering cross sections (appropriate second moments) are calculated for Gaussian statistics of roughness. Some conclusions are presented in section 7.

2. Introduction of the S-matrix

Let us choose the right Cartesian coordinate system with the z axis directed downward. Let the boundary \( z = h(r), \quad r = (x, y) \) separate two homogeneous half-spaces with permittivities \( \varepsilon_1 \) (upper half-space, \( z < 0 \)) and \( \varepsilon_2 \) (lower half-space, \( z > 0 \)). Suppose that the boundary is non-compact and located in some horizontal layer at about the level \( z = 0 \); i.e. it is in a sense plane on the average.

We shall consider waves of frequency \( \omega \) only and the factor exp\((-i\omega t)\) will be omitted in the following. Propagation of such waves in homogeneous dielectric media is described by the Maxwell equations

\[
\begin{align*}
\imath \omega H &= c \text{ rot } E \\
\imath \omega \varepsilon E &= -c \text{ rot } H.
\end{align*}
\]
Here $\epsilon$ is the permittivity of the media (which generally has complex values, i.e., $\text{Im}\ \epsilon \geq 0$), $H$ is the magnetic-field strength, $E$ is the electric-field strength and $c$ is the speed of light. Equations (2.1) have the following plane wave solutions

$$E_\alpha = e_\alpha^+(k) \exp(ikr + iq_k z)$$

$$\epsilon^{-1/2} H_\alpha = h_\alpha^+(k) \exp(ikr + iq_k z).$$

(2.2)

Here $\alpha = 1, 2$ is the index describing the vertical and horizontal polarizations of EM waves, respectively; $(k, q_k)$ are the horizontal and vertical projections of the wavevector and $q_k$ here is a generally complex function of the real vector $k$

$$q_k = (K^2 - k^2)^{1/2} \quad K = \epsilon^{1/2} \omega / c.$$  

(In what follows the notation $k^2 = k^2$ is assumed.) It is also assumed that the value of $q_k$ is in the first quadrant: $\text{Re} \ q_k \geq 0, \text{Im} \ q_k \geq 0$. To simplify the notation, let us set a convention that in all cases when the argument of the function $q$ is not indicated explicitly it is supposed to be equal to $k : q = q_k$. Analogously the value $q_0$ assumes the argument equal to $k_0 : q_0 = q_{k_0}$.

The vectors $e_\alpha^+(k), h_\alpha^+(k)$ depend on $k$ and are as follows:

$$e_1^+(k) = (k^2 N - q_k k) / (K k)$$

$$h_1^+(k) = -N \times k / k$$

and

$$e_2^+(k) = -h_1^+(k) \quad h_2^+(k) = e_1^+(k).$$

(2.3)

Here $k = |k|$, and $N = (0, 0, 1)$ is a positive unit vector along the $z$ axis (which is thus directed from the first into the second medium).

The superscript $+$ in (2.2) indicates that the given set describes plane waves travelling in the direction of positive $z$. An analogous set of plane waves propagating to negative $z$ is

$$E_\alpha = e_\alpha^-(k) \exp(ikr - i\bar{q}_k z)$$

$$\epsilon^{-1/2} H_\alpha = h_\alpha^-(k) \exp(ikr - i\bar{q}_k z)$$

(2.4)

with

$$e_1^-(k) = (k^2 N + q_k k) / (K k) \quad h_1^-(k) = -N \times k / k$$

and

$$e_2^-(k) = -h_1^-(k) \quad h_2^-(k) = e_1^-(k).$$

It is obvious that both the electric- and magnetic-field vectors in the plane waves are orthogonal to the wavevector:

$$e_\alpha^\pm(k) \cdot (k \pm q_k N) = h_\alpha^\pm(k) \cdot (k \pm q_k N) = 0.$$
Substituting (2.2) and (2.4) into (2.1), one can easily see that these expressions are really solutions of the Maxwell equations for any $k$.

To distinguish different values related to the first and second medium, we shall use superscripts (1) and (2), respectively; e.g., $e^{(1)}_a, q^{(2)}_k, \ldots$. We exclude the values $\varepsilon_1, \varepsilon_2$ and $K_1, K_2$, for which the number of the medium is indicated by a subscript.

Suppose that the incident field is a plane wave. Consider first the case when this wave travels in the positive $z$ direction, i.e. from the side of the first medium. Then, outside the layer $\min h(r) < z < \max h(r)$ the total field can be represented in the following form:

$$E = E_{in} + E_{sc} \quad H = H_{in} + H_{sc}$$  \hspace{1cm} (2.5)

with

$$E_{in} = E_{in}^{(1)} = e^{+}_{a_0}(k_0)(q^{(1)}_0)^{-1/2} \exp(i k_0 r + i q^{(1)}_0 z) \quad \text{at } z < \min h(r)$$  \hspace{1cm} (2.6)

and

$$H_{in} = H_{in}^{(1)} = h^{+}_{a_0}(k_0)(q^{(1)}_0)^{-1/2} \exp(i k_0 r + i q^{(1)}_0 z)$$  \hspace{1cm} (2.7)

with

$$E_{sc}^{(1)} = \sum_{\alpha} \int dk e^{-1/2}(k)q^{(1)}_k \exp(i k r - i q^{(1)}_k z) \cdot S^{11}_{a_0a_0}(k, k_0) \quad \text{at } z < \min h(r)$$

$$\varepsilon^{-1/2}_{\alpha} H_{sc}^{(1)} = \sum_{\alpha} \int dk h^{-1/2}(k)q^{(1)}_k \exp(i k r - i q^{(1)}_k z) \cdot S^{11}_{a_0a_0}(k, k_0)$$

$$E_{sc}^{(2)} = \sum_{\alpha} \int dk e^{+2}(k)q^{(2)}_k \exp(i k r + i q^{(2)}_k z) \cdot S_{a_0a_0}^{21}(k, k_0) \quad \text{at } z > \max h(r)$$

$$\varepsilon^{-1/2}_{\alpha} H_{sc}^{(2)} = \sum_{\alpha} \int dk h^{+2}(k)q^{(2)}_k \exp(i k r + i q^{(2)}_k z) \cdot S_{a_0a_0}^{21}(k, k_0)$$

The factors $(q^{(N)}_0)^{-1/2}, (q^{(N)}_k)^{-1/2}$, that are introduced into (2.6) and (2.7), make the resulting formulae more symmetric. This normalization of the amplitudes of the plane waves corresponds to the unit energy flux in the $z$ direction.

To obtain analogous expressions that describe the scattered field for incident waves from the side of the second medium, one can formally exchange superscripts in (2.6) and (2.7), i.e. (1) $\leftrightarrow$ (2), and the sign indexes $+/-$, i.e. $(e^{+}_a, h^{+}_a) \leftrightarrow (e^{-}_a, h^{-}_a)$, and change the sign of a vertical coordinate $z$.

The value $S_{a_0a_0}^{N,N}(k, k_0)$, where $N, N_0 = 1, 2$ and $\alpha, \alpha_0 = 1, 2$ represents the $\text{SA}$ of the incident plane wave of polarization $\alpha_0$ with the horizontal projection of the wavevector equal to $k_0$ propagating in the medium $N_0$ to the plane wave of polarization $\alpha$ and with the horizontal projection of the wavevector equal to $k$ propagating away from the boundary in the medium $N$.

Expressions (2.6) and (2.7), with arbitrary $S$, obviously represent general solutions of the Maxwell equations with the scattered field satisfying radiation conditions. To determine the $\text{SA}$, one must use boundary conditions that in our case correspond to the continuity of the tangential components of the electric and magnetic fields at the boundary

$$n \times E^{(1)} = n \times E^{(2)} \quad z = h(r)$$

$$n \times H^{(1)} = n \times H^{(2)} \quad z = h(r)$$

$$n = N - \nabla h.$$  \hspace{1cm} (2.8) (2.9)
Here \( \nabla = (\partial_x, \partial_y) \) is a horizontal gradient and \( n \) is a normal vector (not a unit one) to the boundary \( z = h(r) \).

The set of 16 quantities \( S^{NN_0}_{\alpha\alpha_0} \) can be naturally represented as the following 4×4 matrix with a block structure

\[
\hat{S}(k, k_0) = \begin{pmatrix}
\hat{S}_{11}^{N_0} & \hat{S}_{12}^{N_0} \\
\hat{S}_{21}^{N_0} & \hat{S}_{22}^{N_0}
\end{pmatrix}(k, k_0).
\]

(2.10)

Each block \( \hat{S}^{N_0}_{\alpha\alpha_0} \) of this matrix is, in turn, a 2×2 matrix related to indexes \( \alpha, \alpha_0 = 1, 2 \)

\[
\hat{S}^{N_0}_{\alpha\alpha_0}(k, k_0) = \begin{pmatrix}
S_{11}^{N_0} & S_{12}^{N_0} \\
S_{21}^{N_0} & S_{22}^{N_0}
\end{pmatrix}(k, k_0)
\]

(2.11)

where the hat (\(^\hat{\cdot}\)) indicates matrix quantities. The set \( S^{N_0}_{\alpha\alpha_0}(k, k_0) \), considered for all values of the arguments \((N, \alpha, k)\) and \((N_0, \alpha_0, k_0)\), constitutes the \( S \)-matrix. However, we shall often also refer to the 4×4 matrix \( \hat{S}(k, k_0) \) with the fixed arguments \( k \) and \( k_0 \) as the \( S \)-matrix.

3. Reciprocity relations and unitarity of scattering

The matrix \( \hat{S}(k, k_0) \) irrespective of the form of the boundary should a priori obey two restrictions. The first is the reciprocity relation which is valid for arbitrary complex \( \epsilon \), and the second is unitarity of scattering (i.e. energy conservation), which is valid for real \( \epsilon \).

Let \((E, H)\) and \((\hat{E}, \hat{H})\) represent two arbitrary solutions of the Maxwell equations in the first medium. Applying Gauss’s theorem to the vector field \((E \times H - \hat{E} \times \hat{H})\) and to the area that is enclosed between the boundary \( z = h(r) \) and some plane \( z = z_1, z_1 < \min h(r) \), we have

\[
\int_{z=h(r)} n(E \times H - \hat{E} \times \hat{H}) \, dr - \int_{z=z_1 < h(r)} N(E \times H - \hat{E} \times \hat{H}) \, dr = \int \text{div}(E \times H - \hat{E} \times \hat{H}) \, dr \, dz
\]

(3.1)

where the normal vector \( n \) is given by (2.9). One can easily check that, as a consequence of the well-known identity

\[
\text{div}(E \times H) = H \text{rot} E - E \text{rot} H
\]

and equations (2.1), the right-hand side of equation (3.1) becomes zero. Hence

\[
\int_{z=z_1 < h(r)} N(E \times H - \hat{E} \times \hat{H}) \, dr = \int_{z=h(r)} n(E_r \times H_r - \hat{E}_r \times \hat{H}_r) \, dr.
\]

(3.2)

The values of the electric and magnetic field at the boundary have been replaced in (3.2) by the corresponding tangential components \( E_r \) and \( H_r \).
The same relation (3.2) is also valid for the second medium. Hence, because of the boundary conditions (2.8), \( E_t^{(1)} = E_t^{(2)} \), \( H_t^{(1)} = H_t^{(2)} \) and we find that

\[
\int_{z=z_1<h(r)} N(1)\left( E \times H^{(1)} - E^{(1)} \times H \right) dr = \int_{z=z_2>h(r)} N(2)\left( E \times H^{(2)} - E^{(2)} \times H \right) dr.
\] (3.3)

To obtain the reciprocity relations one should substitute into (3.3), in place of \((E, H)\) and \((E, H)\), the solutions of the form (2.5) with \(k_0 = k_1\) and \(k_0 = k_2\), respectively, and analogous representations of the fields for incident waves from the side of the second half-space. Simple calculations give the following result:

\[
\begin{align*}
\hat{\sigma}_3 \hat{S}^{11}(k_1, k_2) &= \hat{S}^{11T}(-k_2, -k_1)\hat{\sigma}_3 \\
\hat{\sigma}_3 \hat{S}^{22}(k_1, k_2) &= \hat{S}^{22T}(-k_2, -k_1)\hat{\sigma}_3 \\
\hat{\sigma}_3 \hat{S}^{12}(k_1, k_2) &= \hat{S}^{12T}(-k_2, -k_1)\hat{\sigma}_3.
\end{align*}
\] (3.4)

Here the superscript \(T\) means the transpose of the matrix and \(\hat{\sigma}_3\) is one of the Pauli matrices that we shall also use further:

\[
\hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (3.5)

Equations (3.4) can also be written as

\[
\begin{align*}
\hat{S}^{NM}_{11}(k_1, k_2) &= \hat{S}^{MN}_{11}(-k_2, -k_1) \\
\hat{S}^{NM}_{22}(k_1, k_2) &= \hat{S}^{MN}_{22}(-k_2, -k_1) \\
\hat{S}^{NM}_{12}(k_1, k_2) &= -\hat{S}^{MN}_{21}(-k_2, -k_1)
\end{align*}
\]

where \(N, M = 1, 2\) are indexes describing the numbers of the media.

