On Convergence Time and Disturbance Rejection of Super-Twisting Control

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Abstract—Super-twisting algorithm is one of the versions of high-order sliding mode control. The interest to this algorithm is explained by its attractive properties: continuous control input, finite convergence time, disturbance rejection. In this paper, the upper bound of admissible unknown disturbances and low bound of the convergence time are found and shown that both the values can be achieved with any desired accuracy.

Index Terms—Finite time convergence, sliding mode control, super-twisting.

I. INTRODUCTION

SLIDING mode control implies selection of some manifold in system state space and then discontinuous control such that the state trajectories are enforced to this manifold. Then they belong to the manifold and the motion is called sliding mode, governed by reduced-order differential equations with the desired properties. A high-frequency oscillation, called chattering, is the well-known drawback of the sliding mode control. Among great variety of chattering suppression methods [1], so-called high-order sliding mode control [2] has been studied intensively within the last decade.

First of all, let us discuss what type of motion is called the high-order sliding mode in numerous publications. We exclude from our brief discussion twisting and super-twisting algorithms, intended for the second-order cases only. The design idea can be explained easily for the systems governed by equations in the canonical form

\begin{equation}
\dot{x}_i = x_{i+1}, \quad i = 1, \ldots, n-1
\end{equation}

\begin{equation}
\dot{x}_n = f(x) + u, \quad x^T = (x_1, \ldots, x_n)
\end{equation}

The conventional, now called the first-order sliding mode control, studied in the 1960s, implies that the control undergoes discontinuities on the surface \( s(x) = x_n + s_0(x_1, \ldots, x_{n-1}) \).

The transient process consists of two intervals: reaching the surface \( s(x) = 0 \) and then sliding mode governed by

\begin{equation}
\dot{x}_i = x_{i+1}, \quad i = 1, \ldots, n-2
\end{equation}

\begin{equation}
\dot{x}_{n-1} = -s_0(x_1, \ldots, x_{n-1})
\end{equation}

The function \( s_0(x_1, \ldots, x_{n-1}) \) is selected such that the motion in the first-order sliding mode has desired properties, using any method of the control theory arsenal: eigenvalue placement, LQR etc. Another interpretation of this design method: select

\begin{equation}
s_r = x_n - s_0(x_1, \ldots, x_{n-1})
\end{equation}

\begin{equation}
s = s_r^{(r)} + s_0(s_r, s_r, \ldots, s_r^{(r-1)}).
\end{equation}

Of course, if sliding mode is enforced (it can be done, since \( \dot{s} \) depends on discontinuous control), we deal with the conventional first-order sliding mode. Finite time convergence in subspace \((s_r, \dot{s}_r, \ldots, s_r^{(r)})\) can be obtained by a proper choice of function \( s_0 \), then it is called the high order sliding mode [2]. For example, if \( r = 1 \), \( f(x) \leq f_0, u = -M \text{sign}(s), s_0 = a \sqrt{|s_1|} \text{sign}(s_1) \), time derivative \( \dot{s} \) with \( s \) tending to zero is equal to \(-M \text{sign}(s) - f(x) - (a^2)/2 \) and has sign opposite to \( s \) for \( M > f_0 + (a^2)/2 \). It means that sliding mode occurs on \( s = 0 \) governed by equation \( \dot{s}_1 + a \sqrt{|s_1|} \text{sign}(s_1) = 0 \) with finite time convergence of the solution \( s_1(t) = (\sqrt{|s_1(0)|} - (a/2)t) \).

The control in these two cases is labeled as a high order sliding mode control as well. Generalization for \( r \) can be found in [4]: the origin in subspace \((s_r, \dot{s}_r, \ldots, s_r^{(r)})\) can be reached after a finite time intervals without sliding modes of a lower order, but with increasing frequency oscillations. Similar effect without an oscillating component can be obtained using the first-order sliding mode. Indeed the time optimal control for the system \( s_r^{(r+1)} = u, |u| \leq M \) consists of \( r + 1 \) intervals with control \( u = -M \text{sign}({\sqrt{|s_r|}}) \text{sign}(s_r, \ldots, s_r^{(r-1)}). \) The switching surface \( q = 0 \) is reached after the first interval and then \( \dot{q} = 0 \) on the system trajectories [5]. In our case, \( s_r^{(r+1)} = u + f(x) \). For control \( u = -(M + M_1) \text{sign}(q), M_1 > f_0 \) after reaching the switching surface \( \dot{q} = -M_1 \text{sign}(q) + f(t) \). It means that sliding mode occurs on \( q = 0 \) and the state is reduced to zero.
after a finite time interval. To conclude the brief comments on a sliding mode order, note, that the state variable is needed such that its time derivative depends on control for implementation of sliding mode control of the first and higher order. It means that a relative degree is equal to one in all cases.