The calculations can be facilitated by using formulae given in appendix A. In terms of the \(4 \times 4\) matrix \(\hat{S}\), (3.4) can be represented as

\[
\begin{pmatrix} \hat{\sigma}_3 & 0 \\ 0 & \hat{\sigma}_3 \end{pmatrix} \cdot \hat{S}(k_1, k_2) = \hat{S}^{T}(-k_2, -k_1) \begin{pmatrix} \hat{\sigma}_3 & 0 \\ 0 & \hat{\sigma}_3 \end{pmatrix}.
\] (3.6)

The relation (3.6) represents the reciprocity theorem formulated in terms of the \(S\)-matrix.

From the Maxwell equations (2.1) with real \(\epsilon\) and boundary conditions (2.8) it follows that if \((\hat{E}, \hat{H})\) is a solution of the given boundary problem, then \((\hat{E}^*, \hat{H}^*)\), where \(\ast\) denotes the complex conjugate, also represents some solution (a 'time-reversed' one). Hence, along with (3.3) the following relation holds:

\[
\int_{z=z_1<h(r)} N(1)\left( E \times H^{(1)*} + E^{(1)*} \times H \right) dr = \int_{z=z_2>h(r)} N(2)\left( E \times H^{(2)*} + E^{(2)*} \times H \right) dr.
\] (3.7)
The same set of solutions that was used to obtain the reciprocity relations should again be substituted into (3.7). With the help of the relations presented in appendix A, one can derive from (3.7) the following equation:

$$\int dk' \hat{S}^{+}(k', k_1) \cdot \hat{S}(k', k_2) = \delta(k_1 - k_2). \quad (3.8)$$

Here the superscript $+$ designates Hermitian conjugation of the $4 \times 4$ matrix $\hat{S}$. The superscript $(h)$ on the integral sign means integration over only those $k'$ that correspond to homogeneous waves: $k' < K_{N' \nu}$. It is assumed in (3.8) that waves $k_1, k_2$ are also homogeneous. In terms of $2 \times 2$ matrices, the relation (3.8) can be represented as

$$\int_{k < K_1} dk' \hat{S}^{11+}(k', k_1) \cdot \hat{S}^{11}(k', k_2) + \int_{k' < K_2} dk' \hat{S}^{21+}(k', k_1) \cdot \hat{S}^{21}(k', k_2) = \delta(k_1 - k_2)$$

with $k_1 < K_1$, $k_2 < K_1$ and

$$\int_{k' < K_1} dk' \hat{S}^{11+}(k', k_1) \cdot \hat{S}^{12}(k', k_2) + \int_{k' < K_2} dk' \hat{S}^{21+}(k', k_1) \cdot \hat{S}^{22}(k', k_2) = 0$$

with $k_1 < K_1$, $k_2 < K_2$ etc.

The reciprocity relations (3.6) and the unitarity condition (3.8) each represent 16 scalar relations.

4. Power series expansion for the $S$-matrix

For small enough elevations, the power series expansion of the $S$-matrix can be of interest. Generally, it can be represented in the following form:

$$S_{a0}^{N_0}(k, k_0) = V_{a0}^{N_0}(k, k_0) \cdot \delta(k - k_0) + 2i(q^{(N)}_k q^{(N_0)}_{k_0})^{1/2} B_{a0}^{N_0}(k, k_0) h(k - k_0)$$

$$+ (q^{(N)}_k q^{(N_0)}_{k_0})^{1/2} \sum_{m=1}^{\infty} \int (B_{m+1})^{N_0}_{a0}(k, k_0; \xi_1, \ldots, \xi_m)$$

$$\times h(k - \xi_1) \ldots h(\xi_m - k_0) \, d\xi_1 \ldots d\xi_m. \quad (4.1)$$

Here

$$h(k) = \int e^{-ikr} h(r) \, \frac{dr}{(2\pi)^2}$$

is the Fourier transform of the roughness shape. We will not introduce special notation for the Fourier components of the elevations because their arguments will always be designated by $k, k', k_1, \ldots$ or $\xi, \xi', \xi_1, \ldots$ in contrast to arguments of the elevations themselves: $r, r', r_1, \ldots$.

The first term in (4.1) describes specular reflected or refracted plane waves that possess the same horizontal component of the wavevector as the incident wave. The matrix $\hat{V}$ consists of the corresponding Fresnel coefficients. The factor $(q^{(N)}_k q^{(N_0)}_{k_0})^{1/2}$ is separated in (4.1) for convenience; the matrix $\hat{B}$ is dimensionless. Matrices $\hat{B}, \hat{B}_2, \hat{B}_3, \ldots$ in (4.1) are
coefficients of the power expansion. They do not depend on elevations and are functions of the horizontal components of wavevectors and permittivities $\varepsilon_1$ and $\varepsilon_2$.

The structure of the arguments of the functions $h(k)$ in (4.1) is related to the transformational property of the SA that if one shifts the boundary $z = h(r)$ in the horizontal direction by the vector $d$ then, as easily follows from geometrical arguments, the SA is subjected to the following transformation:

$$h(r) \rightarrow h(r - d) \quad \text{then} \quad \hat{S}(k, k_0) \rightarrow \hat{S}(k, k_0) \cdot e^{-i(k - k_0) \cdot d}. \quad (4.2)$$

Because a horizontal shift of $h(r)$ results in a multiplication of the Fourier components $h(k)$ by $\exp(-ikd)$, then the structure of the arguments of functions $h(k)$ in (4.1) ensures the transformational property (4.2).

To determine $V$, $B$, $B_2$, ..., one should apply to the given boundary-problem perturbative analysis. A straightforward approach proposes using exact-integral equations for the boundary values of the fields and then obtaining approximate solutions of these equations. This approach requires tedious calculations. A far more simple approach is the following. Let us calculate the fields at the boundary using representations (2.5)–(2.7) without paying attention to the restrictions $z > \max h(r)$, $z < \min h(r)$ and then substitute the result into the boundary conditions (2.8). After that we shall seek the solution of the resulting formal equations for $S_{\text{CM}}^{N_6}(k, k_0)$ in the form of the power expansion (4.1). All exponentials containing elevations $h(r)$ should also be replaced by corresponding power expansions. By collecting $O(1)$, $O(h)$, $O(h^2)$, ... terms we consequently determine $\hat{V}$, $\hat{B}$, $\hat{B}_2$, ...

This approach corresponds exactly to the procedure that was first used by Rayleigh [11] for the scalar problem and then by Rice [3] for the EM case. One important remark should be made concerning this procedure. The possibility of calculating the fields at the boundary with the help of representations of the form (2.5)–(2.7) is usually called the Rayleigh hypothesis. It is very well known that this hypothesis is generally wrong. On this basis, it is sometimes suggested that the procedure outlined above for determining the power expansion of $\hat{S}$ can lead to erroneous results. However, this is not the case. The statement that the Rayleigh hypothesis does not hold means only that if one uses exact values of SA in the representations (2.5)–(2.7) then the resulting integrals over $k$ could appear to be divergent for some $z > \min h(r)$, but because we take into account only the final-order terms with respect to $h$, both in $\hat{S}$ and in $\exp(\pm ih\theta)$, we never obtain divergent integrals. We consider the representation (4.1) as an asymptotic expansion only and not as a convergent series (although this series is, in fact, probably convergent for small enough $h$).

The formal justification of the procedure outlined above is as follows. Consider a set of elevations of the form $z = ah(r)$ with $h(r)$ being an analytic function and $a$ some parameter. It was established [12], that for small enough $a$, the integrals in (2.5)–(2.7) are convergent and provide analytic continuation of the field from the area $z < \min h(r)$, $z > \max h(r)$ into the area $\min h(r) < z < \max h(r)$. Thus, for such elevations the procedure described above is completely correct. Because the coefficients of the asymptotic expansion (4.1) $\hat{V}$, $\hat{B}$, $\hat{B}_2$, ... are uniquely determined, then any other "fair" methods of calculating these coefficients must give the same results.

Because the statement cited above is very important for establishing the validity of the Raleigh–Rice procedure (RRP) for calculating expansion (4.1), the essential steps in the corresponding proof are reproduced in appendix C.

Another substantiation of the validity of the RRP is as follows. Let us use in expansion (4.1), which was calculated according to the RRP, some finite number of terms $M$. Assume
that the elevations \( h(r) \) are smooth enough and that the spectrum \( h(k) \) tends to zero at \( k \to \infty \) faster than the exponential does (e.g. \(|h(k)| < C \exp(-\gamma k^2), |k| \to \infty\). Substituting this final sum (instead of \( S \)) into (2.7) provides convergent integrals for any \( z \) and for \( z = h(r) \) in particular. Substituting scattered fields calculated in this way into the boundary conditions (2.8) results in a \( O(h^{M+1}) \) mismatch (because (4.1) was calculated according to the RRP exactly from this requirement). This mismatch is equivalent to some external sources which are situated at the boundary and have intensities proportional to \( O(h^{M+1}) \). Thus, the approximate solution to the boundary problem differs from the exact one at any point by \( O(h^{M+1}) \). As a result, we have shown that (4.1) (calculated according to the RRP) provides an asymptotic expansion of scattering amplitude. This asymptotic expansion generally should not, of course, represent a convergent series with \( h \) fixed as \( M \to \infty \).

The proposed justification of the procedure for calculating the power expansion of the SA could appear somewhat speculative. Thus, a direct comparison between two power expansions of the form (4.1) was made in [13] for the scalar Dirichlet problem. The first expression was obtained with the help of the Rayleigh hypothesis as described above and the second with the help of perturbative analysis applied to the exact-integral equation for the surface sources of the field. The two expansions were in agreement in all orders with respect to \( h \), as expected. Analogous observations for the EM case for lower-order terms were made in particular in [14,15].

Now we will proceed to the calculations. For an incident wave from the first medium, substitution of the representations (2.6) and (2.7) into the boundary conditions (2.8) gives

\[
(N - \nabla) \times e^{(1)}_{\alpha_0} (k_0) \hat{q}_0^{(1)} - \frac{1}{2} \exp(ik_0 r + i\hat{q}_0^{(1)} h(r)) + \sum_\alpha \int dk \exp(ikr)(N - \nabla h) \\
\times (e^{-1}_{\alpha_1} (k) \hat{q}_k^{(1)} - \frac{1}{2} \exp(-i\hat{q}_k^{(1)} h(r)) \times S_{\alpha_0 \alpha_1}^{(1)} (k, k_0) - e^{+2}_{\alpha_1} (k) \hat{q}_k^{(2)} - \frac{1}{2} \\
\times \exp(i\hat{q}_k^{(2)} h(r)) \cdot S_{\alpha_0 \alpha_1}^{(2)} (k, k_0) = 0.
\]

\[
K_1 (N - \nabla) \times h_{\alpha_0}^{(1)} (k_0) \hat{q}_0^{(1)} - \frac{1}{2} \exp(ik_0 r + i\hat{q}_0^{(1)} h(r)) + \sum_\alpha \int dk \exp(ikr) (N - \nabla h) \\
\times (K_1 h_{\alpha_1}^{(1)} (k) \hat{q}_k^{(1)} - \frac{1}{2} \exp(-i\hat{q}_k^{(1)} h(r)) \times S_{\alpha_0 \alpha_1}^{(1)} (k, k_0) - K_2 h_{\alpha_1}^{+2} (k) \hat{q}_k^{(2)} - \frac{1}{2} \\
\times \exp(i\hat{q}_k^{(2)} h(r)) \cdot S_{\alpha_0 \alpha_1}^{(2)} (k, k_0) = 0.
\]

(Note that it is inexplicitly assumed in (4.3) that all exponentials of the form \( \exp(\pm i\hat{q} h) \) are replaced by the corresponding power expansions to some final order, and hence no divergent integrals arise.) An analogous set of equations could be written for an incident wave from the second medium. One can easily check that the latter equations result from (4.3) after the formal replacements

\[
N \to -N, h \to -h \text{ and exchange of medium indexes (1) } \leftrightarrow \text{ (2).}
\]

This avoids repeating the calculations for this case and can be performed (4.4) in the resulting expressions for SA.

Let us consider first the zero-order term. For the plane boundary \( h(r) = 0 \)

\[
\hat{S} = \hat{V} \delta (k - k_0)
\]
where \( \hat{V} \) is a \( 4 \times 4 \) matrix. Let us assume an incident wave onto the boundary from the first medium. Then the boundary conditions (2.8) become

\[
N \times \left( e^{+1}_{\alpha_0}(k_0)q_0^{(1)-1/2} + \sum_{\alpha} e^{-(1)}_{\alpha}(k_0) V_{a_{\alpha_0}}^{11} q_0^{(1)-1/2} \right) = N \times \sum_{\alpha} e^{+2}_{\alpha}(k_0) V_{a_{\alpha_0}}^{21} q_0^{(2)-1/2} + \sum_{\alpha} h^{+1}_{\alpha}(k_0) V_{a_{\alpha_0}}^{11} q_0^{(1)-1/2}
\]

\[
e_{1}^{1/2} N \times \left( h^{+1}_{\alpha_0}(k_0)q_0^{(1)-1/2} + \sum_{\alpha} h^{-(1)}_{\alpha}(k_0) V_{a_{\alpha_0}}^{11} q_0^{(1)-1/2} \right)
\]

\[
= e_{2}^{1/2} N \times \sum_{\alpha} h^{+2}_{\alpha}(k_0) V_{a_{\alpha_0}}^{21} q_0^{(2)-1/2}.
\]

(4.5)

Let \( \alpha_0 = 1 \). Projecting equations (4.5) on the vectors \( N \times k_0/k_0 \) and \( k_0/k_0 \), we find

\[
\begin{align*}
q_0^{(1)1/2} K_1^{-1} (1 - V_{11}^{11}) &= q_0^{(2)1/2} K_2^{-1} V_{11}^{21} \\
q_0^{(1)-1/2} V_{21}^{11} &= q_0^{(2)-1/2} V_{21}^{21} \\
q_0^{(1)1/2} V_{21}^{11} &= -q_0^{(2)1/2} V_{21}^{21} \\
\epsilon_1^{1/2} q_0^{(1)-1/2} (1 + V_{11}^{11}) &= \epsilon_2^{1/2} q_0^{(2)-1/2} V_{11}^{21}.
\end{align*}
\]

(4.6)

Equations (4.6) split into two pairs of independent equations, as shown, and can easily be solved. The result is

\[
\begin{align*}
\epsilon_2 q^{(1)} - \epsilon_1 q^{(2)} \end{align*}
\]

\[
= \epsilon_2 q^{(1)} + \epsilon_1 q^{(2)}
\]

\[
V_{11}^{11} = a = \frac{(\epsilon_2 q^{(1)} - \epsilon_1 q^{(2)})}{(\epsilon_2 q^{(1)} + \epsilon_1 q^{(2)})}
\]

\[
V_{11}^{21} = c = \frac{2(\epsilon_2 q^{(1)} q^{(2)})}{(\epsilon_2 q^{(1)} + \epsilon_1 q^{(2)})}
\]

(4.7)

If the dielectric permittivity of one of the media takes negative values then the denominators in formulae (4.7) can vanish. This corresponds to the surface wave, i.e. the wave propagating horizontally and exponentially decaying at \( |z| \to \infty \). In this case the scheme of solving the problem should be modified and this surface wave should be considered from the very beginning. Here we shall not analyse this situation and shall assume the absence of the surface wave.

The case \( \alpha_0 = 2 \) is examined quite analogously; equations (4.6) are replaced by the following relations:

\[
\begin{align*}
q_0^{(1)1/2} K_1^{-1} V_{12}^{11} &= q_0^{(2)1/2} K_2^{-1} V_{12}^{21} \\
q_0^{(1)-1/2} (1 + V_{22}^{11}) &= q_0^{(2)-1/2} V_{22}^{21} \\
q_0^{(1)1/2} (1 - V_{22}^{11}) &= q_0^{(2)1/2} V_{22}^{21} \\
\epsilon_1^{1/2} q_0^{(1)-1/2} V_{12}^{11} &= \epsilon_2^{1/2} q_0^{(2)-1/2} V_{12}^{21}.
\end{align*}
\]

(4.8)

The solutions of these equations are

\[
\begin{align*}
V_{22}^{11} &= b = (q^{(1)} - q^{(2)})/(q^{(1)} + q^{(2)}) \\
V_{22}^{21} &= d = 2(q^{(1)} q^{(2)})^{1/2}/(q^{(1)} + q^{(2)}) \\
V_{12}^{11} &= V_{12}^{21} = 0.
\end{align*}
\]
The results for the case of an incident primary wave from the second medium can be obtained with the help of the substitution (4.4). Hence for matrix $\hat{V}$ in (4.1) we have

$$\hat{V} = \begin{pmatrix} a & 0 & c & 0 \\ 0 & b & 0 & d \\ c & 0 & -a & 0 \\ 0 & d & 0 & -b \end{pmatrix}$$ (4.9)

Because, as is easily seen, $a^2 + c^2 = b^2 + d^2 = 1$, matrix $\hat{V}$ is unitary for real $\epsilon$. Because $\hat{V}(k) = \hat{V}^T(k) = \hat{V}(-k)$ the reciprocity theorem (3.6) is satisfied.

Note that in calculating $\hat{V}$ we made use of the horizontal components of (4.3) only, whereas projections of (4.3) on $N$ were equal to zero. This situation is common for all orders of perturbative analysis: the vertical projections of (4.3) to the $n$th order with respect to $h$ are satisfied identically as a consequence of determining the expressions for the SA up to the $(n - 1)$th order.

This observation allows us to establish the following interesting formal relationship for the $S$-matrix (the proof is given in appendix B)

$$\hat{S}_h^{-1} = \begin{pmatrix} \delta_3 & 0 \\ 0 & \delta_3 \end{pmatrix} \hat{S}_{-h} \begin{pmatrix} \delta_3 & 0 \\ 0 & \delta_3 \end{pmatrix}$$ (4.10)

Here $\hat{S}_h$ is the $4 \times 4$ scattering matrix (2.10) calculated for elevations $z = h(r)$, and $\hat{S}_{-h}$ is an analogous matrix calculated for elevations of the form $z = -h(r)$. The superscript $-1$ refers to inversion of the complete scattering matrix $\hat{S}_h$ which includes dependence on arguments $(k, k_0)$. Thus $\hat{S}_h^{-1}$ in (4.10) is defined by the equation

$$\hat{S}_h^{-1} \cdot \hat{S} = \sum_{a',N'} \int dk' \langle S_{a'd'}^{-1}N'N'_{a'd'}(k', k') \cdot S_{a'd'00}(k', k_0) = \delta_{NN_0} \delta_{aa_0} \delta(k - k_0).$$ (4.11)

Substituting into (4.10) the representation (4.1) for $\hat{S}$, one can obtain some relations that the matrices $\hat{B}$, $\hat{B}_2$, ... should obey. Considering, for instance, the $O(1)$ and $O(h)$ terms, we get

$$\begin{pmatrix} \delta_3 & 0 \\ 0 & \delta_3 \end{pmatrix} \hat{V} = \hat{V} \begin{pmatrix} \delta_3 & 0 \\ 0 & \delta_3 \end{pmatrix}$$

$$\begin{pmatrix} \delta_3 & 0 \\ 0 & \delta_3 \end{pmatrix} \hat{V} \hat{B} = \hat{B} \hat{V} \begin{pmatrix} \delta_3 & 0 \\ 0 & \delta_3 \end{pmatrix}$$ (4.12)

To calculate $\hat{V}(k, k_0)$, one must consider in (4.3) first-order terms in $h$. Taking into account relations (4.6) and (4.8) greatly simplifies the appropriate calculations. As a result, equations (4.6) are replaced by

$$\frac{N \cdot k \times k_0}{kk_0} q_0 (2\pi)^2 K_2 q_0^{(2)} \kappa_1 V_{21}^{(1)}(k_0) + 2 q_0^{(1)} q_0^{(1)/2} B_{21}^{(1)}(k, k_0) = -2 q_0^{(2)} q_0^{(1)/2} B_{21}^{(1)}(k, k_0)$$

$$\frac{k \cdot k_0}{kk_0} (2\pi)^2 K_2 q_0^{(2)} \kappa_1 V_{11}^{(1)}(k_0) + 2 K_1 q_0^{(1)/2} B_{11}^{(1)}(k, k_0) = 2 K_2 q_0^{(1)/2} B_{11}^{(1)}(k, k_0)$$

$$B_{21}^{(1)}(k, k_0) = B_{21}^{(1)}(k, k_0)$$

$$kk_0 q_0^{(2)} q_0^{(1)/2} \kappa_1 V_{21}^{(1)}(k_0) + 2 K_2 q_0^{(1)/2} B_{21}^{(1)}(k, k_0) = -2 K_1 q_0^{(2)} q_0^{(1)/2} B_{21}^{(1)}(k, k_0)$$ (4.13)
and equations (4.8) are replaced by

\[
K_1^{-1} q^{(1)} B_{12}^{(1)} (k, k_0) = -K_2^{-1} q^{(2)} B_{12}^{(2)} (k, k_0)
\]

\[
B_{22}^{(1)} (k, k_0) = B_{22}^{(2)} (k, k_0)
\]

(4.14)

\[
\begin{align*}
\frac{N \cdot k \times k_0}{k k_0} q_0 (2) - 1/2 (K^2 - k^2) V_{22}^{(k)} (k_0) + 2 K_1 q_0 (1) 1/2 B_{12}^{(1)} (k, k_0) = 2 K_2 q_0 (1) 1/2 B_{12}^{(2)} (k, k_0) \\
- \frac{k \cdot k_0}{k_0} q_0 (2) - 1/2 (K^2 - k^2) V_{22}^{(k)} (k_0) + 2 q_0 (1) 1/2 B_{22}^{(1)} (k, k_0) = -2 q_0 (1) 1/2 B_{22}^{(2)} (k, k_0).
\end{align*}
\]

It can be immediately checked that the vertical components of (4.3) in the first order are satisfied identically.

Both (4.13) and (4.14) split into pairs of independent equations: the first and the third, the second and the fourth. All these equations are easily solved. To make resulting formulae more compact, let us introduce the following notations:

\[
\begin{align*}
\hat{d}_{1\alpha}^N (k) &= (e_2 - e_1)^{1/2} \\
&= \begin{cases} \\
\frac{e_1^{1/2} q_k (2)}{e_2 q_k (1) + e_1 q_k (2)} & \text{if } \alpha = 1, N = 1 \\
\frac{e_2^{1/2} q_k (1)}{e_2 q_k (1) + e_1 q_k (2)} & \text{if } \alpha = 1, N = 2 \\
\frac{e_2^{1/2} q_k (1)}{\omega/c} & \text{if } \alpha = 2, N = 1, 2.
\end{cases}
\end{align*}
\]

This set can be considered the diagonal matrix:

\[
\hat{d}_1 (k) = (e_2 - e_1)^{1/2} \text{diag} \left( \frac{e_1^{1/2} q_k (2)}{e_2 q_k (1) + e_1 q_k (2)}, \frac{\omega/c}{q_k (1) + q_k (2)}, \frac{e_2^{1/2} q_k (1)}{e_2 q_k (1) + e_1 q_k (2)}, \frac{\omega/c}{q_k (1) + q_k (2)} \right)
\]

\[
(N, \alpha) = (1, 1) \quad (1, 2) \quad (2, 1) \quad (2, 2)
\]

(the lower line here shows the ordering of the indexes that correspond to the representation of the 4 x 4 matrices in the form of (2.10) and (2.11). Further sets are

\[
\hat{d}_2 (k) = \text{diag} \left( e_2 \frac{k}{q_k (2)} , 0 , -e_1 \frac{k}{q_k (1)} , 0 \right)
\]

\[
\hat{d}_3 (k) = \text{diag} \left( -e_2 \frac{\omega}{q_k (2)} , \omega/c , e_1 \frac{\omega}{q_k (1)} , -\omega/c \right)
\]

\[
\hat{Q}_k^{1/2} = \text{diag} (q_k (1)^{1/2} , q_k (1)^{1/2} , q_k (2)^{1/2} , q_k (2)^{1/2}).
\]

(The matrices \( \hat{d}_3 (k) \) and \( \hat{Q}_k^{1/2} \) will be used in the expression for \( \hat{B}_2 \).) With the help of the introduced notation, Bragg's kernel \( \hat{B} (k, k_0) \) in (4.1) can be written in the following form:

\[
B_{aaa}^{N0} (k, k_0) = \hat{d}_{1\alpha}^N (k) \times \frac{N \cdot k \times k_0}{k k_0} (\hat{d}_3)_{aaa} + \frac{N \cdot k \times k_0}{k k_0} (\hat{d}_1)_{aaa} \\
- \frac{1}{e_1 e_2} \hat{d}_{2\alpha}^N (k) \cdot \frac{1}{2} (\hat{d}_0 + \hat{d}_3)_{aaa} \cdot d_{2\alpha}^N (k_0) \}
\]

(4.15)
This formula can obviously be represented in the matrix notations if one introduces the following $4 \times 4$ matrices

$$\hat{\sigma}_j = \begin{pmatrix} \hat{\sigma}_j & \tilde{\sigma}_j \\ \tilde{\sigma}_j & \tilde{\sigma}_j \end{pmatrix}$$

where $\hat{\sigma}_j$ are $2 \times 2$ Pauli matrices (see (3.5)) and underlined $\tilde{\sigma}_j$ refers to $4 \times 4$ matrices. One can easily check that equations (4.12) really hold for expressions (4.9) and (4.15).