The two control methods—twisting and super-twisting—can be singled out of high-order sliding mode control design approaches [6]. Both of them are intended for designing the second-order sliding mode, both of them imply an additional integrator in the system input, and the second-order sliding mode appears after a finite time interval without the first-order sliding mode in the reaching stage. If the twisting algorithm needs an additional differentiator (preserving the structural requirement for the common first-order sliding mode), the super-twisting does not need it. The remarkable properties of the super-twisting algorithm should be underlined:

- The sliding manifold is reached after a finite time interval.
- Control input is a continuous (non-Lipschitzian) state function.
- Time derivative of the output is not needed.
- Unknown bounded disturbance is rejected.

These properties explain high level of research activity related to stability analysis, estimation of the convergence time, estimation of the admissible range of disturbances. Different methods were utilized to study the problems: the author of the super-twisting algorithm Levant proved finite time convergence using point-to-point method for state trajectories [2], [7], Lyapunov function was found and analyzed in [8] and [9], Lyapunov function with finite convergence time as a solution to partial differential equation was offered in [10], [11]. In this paper the analysis is performed in time domain directly.

II. PROBLEM STATEMENT

Super-twisting control algorithm is intended for systems with scalar control and one sliding surface \( x = 0, x \) is a continuous differentiable function of the system state. The relative degree of the system is assumed to be equal to one. It means that \( \dot{x} = g(x)u + F(x) \), \( g(x) \neq 0 \), and \( F(x) \) are state functions, \( u \) is control. It is assumed also that time derivative of function \( F(x) \) is bounded. Following the super-twisting methodology an additional integrator with a discontinuous input \( M_0 \text{sign}(x) \) is added and control is designed as a continuous state function \( u = -g^{-1}a\sqrt{x}\text{sign}(x) \) added to the output of the integrator, which yields the motion equations

\[
\begin{align*}
\dot{x} &= -a\sqrt{x}\text{sign}(x) + y \quad (4) \\
\dot{y} &= -M_0\text{sign}(x) + f(t), \\
\end{align*}
\]

\( f(t) \leq f_0, M_0 > f_0, m - M_0 - f_0, M - M_0 + f_0; a, M_0 \), and \( f_0 \) are constant positive parameters. The control (the right-hand side in (4) is a continuous state function, while the input of the additional integrator is discontinuous. As we discussed the discontinuities result in chattering, therefore it is desirable to have the properties listed in the Introduction along with a low level of discontinuity magnitude \( M_0 \).

The methodologies offered in [6], [8], [11] let the authors prove finite time stability and find the upper estimate for the disturbance. It proved to be less than \( 0.5M_0 \). The result of [7] and [9]: for any small \( \mu_0 \) finite time convergence can be reached by increasing gain \( a \). This result should be complemented by the upper estimate of the convergence time as a function of parameter \( a \). In [9], the upper estimate tends to infinity with increasing \( a \).

It is evident that inequality \( m > 0 \) is necessary condition for asymptotic stability. Indeed, if for \( m < 0 \) the values of \( \alpha \) and \( f(t) \), \( |f(t)| \) in (5) have the same signs then \( y \) is constant or tends to infinity. It is evident also that the convergence time cannot be less than \( y(0)/m \)—the minimal possible time needed to reduce \( y(t) \) to zero.

The objective of the paper is to show that for any \( m > 0 \) and \( b > 0 \) there exists \( a^* \) such that the state is reduced to zero in finite time less than \( y(0)/m + \delta \), if \( a > a^* \). The proposed analysis method does need equations of state trajectories or search of an appropriate Lyapunov function. The validity of the above statements will be shown in terms of time solutions to (4), (5) directly.

III. FINITE TIME CONVERGENCE

A. Solution Outline

As it was mentioned on Section II, the convergence time cannot be less than \( y(0)/m \). First the system behavior will be analyzed for \( y(0) \neq 0 \). It will be shown that for any initial value of \( x(0) \) the state component \( y(t) \) will change sign after a finite time interval and this interval tends to \( y(0)/m \) with a tendency to infinity. Two cases (Case 1 and 2) will be treated separately with \( x(0) \) and \( y(0) \) having the same and opposite signs. Then further system behavior with \( y(0) = 0 \) (Case 3) will be studied and shown that the system state will converge to zero with convergence time tending to zero if \( a \) tends to infinity.