Let roughness have the form $\hat{h}(r) = H = \text{constant}$. In this case scattering reduces to specular reflection and refraction occurring without change of the horizontal component of the wavevector: $\hat{k} = \hat{k}_0$. Let $H \to 0$. Then using the transformational property of $\hat{S}_A$ under vertical shifts (see (5.3) below), we obtain the equation that enables us to express $\hat{\nu}$ by means of $\hat{B}$

$$\hat{\nu}^{MN}(k) = -\frac{2(q^{(N)}q^{(N)})^{1/2}}{(-1)^Nq^{(N)} + (-1)^N_0q^{(N)}} \hat{B}^{N_0}(k, k). \quad (4.16)$$

The validity of (4.16) can be checked by direct calculation as well.

To calculate $\hat{B}_2$, it is necessary to perform one more iteration of equations (4.3). Appropriate calculations do not present any great difficulties and require only patience and carefulness. The corresponding formula for the $4 \times 4$ matrix $\hat{B}_2$ can be represented as follows.

$$\hat{B}_2(k, k_0; \xi) = -2\hat{B}(k, \xi) \hat{Q}^{1/2}_N \hat{\nu} \hat{Q}^{1/2}_N \hat{B}(\xi, k_0) + \hat{\delta}_1(k) \cdot \left\{ \begin{array}{l} \frac{2q^{(1)}_N + q^{(2)}_N}{\epsilon_2q^{(1)}_N + \epsilon_1q^{(2)}_N} \\ \frac{k_0 \cdot \frac{\hat{\sigma}}{2} (\hat{\sigma} + \hat{\sigma}) + \frac{N \cdot \hat{k} \times \xi}{k} \cdot \frac{1}{2} (\hat{\sigma} - i\hat{\sigma})}{\hat{\delta}_3(k_0)} \\
\hat{\delta}_3(k) \cdot \left( \frac{\xi k_0}{k_0} \frac{1}{2} (\hat{\sigma} + \hat{\sigma}) + \frac{N \cdot \hat{k} \times \xi}{k_0} \cdot \frac{1}{2} (\hat{\sigma} - i\hat{\sigma}) \right) \right\} \hat{\delta}_1(k_0). \quad (4.17)$$

Some other possible representations of this matrix, as well as explicit formulae for the matrix elements, are given in appendix D.

Suppose that the second medium is a perfect conductor. In this case the fields do not penetrate into the second medium and the scattering matrix describing the mutual transformations of the waves of the $V$ and $H$ polarizations coincides with $\hat{S}^{11}$ (see (2.10) and (2.11)). Assuming in (4.15) that $\epsilon_2 \to \infty$, we find

$$\hat{B}^{11}(k, k_0) = \begin{pmatrix}
K_1^2(kk - k_0^2) & K_1N \cdot k \times k_0 \\
q^{(1)}_kq^{(1)}_0kk & q^{(1)}_kkk_0 \\
K_1N \cdot k \times k_0 & q^{(1)}_kkk_0 \\
q^{(1)}_kkk_0 & q^{(1)}_kkk_0
\end{pmatrix}. \quad (4.18)$$
The limit \( \varepsilon_2 \to \infty \) in (4.17) for the block \( \hat{B}_2^{11}(k, k_0; \xi) \) can be easily calculated. It is interesting that in this case this matrix can be directly expressed through \( \hat{B}^{11} \) given in (4.18)

\[
\hat{B}_2^{11}(k, k_0; \xi) = -2q_3^{(1)} \hat{B}^{11}(k, \xi) \bar{\sigma}_3 \hat{B}^{11}(\xi, k_0) \quad \varepsilon_2 \to \infty.
\]

(4.19)

Note that for \( \varepsilon_2 \to \infty \)

\[ \hat{V}^{11}(k) = \hat{B}^{11}(k, k) = \bar{\sigma}_3. \]

Hence, one can represent (4.19) in the following form also

\[ \hat{B}_2(k, k_0; \xi) = -2q_3^{(1)} \hat{B}^{11}(k, \xi) \hat{B}^{11}(\xi, \xi) \hat{B}^{11}(\xi, k). \]

Taking into account the commutative relations for the Pauli matrices, it is possible to confirm that the following relations hold

\[
\hat{B}(k, k_0) = \begin{pmatrix}
\bar{\sigma}_3 & 0 \\
0 & \bar{\sigma}_3
\end{pmatrix} \hat{B}^T(-k_0, -k) \begin{pmatrix}
\bar{\sigma}_3 & 0 \\
0 & \bar{\sigma}_3
\end{pmatrix}
\]

\[ \hat{B}_2(k, k_0; \xi) = \begin{pmatrix}
\bar{\sigma}_3 & 0 \\
0 & \bar{\sigma}_3
\end{pmatrix} \hat{B}^T_2(-k_0, -k; -\xi) \begin{pmatrix}
\bar{\sigma}_3 & 0 \\
0 & \bar{\sigma}_3
\end{pmatrix} \]

(4.20)

As a consequence, the reciprocity relations (3.6) are fulfilled to the second order of accuracy in \( \hbar \), as expected.

Moreover, the matrices \( \hat{B}, \hat{B}_2 \) also obey the relations of the crossing symmetry type

\[ \hat{B}(k, k_0) = \hat{B}(-k, -k_0) \]

\[ \hat{B}_2(k, k_0; \xi) = \hat{B}_2(-k, -k_0; -\xi). \]

Expressions (4.1), (4.9), (4.15), and (4.17) give the solution to our scattering problem to the second order of accuracy with respect to \( \hbar \).

5. The small-slope approximation

Expansion (4.1) can be practically used for the case of small elevations only when the condition

\[ q_k h \ll 1 \]

holds both for \( k = k_0 \) and all \( k \) corresponding to scattered waves with significant amplitudes. However, when functions \( \hat{B}, \hat{B}_2, \ldots \) are known, one can proceed in a standard way to SSA [10], which is valid if

\[ \nabla h \ll \frac{q_k}{k} \]

for all waves possessing significant amplitudes. In other words, slopes of irregularities should be small enough and at least be smaller than the grazing angles of the incident and scattered waves. The SSA method was checked in numerical experiments [16] and
demonstrated good results. In this section we shall describe the procedure of transition from MSP to SSA without detailed discussion of the foundation of this method which can be found in [10] and [17].

The main ansatz of the SSA is to seek the SA in the form

\[ S^{N_0}(k, k_0) = \int \frac{dr}{(2\pi)^2} \exp \left[ -i(k - k_0) r - i((-1)^N q_{k_0}^{N_0}) h(r) \right] \]

\[ \times \hat{\Phi}^{N_0}(k, k_0; r; \{ h \}) \]

(5.1)

where

\[ \hat{\Phi}^{N_0}(k, k_0; r; \{ h \}) = \hat{\Phi}_0^{N_0}(k, k_0) + \int \hat{\Phi}_1^{N_0}(k, k_0; \xi) h(\xi) e^{i \xi r} d\xi \]

\[ + \int \hat{\Phi}_2^{N_0}(k, k_0; \xi_1, \xi_2) h(\xi_1) h(\xi_2) e^{i(\xi_1 + \xi_2)r} d\xi_1 d\xi_2 + \cdots \]

(5.2)

Here \( \hat{\Phi} \) is some arbitrary functional of elevations represented in the form of an integral power series. The functions \( \hat{\Phi}_2, \hat{\Phi}_3, \ldots \) are symmetric with respect to variables \( \xi_1, \xi_2, \ldots \). As in the previous section, one can easily see that the transformational property (4.2) is fulfilled. Moreover, the exponential factor related to elevations is extracted in (5.1), which ensures the transformational property of the SA when the boundary is shifted in the vertical direction as a whole. If \( h(r) \rightarrow h(r) + H \), where \( H = \text{constant} \), then

\[ \hat{S}^{N_0}(k, k_0) \rightarrow \hat{S}^{N_0}(k, k_0) \exp[ -i Q_{kk_0}^{N_0} h(r) ] \]

(5.3)

Here we introduced the following notation for the appropriate sum of vertical wavenumbers

\[ Q_{kk_0}^{N_0} = (-1)^N q_{k_0}^{N_0} + (-1)^N q_{k_0}^{N_0} \]

Thus, the functional \( \hat{\Phi} \) should be invariant with respect to the transformation

\[ h(\xi) \rightarrow h(\xi) + H \delta(\xi) \]

Hence, functions \( \hat{\Phi}_1, \hat{\Phi}_2, \ldots \) should be equal to zero if one of the \( \xi \)-arguments of the functions turns out to be zero. Taking into account the symmetry of these functions we can represent them in the following form

\[ \Phi_1^{N_0}(k, k_0; \xi) = \sum_\alpha \xi_\alpha \cdot \hat{\Phi}_1^{N_0,\alpha}(k, k_0; \xi) \]

\[ \Phi_2^{N_0}(k, k_0; \xi_1, \xi_2) = \sum_{\alpha_1, \alpha_2} \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdot \hat{\Phi}_2^{N_0,\alpha_1,\alpha_2}(k, k_0; \xi_1, \xi_2) \]

(5.4)

where \( \alpha_i = (1, 2) = (x, y) \) and functions \( \hat{\Phi}_n \) are regular at \( \xi_i = 0 \) (to simplify notation it is assumed in what follows that the quantities \( \hat{\Phi}, \hat{\Phi}_n, \ldots \) are matrices although the hat (') is omitted).

It is important to mention that the functional \( \Phi \) in (5.2) is determined ambiguously and can be subjected to the gauge transformation

\[ \Phi \rightarrow \Phi + \nabla g - i(k - k_0 + Q_{kk_0}^{N_0} \nabla h(r)) g \]

(5.5)
with the arbitrary function $g = g(k, k_0; r)$. In this transformation the uniquely determined value of the SA obviously remains invariant because the terms proportional to $g$ form a gradient in (5.1) and become zero after integration over $r$. The existence of the gauge transformation (5.5) explains why the function $\Phi$ in (5.1), which is due to surface sources and at first glance should depend on parameters of the incident wave only, can depend on $(k, \alpha)$ as well.

Now we describe some simple transformations (which are based on no more than integration by parts) that allow us to exclude from expansion (5.1) the arbitrary term with $n > 1$. Consider, for instance, the first-order term. Let us set

$$\Phi^{N_0, \alpha} (k, k_0; \xi) = \Phi^{N_0, \alpha}_1 (k, k_0; k - k_0) + [\Phi^{N_0, \alpha}_1 (k, k_0; \xi) - \Phi^{N_0, \alpha}_1 (k, k_0; k - k_0)].$$

(5.6)

Because the first term in (5.6) does not depend on $\xi$, it provides the following contribution to $\Phi$:

$$\int \frac{dr}{(2\pi)^2} \frac{\Phi^{N_0, \alpha}_1 (k, k_0; k - k_0)}{Q^{N_0}_{kk_0}} e^{-i(k-k_0)r} \nabla \Phi^{N_0, \alpha}_1 (k, k_0; k - k_0) h(r).$$

Substituting this result into (5.1), we have

$$\int \frac{dr}{(2\pi)^2} \frac{\Phi^{N_0, \alpha}_1 (k, k_0; k - k_0)}{Q^{N_0}_{kk_0}} e^{-i(k-k_0)r} \nabla \Phi^{N_0, \alpha}_1 (k, k_0; k - k_0) h(r)$$

and integrating by parts and taking into account (5.4) we can continue this equation as follows:

$$= \int \frac{dr}{(2\pi)^2} \frac{i \Phi^{N_0}_1 (k, k_0; k - k_0)}{Q^{N_0}_{kk_0}} e^{-i(k-k_0)r} \exp[-iQ^{N_0}_{kk_0} h(r)].$$

(5.7)

Summation over repeating $\alpha$ indexes is assumed here. Thus, the first term in (5.6) can be transformed into the expression corresponding to the zeroth-order term in (5.2) and incorporated into $\Phi_0$, resulting in

$$\Phi^{N_0}_0 (k, k_0) \rightarrow \Phi^{N_0}_0 (k, k_0) + \frac{i \cdot \Phi^{N_0}_1 (k, k_0; k - k_0)}{Q^{N_0}_{kk_0}}.$$
with the function $f_1$ regular at $\xi = k - k_0$. Substituting (5.8) into (5.2), we obtain
\[
\int \frac{dr}{(2\pi)^2} \exp[-i(k - k_0)r - iQ_{kk_0}^{N_0}h(r)]\xi^\alpha h(\xi) \cdot e^{i\xi r}(k - k_0 - \xi)^\beta f_1^{N_0,\alpha\beta}(k, k_0; \xi) d\xi
\]
\[
= \int \frac{dr}{(2\pi)^2} \exp[-i(k - k_0 - \xi)r] \times \exp[iQ_{kk_0}^{N_0}h(r)]\xi^\alpha h(\xi) \cdot f_1^{N_0,\alpha\beta}(k, k_0; \xi) d\xi
\]
\[
= \int \frac{dr}{(2\pi)^2} \exp[-i(k - k_0)r - iQ_{kk_0}^{N_0}h(r)] \times f_1^{N_0,\alpha\beta}(k, k_0; \xi)\xi^\alpha h(\xi) e^{i\xi r} \cdot \nabla h(r).
\]
\[
(5.9)
\]
If we substitute the representation of $h(r)$ as a Fourier transform into (5.9) and perform symmetrization with respect to the $\xi$-arguments, we find that the result has the form of the second-order term in (5.2) and can be incorporated into it. As a result $\Phi_2^{N_0}$ acquires the following addition
\[
\Phi_2^{N_0}(k, k_0; \xi_1, \xi_2) \rightarrow \Phi_2^{N_0}(k, k_0; \xi_1, \xi_2) + \frac{1}{2}Q_{kk_0}^{N_0}(f_1^{N_0,\alpha_1\alpha_2}(k, k_0; \xi_1) + f_1^{N_0,\alpha_2\alpha_1}(k, k_0; \xi_2)).
\]
\[
(5.10)
\]
Thus, we found that with the help of the two transformations described above we can exclude the first-order term from expansion (5.2). The term of any given order $n$, $n \geq 1$, can be excluded using the same transformations by 'transferring' it to terms of the $(n - 1)$th and $(n + 1)$th order.

This allows us to immediately express functions $\Phi_0, \Phi_1, \ldots$ in (5.2) in terms of the functions $\tilde{B}, \tilde{B}_2, \ldots$ considering the limit $h \rightarrow 0$. To do this we assume that the first-order term in (5.2) is excluded, i.e. $\Phi_1 = 0$. Let us expand the corresponding exponential in (5.1) and compare the result with the first-order term in (4.1). This immediately gives us
\[
\Phi_0^{N_0}(k, k_0) = \frac{2(q_k^{(N_0)}q_{k_0}^{(N_0)})^{1/2}}{Q_{kk_0}^{N_0}} \tilde{B}^{N_0}(k, k_0).
\]
\[
(5.11)
\]
It is easily seen that because of relation (4.16) the zeroth-order terms of both expansions also agree.

As a result we obtain the expression for $\text{SA}$ in the first approximation of the small-slope approach:
\[
\hat{S}^{N_0}(k, k_0) = -\frac{2(q_k^{(N_0)}q_{k_0}^{(N_0)})^{1/2}}{Q_{kk_0}^{N_0}} \int \frac{dr}{(2\pi)^2} \exp[-i(k - k_0)r - iQ_{kk_0}^{N_0}h(r)]
\]
\[
(5.12)
\]
where for the EM case $\tilde{B}^{N_0}$ is given by (4.15). This result was first obtained for EM waves by Bahar [9]. This expression can also be rewritten as
\[
\hat{S}^{N_0}(k, k_0) = \hat{V}^{N_0}(k) \cdot \delta(k - k_0) - \frac{2(q_k^{(N_0)}q_{k_0}^{(N_0)})^{1/2}}{Q_{kk_0}^{N_0}} \tilde{B}^{N_0}(k, k_0)
\]
\[
\times \int \frac{dr}{(2\pi)^2} \exp[-i(k - k_0)r][\exp(-iQ_{kk_0}^{N_0}h(r)) - 1].
\]
It is obvious that it remains regular when \( Q_{kk_0}^{NN_0} \to 0 \).

We will now calculate the correction of the second-order term in (5.2). Assume, as previously, that \( \Phi_1 = 0 \) and compare in (5.1) and (4.1) the terms of the second order in \( h \). As a result, we obtain

\[
\int [\hat{\Phi}_2^{NN_0}(k, k_0; \xi_1, \xi_2) - \frac{1}{2} (Q_{kk_0}^{NN_0})^2 \hat{\Phi}_0^{NN_0}(k, k_0)] h(\xi_1) h(\xi_2) \delta(k - k_0 - \xi_1 - \xi_2) \, d\xi_1 \, d\xi_2
\]

\[
= \frac{1}{2} (q_k^{(N)} q_{k_0}^{(N)})^{1/2} \int [\hat{B}_2^{NN_0}(k, k_0; k - \xi_1) + \hat{B}_2^{NN_0}(k, k_0; k - \xi_2)]
\]

\[
\times h(\xi_1) h(\xi_2) \delta(k - k_0 - \xi_1 - \xi_2) \, d\xi_1 \, d\xi_2.
\]

Because of the arbitrariness of \( h(\xi) \) and the symmetry of kernels with respect to \( \xi_1, \xi_2 \), we find

\[
\hat{\Phi}_2^{NN_0}(k, k_0; \xi_1, \xi_2) = \frac{1}{2} (Q_{kk_0}^{NN_0})^2 \hat{\Phi}_0^{NN_0}(k, k_0) + \frac{1}{2} (q_k^{(N)} q_{k_0}^{(N)})^{1/2} (\hat{B}_2^{NN_0}(k, k_0; k - \xi_1)
\]

\[
+ \hat{B}_2^{NN_0}(k, k_0; k - \xi_2)) + \hat{R}^{NN_0}(k, k_0; \xi_1, \xi_2)
\]

(5.13)

where \( \hat{R}^{NN_0} \) is an arbitrary function that becomes zero on the manifold

\[ k - k_0 - \xi_1 - \xi_2 = 0. \]

But, as we have ensured, in this case the corresponding contribution with the help of transformation (5.9) can be transformed into the third-order term in (5.2). Hence in (5.13) one can set \( \hat{R}^{NN_0} = 0 \) with the \( O((\nabla h)^3) \) accuracy.

Thus the functions \( \hat{\Phi}_0 \) and \( \hat{\Phi}_2 \), with the help of relations (5.11) and (5.13), are expressed through the functions \( \hat{B} \) and \( \hat{B}_2 \); hereby \( \Phi_1 = 0 \). Let us now simplify the resulting expression by transforming the already determined second-order term with the help of transformations (5.6), (5.7) and (5.9) into the sum of the first- and third-order terms. As a matter of fact, we are not interested in the exact value of the corresponding third-order contribution because all calculations were performed with the \( O((\nabla h)^3) \) accuracy. The corresponding contribution of the term \( \hat{\Phi}_2 \) to the first-order term can be calculated as in (5.7)

\[
\Delta \hat{\Phi}_1^{NN_0}(k, k_0; \xi) = \frac{i \hat{\Phi}_2^{NN_0}(k, k_0; \xi, k - k_0 - \xi)}{Q_{kk_0}}.
\]

Putting together formulae (5.14), (5.13) and (5.11), we obtain the following expression for the SA which has second-order accuracy with respect to slope \( (\nabla h)^2 \)

\[
\hat{S}^{NN_0}(k, k_0) = -\frac{2(q_k^{(N)} q_{k_0}^{(N)})^{1/2}}{Q_{kk_0}^{NN_0}} \int \frac{dr}{(2\pi)^2} \left[ \hat{M}^{NN_0}(k, k_0; \xi) h(\xi) e^{ikr} \, d\xi \right]
\]

\[
\times \exp[-i(k - k_0) r - i Q_{kk_0}^{NN_0} h(r)]
\]

(5.15)

where

\[
\hat{M}^{NN_0}(k, k_0; \xi) = \hat{B}_2^{NN_0}(k, k_0; k - \xi) + \hat{B}_2^{NN_0}(k, k_0; k_0 + \xi) - 2 Q_{kk_0}^{NN_0} \hat{B}^{NN_0}(k, k_0).
\]

(5.16)
Because of (4.20), it is obvious that the reciprocity relations (3.6) for the scattering matrix determined by (5.15) are satisfied. On the basis of formulae (4.15) and (4.17), it is possible to ensure that

$$\hat{M}^{NN_0}(k, k_0; 0) = \hat{M}^{NN_0}(k, k_0; k - k_0) = 0.$$  

(5.17)

Hence, the coefficient of $h(\xi)$ in the internal integral in (5.15) becomes zero at $\xi = 0$ and $\xi = k - k_0$. As a result, the second term in the first set of square brackets in the integrand of (5.15) really represents a second-order correction with respect to $\nabla h$.

The method of obtaining approximations (5.12) and (5.15) implies that these expressions agree as $h \to 0$ with the perturbative formula (4.1) to an $O(h)$ and $O(h^2)$ accuracy, respectively. This can be immediately checked as well.

Consider now the high-frequency limit for $\text{SA}$ (5.15). Taking into account that, as $k, k_0 \to \infty$, integration over $\xi$ for smooth elevations is restricted to the relatively small $\xi \ll k, k_0$, one can use the linear approximation for $M$ in (5.15)

$$\hat{M}^{NN_0}(k, k_0; \xi) \approx \frac{\hat{M}^{NN_0}(k, k_0; 0)}{d\xi} \cdot \xi.$$  

In this approximation, integration over $\xi$ in (5.15) gives $\nabla h$, and the resulting term can be integrated by parts and incorporated into the zeroth-order term (just as in transformation (5.7))

$$S^{NN_0}(k, k_0) = -\frac{2\epsilon_0 d^{(N)} q^{(N)}_{k_0}}{Q^{NN_0}_{kk_0}} \left[ \hat{B}^{NN_0}(k, k_0) + \frac{1}{4} k - k_0 \frac{\hat{M}^{NN_0}(k, k_0; 0)}{d\xi} \right]$$

$$\times \int \frac{d\tau}{(2\pi)^2} \exp[-i(k - k_0)\tau - iQ^{NN_0}_{kk_0}h(\tau)].$$  

(5.18)

The expression for the SA in the Kirchhoff approximation has exactly the same structure as expressions (5.12) and (5.18), and we represent it here in a similar form [10]

$$S_{(Kirch)}^{NN_0}(k, k_0) = -\frac{2\epsilon_0 d^{(N)} q^{(N)}_{k_0}}{Q^{NN_0}_{kk_0}} \hat{B}_{(Kirch)}^{NN_0}(k, k_0) \times \int \frac{d\tau}{(2\pi)^2} \exp[-i(k - k_0)\tau - iQ^{NN_0}_{kk_0}h(\tau)]$$  

(5.19)

where the expression for the function $B_{(Kirch)}^{NN_0}(k, k_0)$ can be easily calculated. At $k \neq k_0$ it generally differs both from $\hat{B}(k, k_0)$ and from the corresponding expression in square brackets in (5.18). But for smooth surfaces, the integral in (5.12), (5.18) and (5.19) significantly differs from zero for $k \approx k_0$ only and is exponentially small otherwise. For $k = k_0$, all $\hat{B}^{NN_0}$-type pre-integral factors mentioned above agree and are equal to $\hat{V}^{NN_0}$. However, if elevations $h(\tau)$ contain small-scale components and scattering at large angles occurs, then the Kirchhoff approximation (5.19) gives the wrong results (e.g., it fails to reproduce (4.1) in the limiting case $h \to 0$) and one should use (5.12) or even the more accurate formula (5.15).

Expressions (5.12) and (5.15) are the main results of this section. Note that these formulae are not related to the EM case only and are valid for waves of an arbitrary nature. The only restriction arises here from the fact that wave speeds were assumed to be polarization independent. However, formulae (5.12) and (5.15) will be applicable to the polarization-dependent case as well if one formally includes the appropriate dependency on polarization $q^{(N)}_k \to q^{(N,a)}_k$ in the vertical wavenumbers.
6. Statistical case

If elevations of the boundary are of a statistical character, then the two first statistical moments of the SA are of practical importance. These moments are immediately related to experimentally measured values. In this section we shall assume that the statistical ensemble of elevations is homogeneous in space (i.e. the set \( \{h(r)\} \) remains invariant under horizontal translations, \( \forall d : \{h(r)\} = \{h(r-d)\} \)). For the first statistical moment we have

\[
(\hat{S}(k, k_0)) = \hat{V}(k)\delta(k - k_0).
\]

The \( \delta \)-function in (6.1) is a direct consequence of the spatial homogeneity of the statistical ensemble \( \{h(r)\} \) and easily follows from (4.2). The matrix \( \hat{V} \) is related to the average field; it is usually referred to as the average reflection coefficient and can be measured experimentally.