B. Case 1 (Fig. 1)

\( x(0) = x_0 > 0, y(0) = -y_0, y_c > 0 \). As follows from (5) \( y(t) \) is decreasing and cannot change sign while \( x(t) \) is positive. \( x(t) \) is decreasing as well (4) and changes sign at \( t = t_1 \), \( x(t_1) = 0 \). The solution to the equation \( \dot{z} = -a\sqrt{z}, z_0 = x_0 \) is of form \( z(t) = (\frac{\sqrt{x_0} - \frac{a}{2}}{a})^2, z(t') < 0, t' = 2(\frac{\sqrt{x_0}}{a}) \). Since \( \dot{x} > 0, t_1 < t' = 2(\frac{\sqrt{x_0}}{a}) \), \( y(t) \) is decreasing for \( 0 < t < t_1 \) and \( 0 > y(t_1) > -y_0 - 2M(\sqrt{x_0}/a) \).

For \( t > t_1 \):

- \( x(t) \) becomes negative and cannot change sign while \( y(t) \) is negative.
- \( y(t) \) starts increasing and \( y(t_1 + T) = 0, T' < (y_0 + 2M(\sqrt{x_0}/a))/m \).
- \( x(t) \) is decreasing, reaches the minimum point at time \( t + t_1', a\sqrt{x(t_1 + t_1')} = y(t_1 + t_1') \) and then starts increasing.

Since \( [y(t_1 + t_1')] < y_0 + 2M(\sqrt{x_0}/a), x(t_1 + T) < ((y_0 + 2M(\sqrt{x_0}/a))^2/a^2), t_1 + T < 2(\sqrt{x_0}/a) + (y_0 + 2M(\sqrt{x_0}/a))/m \).

C. Case 2-A (Fig. 2)

\( x(0) = -x_0 > 0, y(0) = y_0 > 0, a\sqrt{x_0} \geq y_0 \). \( x(t) \) starts decreasing and remains positive until \( y(t) \) changes its sign at time \( T, y(T) = 0, T \leq y_0/m \), \( y(t) \) is decreasing monotonously on \( 0 < t < T \). The right-hand side \( \dot{y} = -a\sqrt{y} + y \) in (4) \( \dot{x} = e \)
for $0 < t < T$ is governed by $\varepsilon = -(\alpha/2x)\varepsilon - M_0 + f$. $\varepsilon(0) < 0$, $\dot{\varepsilon} < 0$ for $\varepsilon = 0$; therefore, $\varepsilon(t)$ is negative and $x(t)$ is monotonously decreasing; hence, $0 < x(T) < x_0$.

**D. Case 2-B (Fig. 3)**

$x(0) = x_0 > 0$, $y(0) = y_0 > 0$, $\alpha \sqrt{x_0} \leq y_0$. Further behavior is similarly to Case 1 for $t > t_1$. $x(t)$ for $t > 0$ in our case is increasing, reaches the maximum point at time $t = t_1$, $a_1 \sqrt{x(t_1)} = y(t_1)$ and then starts decreasing. As in Case 2-A, $x(t)$ is monotonously decreasing for $t > t_1$, $y(t_1) < y_0$ ($y(t)$ is decreasing monotonously while $x(t)$ is positive); therefore, $x(T) < y_0/a^2$, $y(T) = 0$, $T < y_0/m$. The above analysis demonstrated that after a finite time interval:

$$T^* < 2 \frac{\sqrt{x_0}}{a} + \frac{y_0 + 2M_0 \sqrt{x_0}}{m}, y(T^*) = 0.$$  \hspace{1cm} (6)

The upper estimate of $|x_1(T^*)|$ 

$$x_1(T^*) < \max \left( \frac{(y_0 + 2M_0 \sqrt{x_0}/m)^2}{a^2}, x_0 \right)$$  \hspace{1cm} (7)

tends to the initial value $x_0$ with $a$ tending to infinity. The upper estimate of $T^*$ cannot be less than $y_0/m$. As follows from (5), this time is needed for $y$ to change sign, if $x(t)$ and $f(t)$ have the same signs and $f(t) = f_0$.