We shall further assume, for simplicity, that the statistical ensemble of elevations is Gaussian. Expression (5.15) can be easily averaged and as a result one finds

\[
\hat{V}^{NN_0}(k) = \exp\left(-1/2Q_0^2\sigma^2\right) \cdot \left(\hat{V}^{NN_0}(k) + \frac{2(q_k^{(N)}q_k^{(N_0)})^{1/2}}{Q_0}, \hat{F}^{NN_0}(k, k; 0)\right).
\]

Here

\[
\hat{F}^{NN_0}(k, k_0; r) = \frac{Q}{4} \int \hat{M}^{NN_0}(k, k_0; \xi)S(\xi)e^{i\xi r} d\xi.
\]

To simplify the formulae, we omit the indices for \( Q \) and set

\[
Q = Q^{NN_0} = (-1)^N q_k^{(N)} + (-1)^{N_0} q_k^{(N_0)} \quad Q_0 = Q^{NN_0}_0.
\]

The function \( \hat{M} \) in (6.3) is given by (5.16) and \( S(\xi) \) is the spectrum of elevations

\[
S(\xi) = \int W(r)e^{-i\xi r} \frac{dr}{(2\pi)^2}
\]

with

\[
W(r) = (h(r + \rho) \cdot h(\rho))
\]

being the correlation function and

\[
\sigma^2 = W(0) = \int S(\xi) d\xi.
\]

The second term in (6.2) depends on the roughness spectrum and represents the small correction to \( V \) that is proportional to the mean square of the slopes. This correction could be of interest, in particular, when the Rayleigh parameter is small: \( \sigma^2 Q_0^2 \ll 1 \).

The energy characteristics of the scattered field are related to the second moments of the SA. Let us assume that

\[
\Delta \hat{S}(k, k_0) = \hat{S}(k, k_0) - (\hat{S}(k, k_0))
\]
and introduce the following quantity

$$
\left( \Delta S_{\alpha_0}^{N_0} \left( k - \frac{1}{2} d, k_0 - \frac{1}{2} d_0 \right) \right) \left( \Delta S_{\alpha'0}^{N_0'} \left( k + \frac{1}{2} d, k_0 + \frac{1}{2} d_0 \right) \right) = \Delta \Pi_{\alpha_0, \alpha_0'}^{N_0, N_0'}(k, k_0; d) \cdot \delta(d - d_0).
$$

(6.5)

The \( \delta \)-function in (6.5) is again a direct consequence of the spatial homogeneity of the statistics of roughness and follows from (4.2). Generally, the value \( \Delta \Pi \) has 256 components that describe cross-correlations of all possible scattering processes. However, because of the reciprocity relations (3.6) not all of them are independent. Practically, the most important is the case of autocorrelation when

$$
N = N', \quad N_0 = N_0', \quad \alpha = \alpha', \quad \alpha_0 = \alpha_0'.
$$

Moreover, if, as usual, the far field is under consideration, then only the value of \( \Delta \Pi \) at \( d = 0 \) should be taken into account. Let us set

$$
\sigma_{\alpha_0 \alpha_0'}^{N_0, N_0'}(k, k_0) = q_{N_0}^{(N)} q_{N_0'}^{(N)} \cdot \Delta \Pi_{\alpha_0 \alpha_0, \alpha_0' \alpha_0'}^{N_0, N_0'}(k, k_0; 0).
$$

(6.6)

The value \( \sigma \) in (6.6) agrees with the conventional definition of the dimensionless scattering cross section for the incoherent component of the scattered field [17,18], which is usually measured experimentally.

Let us substitute the representation of the SSA in the form

$$
S_{\alpha_0 \alpha_0'}^{N_0, N_0'}(k, k_0) = S_{\alpha_0 \alpha_0'}^{N_0, N_0'}(k) \delta(k - k_0) + \Delta S_{\alpha_0 \alpha_0'}^{N_0, N_0'}(k, k_0)
$$

into unitarity relation (3.8)

$$
\sum_{\alpha_0', \alpha_0''} \int_{k' < K_{\inf}} dk' S_{\alpha_0 \alpha_0'}^{N_0, N_0'}(k - \frac{1}{2} d, k') \cdot (S^{N_0, N_0'}_{\alpha_0' \alpha_0''}(k + \frac{1}{2} d, k')) = \delta(d).
$$

Averaging this expression and taking into account definitions (6.1) and (6.5), we find

$$
\sum_{\alpha_0', \alpha_0''} \left( \int V_{\alpha_0 \alpha_0'}^{N_0, N_0'}(k) \right)^2 + \int_{k' < K_{\inf}} dk' \sigma_{\alpha_0 \alpha_0'}^{N_0, N_0'}(k, k') = 1.
$$

This relation describes energy conservation in the aggregate of coherent and incoherent components of the wave field (in the case of non-dissipative media; in our case for real \( \epsilon \)).

The correlator in (6.5) for the Gaussian ensemble of roughness and for the SSA given by (5.15) can be calculated immediately. After simple but tedious manipulations, one finds

$$
\sigma_{\alpha_0 \alpha_0}^{N_0, N_0}(k, k_0) = \left( \frac{2q_k^{(N)} q_{k_0}^{(N)}}{Q^{N_0}_{k_0}} \right)^2 \int e^{-[(k - k_0)^{\alpha} - W(\xi)]} R_{\alpha_0 \alpha_0}^{N_0, N_0}(k, k_0; r) \frac{dr}{(2\pi)^2}
$$

(6.7)

where

$$
R_{\alpha_0 \alpha_0}^{N_0, N_0}(k, k_0; r) = -e^{-Q^2 \sigma^2} \left| B_{\alpha_0 \alpha_0}^{N_0, N_0}(k, k_0) - F_{\alpha_0 \alpha_0}^{N_0, N_0}(k, k_0; 0) \right|^2
$$

$$
+ e^{-Q^2 \sigma^2 W(r)} \left( \frac{1}{16} \int |M_{\alpha_0 \alpha_0}^{N_0, N_0} (k, k_0; \xi)|^2 S(\xi) e^{i\xi r} d\xi \right)
$$

$$
+ (B_{\alpha_0 \alpha_0}^{N_0, N_0}(k, k_0) - F_{\alpha_0 \alpha_0}^{N_0, N_0}(k, k_0; 0) + F_{\alpha_0 \alpha_0}^{N_0, N_0}(k, k_0; r))
$$

$$
\times (B_{\alpha_0 \alpha_0}^{N_0, N_0}(k, k_0) - F_{\alpha_0 \alpha_0}^{N_0, N_0}(k, k_0; 0) + F_{\alpha_0 \alpha_0}^{N_0, N_0}(k, k_0; r))^*.
$$

(6.8)
The functions \( M \) and \( F \) here are determined by (5.16) and (6.3), and \( S(\xi) \) is the spectrum of roughness (6.4). It is obvious that both functions \( F(k, k_0; \tau) \) and \( R(k, k_0; \tau) \) tend to zero when \( |\tau| \to \infty \); thus the integrals in (6.7) and (6.8) are both well defined.

Taking into account (5.17), one can observe that the value of the function \( F \) is proportional to \( (Vh)^{\frac{1}{2}} \). Corresponding quantities for non-grazing angles are generally small corrections to the value \( B(k, k_0) \), which has the order of unity. If they can be neglected, expression (6.8) simplifies to

\[
R_{a\omega}^{N_0}(k, k_0; \tau) = |B_{a\omega}^{N_0}(k, k_0)|^2 (e^{-Q^2\sigma^2}) - e^{-Q^2\sigma^2}
\]

and (6.7) takes the form that corresponds to the first order of the SSA:

\[
\hat{a}_{a\omega}^{N_0}(k, k_0) = (2q_k^{(N)} q_{k_0}^{(N)})^2 \cdot |B_{a\omega}^{N_0}(k, k_0)|^2 
\times \int e^{-i(k-k_0)\tau} e^{-Q^2\sigma^2} \left( \exp(Q^2W(\tau)) - 1 \right) \frac{dr}{(2\pi)^2}.
\]

Expressions (6.2), (6.7) and (6.8) (or (6.9)) are the principal results of the SSA for Gaussian space-homogeneous statistical ensembles of elevations. As in the previous section, the results obtained in this section are not related to the EM problem only and are valid for arbitrary waves (see also the final remark of section 5).

7. Conclusions

The SSA for the scattering of EM waves at the rough interface of two dielectric half-spaces is presented. The SSA unifies two classical approximate approaches to the problem: the method of small perturbations and the Kirchhoff approximation. If the spectrum of roughness is broad enough, then application of the SSA enables one to avoid using the ambiguous two-scale model.

In contrast to classical approaches, the validity of the SSA is wavelength independent and is related to the smallness of the slopes of the roughness only. To be more accurate, the slopes should be small compared with the incident angle and the scattering angles of all waves with significant amplitudes. This condition should hold for intermediate waves also. That means that if the function \( \hat{M} \) in (5.16) becomes large for some \( \xi \), then the contribution of the corresponding integral in (5.15) should nevertheless be small. For instance, for EM waves scattering at the rough boundary of an ideally conducting surface \((\varepsilon_2 \to \infty)\), the function \( \hat{M} \) has integrable singularity at \( q_{k-k_0} = 0 \) or \( q_{k_0+\xi} = 0 \), which is related to the excitation of horizontally propagating waves (surface waves). Their amplitudes should thus remain, in a sense, small. If, for instance, \( h(\tau) \) is a periodic function with period \( L \) and the resonance condition

\[
k_0 + n(2\pi/L) = K_1
\]

holds for some \( n = \pm 1, \pm 2, \ldots \), then expression (5.15) tends to infinity. Neither SSA nor the conventional MSP can be used to consider this case.

We must emphasize, that the results obtained in sections 5 and 6 are quite general and can be applied to waves of arbitrary nature. The concrete scattering problem is related to functions \( \hat{B} \) and \( \hat{B}_2 \) (see (4.1), (5.15) and (5.16)) which are calculated within the framework of the conventional MSP. For the EM case, the corresponding expressions are provided by (4.15) and (4.17) and the formulae of appendix D.

To calculate the functions mentioned above in the general case one can use the procedure based on the Rayleigh hypothesis. As we pointed out, this procedure leads to correct results (although the Rayleigh hypothesis as such is, in general, wrong).
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Appendix A. Auxiliary formulae

When performing calculations to obtain reciprocity relations (3.4) and (3.6), it is convenient to use the set of formulae that easily follows from definitions (2.3) and (2.4)

\[
N \cdot e_{\alpha_1}^+(k) \times h_{\alpha_2}^+(-k) = -\frac{q_k}{K} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha_1,\alpha_2}
\]

\[
N \cdot e_{\alpha_1}^-(k) \times h_{\alpha_2}^+(-k) = \frac{q_k}{K} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha_1,\alpha_2}
\]

\[
N \cdot e_{\alpha_1}^+(k) \times h_{\alpha_2}^-(-k) = -\frac{q_k}{K} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\alpha_1,\alpha_2}
\]

\[
N \cdot e_{\alpha_1}^-(k) \times h_{\alpha_2}^+(-k) = \frac{q_k}{K} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\alpha_1,\alpha_2}
\]

Analogously, to obtain the unitarity condition the following relations can be used

\[
N \cdot e_{\alpha_1}^+(k) \times (h_{\alpha_2}^+(k))^* = \left( \frac{q_k}{K} 0 \\ 0 \frac{q_k}{K} \right)_{\alpha_1,\alpha_2}
\]

\[
N \cdot e_{\alpha_1}^-(k) \times (h_{\alpha_2}^-(k))^* = \left( -\frac{q_k}{K} 0 \\ 0 \frac{q_k}{K} \right)_{\alpha_1,\alpha_2}
\]

\[
N \cdot e_{\alpha_1}^+(k) \times (h_{\alpha_2}^-(k))^* = \left( \frac{q_k}{K} 0 \\ 0 -\frac{q_k}{K} \right)_{\alpha_1,\alpha_2}
\]

\[
N \cdot e_{\alpha_1}^-(k) \times (h_{\alpha_2}^+(k))^* = \left( -\frac{q_k}{K} 0 \\ 0 -\frac{q_k}{K} \right)_{\alpha_1,\alpha_2}
\]

Here * denotes complex conjugation.

Appendix B. The formal relation for the S-matrix

Let us set

\[
\begin{align*}
E_{\alpha}^{+(N)}(k) &= (-1)^{N+1} q_k^{(N)-1/2} \exp(ikr + iq_k^{(N)} h(r)) \cdot e_{\alpha}^+(k) \\
E_{\alpha}^{-(N)}(k) &= (-1)^{N+1} q_k^{(N)-1/2} \exp(ikr - iq_k^{(N)} h(r)) \cdot e_{\alpha}^-(k) \\
H_{\alpha}^{+(N)}(k) &= (-1)^{N+1} q_k^{(N)-1/2} \exp(ikr + iq_k^{(N)} h(r)) \cdot h_{\alpha}^+(k) \\
H_{\alpha}^{-(N)}(k) &= (-1)^{N+1} q_k^{(N)-1/2} \exp(ikr - iq_k^{(N)} h(r)) \cdot h_{\alpha}^-(k).
\end{align*}
\]
The dependence of these functions on \( r \) is suppressed here. Now, the basic equations (4.3) can be represented as

\[
(N - \nabla h) \times \left( \begin{array}{cc} E_{a_0}^{-1}(k_0) & E_{a_0}^{-2}(k_0) \\ e_1^{1/2} H_{a_0}^{-1}(k_0) & e_2^{1/2} H_{a_0}^{-2}(k_0) \end{array} \right) \text{hor}_1 + \sum_{\sigma=1,2} \int dk \ (N - \nabla h) \times \left( \begin{array}{cc} E_{a_0}^{-1}(k) & E_{a_0}^{-2}(k) \\ e_1^{1/2} H_{a_0}^{-1}(k) & e_2^{1/2} H_{a_0}^{-2}(k) \end{array} \right) \text{hor}_1 \left( \begin{array}{cc} (S_h)_{a_0}^{21} & (S_h)_{a_0}^{12} \\ \text{hor}_1 & \text{hor}_1 \end{array} \right) (k_0) = 0. 
\]

(B.1)

Here \( \times \) designates outer product and index (hor) means that only the horizontal components of the appropriate resulting vectors are considered:

\[ A_{\text{hor}} = A - (AN) \cdot N. \]

One can immediately check that the following relations take place for horizontal components of corresponding vectors

\[
\{(N + \nabla h) \times e_{a}^{+}(k)\}_{\text{hor}} = (-1)^{a} \{(N - \nabla h) \times e_{a}^{-}(k)\}_{\text{hor}}
\]

\[
\{(N + \nabla h) \times h_{a}^{+}(k)\}_{\text{hor}} = (-1)^{a+1} \{(N - \nabla h) \times h_{a}^{-}(k)\}_{\text{hor}}.
\]

(B.2)

Consider now (B.1) for elevations of the form \( z = -h(r) \) and take equations (B.2) in the resulting relations into account. Then instead of (B.1) we obtain

\[
\sum_{a'} (N - \nabla h) \times \left( \begin{array}{cc} E_{a'}^{-1}(k_0) & E_{a'}^{-2}(k_0) \\ e_1^{1/2} H_{a'}^{-1}(k_0) & e_2^{1/2} H_{a'}^{-2}(k_0) \end{array} \right) \text{hor}_1 \left( \begin{array}{cc} (\hat{\sigma}_3)_{a'0} & 0 \\ 0 & (\hat{\sigma}_3)_{0a'} \end{array} \right)
\]

\[
+ \sum_{a,a'} \int dk (N - \nabla h) \times \left( \begin{array}{cc} E_{a'}^{-1}(k_0) & E_{a'}^{-2}(k_0) \\ e_1^{1/2} H_{a'}^{-1}(k_0) & e_2^{1/2} H_{a'}^{-2}(k_0) \end{array} \right) \text{hor}_1 \left( \begin{array}{cc} (\hat{\sigma}_3)_{a'0} & 0 \\ 0 & (\hat{\sigma}_3)_{0a'} \end{array} \right)
\]

\[
\times \left( \begin{array}{cc} (S_h)_{a_0}^{21} & (S_h)_{a_0}^{12} \\ \text{hor}_1 & \text{hor}_1 \end{array} \right) \left( \begin{array}{cc} (S_{-h})_{a_0}^{21} & (S_{-h})_{a_0}^{12} \\ \text{hor}_1 & \text{hor}_1 \end{array} \right) (k_0) = 0. 
\]

(B.3)

Multiplying (B.3) by the matrix

\[
\begin{pmatrix} \hat{\sigma}_3 & 0 \\ 0 & \hat{\sigma}_3 \end{pmatrix}
\]

from the right, we see that the \( 2 \times 2 \) matrices related to the functions \( E, H \) exchanged places and that the matrix \( \hat{S}_{-h} \) acquired two additional factors in (B.3) compared with (B.1). Hence, by definition (4.11), we obtain relation (4.10).

Of course, the developments presented here should be considered to be formal because we do not investigate the existence of the inverse matrix \( \hat{S}^{-1} \) defined by (4.11). We consider relation (4.10) in the asymptotic sense, i.e., to an accuracy of \( O(h^m) \) terms only for any given order \( m \).
Appendix C. Rayleigh hypothesis for analytical profiles

Consider the calculation of the field in the first half-space with the help of the Rayleigh hypothesis. For any field satisfying the wave equation and radiation condition the Helmholtz formula holds

$$\psi(R) = \int d\Sigma_R \left( \psi_{R'} \frac{\partial}{\partial n_{R'}} G_0(R' - R) - \frac{\partial \psi_{R'}}{\partial n_{R'}} G_0(R' - R) \right). \quad (C.1)$$

Here $\Sigma_R = \{r, h(r)\}$ is the boundary, $n_{R'}$ is the unit external normal to the boundary at the point $R' \in \Sigma$ and

$$G_0(R) = \frac{\exp(iKR)}{4\pi R}$$

is a Green function. Applying (C.1) to each component of the electric field $E^{(e)}$ and using the expansion of $G_0$ into plane waves

$$\exp(ikr' + iqkz)\frac{dr'}{q_k},$$

one obtains

$$E^{(e)}(r, z) = \int F_{ao}(r'; k; k_0) \cdot \exp(-ikr + iqkz) \frac{dr'}{(2\pi)^2} \exp(ikr' - iqkz) \frac{dk}{q_k}. \quad (C.2)$$

Here $z < \min h(r')$ and

$$F_{ao}(r'; k; k_0) = \frac{q_k^{-1/2}}{2} \left\{ (q_k + k \cdot \nabla h) E^{(e)} + i \left( \frac{\partial}{\partial r} - (\nabla h \cdot \nabla) \right) E^{(e)} \right\}_{z=h(r')} \quad (C.3)$$

The function $F$ depends on the parameters of the incident field $k_0$ and $ao$, as do $E^{(e)}$ and $\partial_r E^{(e)}$. The following expression for $SA$ follows from (C.2)

$$S_{ao0}(k, k_0) = \int e^{-ikr} F_{ao}(r'; k; k_0) \exp(-ikr' + iqkz) \frac{dk}{q_k}. \quad (C.4)$$

The Rayleigh hypothesis holds if the scattered field can be calculated with the help of expansion into down-going waves not only at $z < \min h(r')$ but at any $z \leq h(r')$. In other words, the integral

$$J_{ao0} = \int S_{ao0}(k, k_0) \exp(ikr - iqkz) \frac{dk}{q_k^{1/2}} \quad (C.5)$$

should converge for any $r, z \leq h(r)$. At $|k| \to \infty$ we have

$$q_k \to ik, \quad k = |k|$$

and the exponential in (C.5) tends to $\exp(kz)$. Convergence of the integral (C.5) is determined by the behaviour of the function $S_{ao0}(k, k_0)$ as $|k| \to \infty$. 
Assume that \( h(r) \) is an analytic function of \( r = (x, y) \). Then the asymptote of \( S_{aa}(k, k_0) \) can be found from (C.4) with the help of the steepest descent method. The equation for the saddle point \( r = r_* \) is

\[
(-ik - k \cdot \nabla h(r))_{r = r_*} = 0
\]

or

\[
\nabla h(r_*) = -i(k/k) k \to \infty. \quad (C.6)
\]

Because the exponential in (C.4) should dwindle in the corresponding half-space of the complex variable \( k \), the solutions of (C.6) satisfying the condition

\[
\text{Re}(ikr_*) > 0 \quad (C.7)
\]

should be chosen. As a result one obtains

\[
S_{aa}(k, k_0) \sim C_k \exp(-ikr_* - kh(r_*)). \quad (C.8)
\]

Here \( C_k \) is some bounded value tending to zero at \( |k| \to \infty \). The solution of equation (C.6) nearest to the real-axis should be used in (C.8). Thus the asymptote of the SA at \( |k| \to \infty \) can, in fact, be determined without knowing the solution of the scattering problem. Taking into account in (C.5) the estimation (C.8), one arrives at the following condition for the validity of the Rayleigh hypothesis:

\[
\max h(r) < \text{Re}(i(k/k) \cdot r_* + h(r_*)) \quad (C.9)
\]

for all directions of the vector \( k \).

Assume now that

\[
h(r) = aH(r)
\]

where \( a > 0 \) is some amplitude parameter and \( \max H(r) = 1 \). Then (C.6) takes the form

\[
\nabla H(r_*) = -\frac{i}{a} \frac{k}{k}.
\quad (C.10)
\]

Because the function \( H(r) \) is bounded for real \( r \), then at \( a \to 0 \) we have

\[
|\text{Im } r_*| \to +\infty. \quad (C.11)
\]

The condition for the validity of the Rayleigh hypothesis can be represented as

\[
a < \text{Re} \left( \frac{k}{k} \cdot r_* + \frac{H(r_*)}{|\nabla H(r_*)|} \right). \quad (C.12)
\]

Assume that the function \( H(r) \) satisfies, for all complex \( r \to \infty \), the condition

\[
\frac{H(r)}{|\nabla H(r)|} < \text{constant}. \quad (C.13)
\]
Any function consisting of any final set of harmonics obeys (C.13)

\[ H(r) = \sum_{n=1}^{N} A_n \cos(\xi_n r + \varphi_n). \]

It is obvious that with the help of functions of this class, one can approximate with any accuracy any \( H(r) \in L_2 \). Now from (C.7) and (C.11), it is obvious that condition (C.12) holds for all small enough \( a \). The class of analytic functions satisfying (C.13) is wide enough to ensure unambiguous calculation of the coefficient functions of the asymptotic expansion (4.1).

When we applied the steepest descent method for evaluation of the integral (C.4), it was assumed that the function \( F_{\text{as}}(r') \) could be continued analytically for complex \( r' \). According to (C.3), that means that appropriate analytic continuation allows the function

\[ E^{(sc)}(r') = E^{(sc)}(R') \bigg|_{R'=(r', h(r'))} \text{ (C.14)} \]

Such an analytic continuation can be performed with the help of the expression (C.1). In this formula, let us first tend the point \( R \) to the surface \( \Sigma : R \rightarrow (r, h(r)) \). Using the continuity of the potential of the simple layer and the well-known formula for the limiting value of the potential of a double layer, one comes to the relation

\[ \frac{1}{2} E^{(sc)}(r) = \text{vp} \int d\Sigma_{R'} E^{(sc)}(r') \cdot \frac{\partial}{\partial n_{R'}} G_0(R' - R) - \int d\Sigma_{R'} (n_{R'} \nabla) E^{(sc)}_{L'} \cdot G_0(R' - R) \text{ (C.15)} \]

where \( E^{(sc)}(r) \) is given by (C.14). Attributing complex values to the argument \( r \) in (C.15) (noting that \( r' \) remains by that real vector \( r' \in \Sigma \)) one obtains the analytic continuation of the function \( E^{(sc)}(r) \). The singularities of this function are related to the singularities of the integrand, which cannot be removed by the deformation of the contour of integration (the 'pinch' of the contour). These singularities originate from the zeros of the argument of \( G_0 \) in (C.15). Thus the singular points must satisfy the equations

\[ (r' - r)^2 + (h(r') - h(r))^2 = 0 \]
\[ (r' - r) + (h(r') - h(r)) \cdot \nabla h(r) = 0 \]

and therefore

\[ \nabla h(r) = -i \frac{r - r'}{\sqrt{(r - r')^2}} \text{ (C.16)} \]

In fact, in the one-dimensional case this equation is equivalent to (C.6), and singularities of the function \( E^{(sc)}(r) \) cannot hinder the deformation of the contour of integration in (C.15). In a two-dimensional situation (C.16) is not equivalent to (C.9). It is obvious, however, that for functions of the type (C.13), the solution of equation (C.16) also satisfies condition (C.11). Hence, if the point \( r' \) that prevents further deformation of the contour of integration arises due to singularities of the function \( E^{(sc)}(r) \), then condition (C.9) will nevertheless be satisfied for small enough \( a \).
Appendix D. S-matrix for EM scattering: reference formulae

For reference purposes we give here explicit formulae for the entries of the matrix \( \hat{B}_2(k, k_0; \xi) \). The complete set of arguments of the corresponding functions includes permittivities \( \varepsilon_1 \) and \( \varepsilon_2 \), which are explicitly shown here. Thus the matrix \( \hat{B}_2 \) for the EM problem under consideration is

\[
\hat{B}_2(k, k_0; \xi) = \hat{B}_2(k, k_0; \xi; \varepsilon_1, \varepsilon_2).
\]

(In the formulae presented below, pay attention to the order of the arguments \( \varepsilon_1, \varepsilon_2 \).) Let us set

\[
b_{11}(k, k_0; \varepsilon_1, \varepsilon_2) = (\varepsilon_2 - \varepsilon_1)(\varepsilon_1 q_k^{(2)} + \varepsilon_2 q_k^{(1)})^{-1}(\varepsilon_1 q_0^{(3)} + \varepsilon_2 q_0^{(1)})^{-1}.
\]

Then

\[
(B_2)_{11}^{11}(k, k_0; \xi; \varepsilon_1, \varepsilon_2) = b_{11}(k, k_0; \varepsilon_1, \varepsilon_2) \left[ -2 \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 q_k^{(2)} + \varepsilon_2 q_k^{(1)}} \right.
\]

\[
\times \left( \varepsilon_1 q_k^{(2)} \frac{k \xi k_0}{k} + \varepsilon_2 k k_0 \xi^{2} \right) + 2 \xi_1 \varepsilon_2 \frac{q_k^{(1)} + q_k^{(2)}}{\varepsilon_1 q_k^{(2)} + \varepsilon_2 q_k^{(1)}}
\]

\[
\times \left( k_0 q_k^{(2)} \frac{k \xi}{k} + k_0 q_k^{(2)} \frac{(k \xi k_0)}{k_0} \right) - \varepsilon_1 (\xi_2 q_k^{(2)} + \xi_2 q_0^{(2)})
\]

\[
+ 2 q_k^{(2)} (q_k^{(1)} - q_k^{(2)})) \frac{k k_0}{k_0}
\]

\[
(B_2)_{11}^{12}(k, k_0; \xi; \varepsilon_1, \varepsilon_2) = b_{11}(k, k_0; \varepsilon_1, \varepsilon_2) (\varepsilon_1 \varepsilon_2)^{1/2} \left[ -2 \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 q_k^{(2)} + \varepsilon_2 q_k^{(1)}} \right.
\]

\[
\times \left( q_k^{(2)} \frac{k \xi \frac{k \xi k_0}{k}}{k_0} - k k_0 \xi^{2} \right) + 2 \frac{q_k^{(1)} + q_k^{(2)}}{\varepsilon_1 q_k^{(2)} + \varepsilon_2 q_k^{(1)}}
\]

\[
\times \left( \varepsilon_1 k_0 q_k^{(2)} \frac{k \xi}{k} - \varepsilon_2 k_0 q_k^{(2)} \frac{(k \xi k_0)}{k_0} \right) - (\xi_2 q_k^{(2)} - \xi_2 q_0^{(1)})
\]

\[
+ 2 q_k^{(2)} q_0^{(1)} (q_k^{(2)} - q_k^{(1)})) \frac{k k_0}{k_0}
\]

\[
(B_2)_{11}^{21}(k, k_0; \xi; \varepsilon_1, \varepsilon_2) = (B_2)_{11}^{12}(k_0, k; \xi; \varepsilon_1, \varepsilon_2)
\]

\[
(B_2)_{11}^{12}(k, k_0; \xi; \varepsilon_1, \varepsilon_2) = (B_2)_{11}^{11}(k, k_0; \xi; \varepsilon_2, \varepsilon_1).
\]

Let us set

\[
b_{12}(k, k_0; \varepsilon_1, \varepsilon_2) = (\varepsilon_2 - \varepsilon_1)(\varepsilon_1 q_k^{(2)} + \varepsilon_2 q_k^{(1)})^{-1}(q_k^{(2)} + q_0^{(1)})^{-1}.
\]
Then

\[
(B_2)_{12}^{11}(k, k_0; \xi; \epsilon_1, \epsilon_2) = b_{12}(k, k_0; \epsilon_1, \epsilon_2) K_1 \left[ -2 \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 q_k^{(2)} + \epsilon_2 q_k^{(1)}} \frac{\ln N \cdot \xi \times k_0}{k k_0} + 2 \epsilon_2 \frac{q_k^{(2)} + q_k^{(1)}}{\epsilon_1 q_k^{(2)} + \epsilon_2 q_k^{(1)}} \frac{N \cdot \xi \times k_0}{k k_0} - (K_2^2 + q_k^{(2)})^2 \right]
\]

\[
(B_2)_{12}^{11}(k, k_0; \xi; \epsilon_1, \epsilon_2) = (B_2)_{12}^{11}(k, k_0; \xi; \epsilon_1, \epsilon_2) + \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 q_k^{(2)} + \epsilon_2 q_k^{(1)}} K_1 q_k^{(2)} \frac{N \cdot k \times k_0}{kk_0}
\]

\[
(B_2)_{12}^{21}(k, k_0; \xi; \epsilon_1, \epsilon_2) = -(B_2)_{12}^{12}(k, k_0; \xi; \epsilon_2, \epsilon_1) \\
(B_2)_{12}^{22}(k, k_0; \xi; \epsilon_1, \epsilon_2) = -(B_2)_{12}^{11}(k, k_0; \xi; \epsilon_2, \epsilon_1).
\]

Let us set

\[
b_{22}(k, k_0; \epsilon_1, \epsilon_2) = (\epsilon_2 - \epsilon_1)(q_k^{(2)} + q_k^{(1)})^{-1} (q_0^{(2)} + q_0^{(1)})^{-1}.
\]

Then

\[
(B_2)_{22}^{11}(k, k_0; \xi; \epsilon_1, \epsilon_2) = b_{22}(k, k_0; \epsilon_1, \epsilon_2) \cdot (\omega^2/c^2) \left[ -2 \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 q_k^{(2)} + \epsilon_2 q_k^{(1)}} \frac{k \xi \cdot k_0}{k k_0} + q_k^{(2)} + q_0^{(2)} + 2(q_k^{(1)} - q_k^{(2)}) \frac{k k_0}{kk_0} \right]
\]

\[
(B_2)_{22}^{11}(k, k_0; \xi; \epsilon_1, \epsilon_2) = (B_2)_{22}^{12}(k, k_0; \xi; \epsilon_1, \epsilon_2) - \frac{\omega^2}{c^2} \frac{\epsilon_2 - \epsilon_1}{q_k^{(2)} + q_k^{(1)}} \frac{k k_0}{kk_0}
\]

\[
(B_2)_{22}^{21}(k, k_0; \xi; \epsilon_1, \epsilon_2) = (B_2)_{22}^{12}(k_0, k; \xi; \epsilon_1, \epsilon_2)
\]

\[
(B_2)_{22}^{22}(k, k_0; \xi; \epsilon_1, \epsilon_2) = (B_2)_{22}^{11}(k, k_0; \xi; \epsilon_2, \epsilon_1).
\]

And finally

\[
(B_2)_{21}^{11}(k, k_0; \xi; \epsilon_1, \epsilon_2) = -(B_2)_{12}^{11}(k_0, k; \xi; \epsilon_1, \epsilon_2)
\]

\[
(B_2)_{21}^{12}(k, k_0; \xi; \epsilon_1, \epsilon_2) = -(B_2)_{12}^{12}(k_0, k; \xi; \epsilon_1, \epsilon_2)
\]

\[
(B_2)_{21}^{21}(k, k_0; \xi; \epsilon_1, \epsilon_2) = -(B_2)_{12}^{11}(k_0, k; \xi; \epsilon_1, \epsilon_2)
\]

\[
(B_2)_{21}^{22}(k, k_0; \xi; \epsilon_1, \epsilon_2) = -(B_2)_{12}^{22}(k_0, k; \xi; \epsilon_1, \epsilon_2).
\]

Note, that in formulae (D.1)–(D.4) the order of the arguments \(\epsilon_1\) and \(\epsilon_2\) changed.

The formulae for matrices \(\hat{B}, \hat{B}_2\) are simplified significantly in the basis where the matrix \(\hat{V}\) has a diagonal form. Let us introduce the following matrix:

\[
\hat{T}(k) = \begin{pmatrix}
  t_1 & 0 & t_3 & 0 \\
  0 & t_2 & 0 & t_4 \\
  t_3 & 0 & -t_1 & 0 \\
  0 & t_4 & 0 & -t_2
\end{pmatrix}
\]
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where

\[ t_1 = \left( \frac{e_2 q_k^{(1)}}{e_1 q_k^{(2)} + e_2 q_k^{(1)}} \right)^{1/2} \]

\[ t_2 = \left( \frac{q_k^{(1)}}{q_k^{(2)} + q_k^{(1)}} \right)^{1/2} \]

\[ t_3 = \left( \frac{e_1 q_k^{(2)}}{e_1 q_k^{(2)} + e_2 q_k^{(1)}} \right)^{1/2} \]

\[ t_4 = \left( \frac{q_k^{(2)}}{q_k^{(2)} + q_k^{(1)}} \right)^{1/2} \]

It is easy to see that

\[ \hat{T}(k) = \hat{T}^{-1}(k). \]  \hfill (D.5)

The matrix \( \hat{T} \) transforms matrix \( \hat{V} \) to the diagonal form

\[ \hat{T}(k) \hat{V}(k) \hat{T}^{-1}(k) = \hat{T}(k) \hat{V}(k) \hat{T}(k) = \text{diag}(1, 1, -1, -1). \]

We define the transformation of the arbitrary matrix \( \hat{C}(k, k_0) \) into the \( V \)-representation by the formula

\[ \hat{C}_V(k, k_0) = \hat{T}(k) \hat{C}(k, k_0) \hat{T}(k_0). \]  \hfill (D.6)

Because of (D.5), the inverse transform \( \hat{C}_V \rightarrow \hat{C} \) is given by the same formula (D.6). Let us consider the perturbative expression (4.1) for the \( S \)-matrix in the \( V \)-representation and write it in the following form

\[ \hat{S}_V(k, k_0) = \left( \begin{array}{cc} \delta_0 & 0 \\ 0 & -\delta_0 \end{array} \right) \delta(k - k_0) + \hat{p}^{1/2}(k) \cdot \{2i \hat{A}(k, k_0) h(k - k_0)

+ \int \hat{A}_2(k, k_0; \xi) h(k - \xi) h(\xi - k_0) d\xi + \cdots \} \hat{p}^{1/2}(k_0) \]  \hfill (D.7)

where \( \hat{p}^{1/2}(k) \) is the following diagonal matrix

\[ \hat{p}^{1/2}(k) = (e_2 - e_1)^{1/2} \times \text{diag} \left( \begin{array}{c} k \\ (e_1 q_k^{(2)} + e_2 q_k^{(1)})^{1/2} \\ (q_k^{(2)} + q_k^{(1)})^{1/2} \end{array} \right) \]

\[ \frac{(q_k^{(1)} q_k^{(2)})^{1/2}}{(e_1 q_k^{(2)} + e_2 q_k^{(1)})^{1/2}} \frac{(q_k^{(1)} q_k^{(2)})^{1/2}}{(q_k^{(2)} + q_k^{(1)})^{1/2}} \]

Calculation immediately shows that matrix \( \hat{A} \) in (D.7) is

\[ \hat{A}(k, k_0) = \left( \begin{array}{cccc} -1 & 0 & N \cdot k & 0 \\ 0 & k k_0 & N \cdot k \times k_0 & 0 \\ 0 & N \cdot k \times k_0 & k k_0 & 0 \\ 0 & 0 & k k_0 & 0 \end{array} \right) \]

Let us introduce another diagonal matrix:

\[ \hat{g}(k) = \text{diag} \left( \frac{(e_1 e_2)^{1/2}}{k}, \frac{\omega}{c}, 1, 1, 1 \right). \]
With the help of newly introduced notation, one can represent the formula for matrix $\hat{A}_2$ in (D.7) as follows:

$$
\hat{A}_2(k, k_0; \xi) = -2\hat{A}(k, \xi)\hat{p}^{1/2}(\xi) \begin{pmatrix}
\hat{\sigma}_0 & 0 \\
0 & -\hat{\sigma}_0
\end{pmatrix} \hat{p}^{1/2}(\xi) \hat{A}(\xi, k_0)
$$

$$
-\frac{\omega}{c} \hat{g}(k) \cdot \left[ \begin{pmatrix}
0 & \hat{\sigma}_3 \\
\hat{\sigma}_3 & 0
\end{pmatrix} \frac{kk_0}{kk_0} + \begin{pmatrix}
\hat{\sigma}_1 & 0 \\
0 & \hat{\sigma}_1
\end{pmatrix} \frac{N \cdot k \times k_0}{kk_0} \right] \hat{g}(k_0)
$$

$$
+ \frac{\omega}{c} \frac{\xi^2}{\xi^2 + q^2(\xi)} \hat{g}(\xi) \left[ \begin{pmatrix}
0 & \frac{N \cdot k \cdot \xi}{kk_0} & \frac{\xi k_0}{kk_0} & 0 \\
\frac{\xi k_0}{kk_0} & 0 & 0 & 0 \\
\frac{N \cdot k \times \xi}{kk_0} & \frac{\xi k_0}{kk_0} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \hat{g}(\xi) \right].
$$

References