The objective of the further analysis is to demonstrate that the system state with initial conditions $y(0) = 0$ and arbitrary value of $x(0)$ will be equal to zero after a finite time interval with infinite number of sign alternations in the both state components.

**E. Case 3 (Fig. 4)**

Without loss of generality, assume that $x(0) = -x_0$, $x_0 > 0$. For $t > 0$ $y(t) > 0$ while $x(t)$ is negative, $x(t)$ is increasing and becomes equal to zero at time $t_1$ [again it follows from (5)].

Compare two systems:

$$\dot{x} = -\frac{\alpha}{\sqrt{x}} x + y$$\hspace{1cm} (8)  

$$\dot{z} = -\frac{\alpha}{\sqrt{x_0}} z + mt,\hspace{1cm} z(0) = -x_0$$

$$y(t) = \int_0^t |M_0 + f(\gamma)| d\gamma > mt.$$\hspace{1cm} (9)

Equation (8) is equivalent to (4).

Since $|x(t)|$ is decreasing, $1/\sqrt{|x(t)|} > 1/\sqrt{|x_0(t)|} = k_0$ for $t > 0$. If $x = z < 0$, then $\dot{x} > \dot{z}$. Hence, $x(t) > z(t)$. It
means that $t_1 < t^*_1$, if $z(t^*_1) = 0$. Time instant $t^*_1$ can be found from the solution to (9)

$$z(t) = \frac{m}{a_k \epsilon} \left( \frac{m}{a_k \epsilon} \right)^{1/2} e^{-a_k t},$$

and

$$z(t) = \frac{m}{a} x_0 \left( \frac{t}{\sqrt{x_0}} - 1 \right) + \left( -x_0 + x_0 \frac{m}{a} \right) e^{-a \sqrt{x_0} t} = 0.$$}

Select

Then solution to (10), where $q = q_1$, is the solution to equation $g = \dot{q} = (a - (m/a)) e^{a \dot{q}}$, $q_1 = (2e/(ma))^{1/3}$, $t_1 = q_1^{1/2} + (1/a)^{1/2}$ and finally

$$t_1 < \sqrt{x_0} f(a), \quad h(a) = \left( \frac{2e}{ma} \right)^{1/2} + \frac{1}{a} \quad (12)$$

$$y(t_1) < M t_1 < \sqrt{x_0} M h(a). \quad (13)$$

$$x(t) \text{ changes sign at } t = t_1, \quad y(t) \text{ starts decaying after that and}$$

$$y(t_1 + T) = 0, \quad T < \sqrt{x_0} M h(a)/m$$

$$T_1 < \sqrt{x_0} \left( 1 + \frac{2M}{m} \right) h(a), \quad T_1 = t_1 + T. \quad (14)$$

The result for initial condition $y(0) = 0$ means finite time convergence for $a > a_1$, and (11). Since $\lim_{x \to 0} h(a) = 0$ the convergence time can be reduced to any desired value by increasing parameter $a$.

As the analysis of Cases 1,2 showed, the time interval preceding the instant $T^*$ (6), when $y = 0$ cannot be less than $y(0)/m$ for any values of $a$. The other state variable $x$ at time $T^*$ can be estimated by inequality (7); therefore, $x_0$ in (17) should be replaced by this value. Finally, the main result can be formulated as follows.

For any bounded disturbance $|f(t)| \leq f_0$, $M_0 > f_0$ there exists $a^* > 0$, such that the state convergences to zero after a finite time interval, if $a \geq a^*$; the convergence time tends to $y(0)/m$ with $a \to \infty$. Note, that inequality $M_0 > f_0$ is necessary stability condition and the convergence time cannot be less than $y(0)/m$.

IV. CONCLUSION AND SHORT DISCUSSION

It is common to give an overview of the paper results in the final section, but it was done in the last two sentences of the previous section: finite time convergence for any $m > 0$ and convergence time $y(0)/m + \delta$ (for any $\delta > 0$ can be provided by a proper choice of $a$ and it cannot be done for $m \leq 0$ and $\delta \leq 0$ with any value of $a$).

The above result leaves one problem open. The finite convergence time for $m \to 0$ can be reached asymptotically with $a \to \infty$. Question: can the same effect be obtained with some finite value of parameter $a$?

The second question stems from the general methodology of the super-twisting control. Chattering caused by discontinuities in control is known as the main drawback of sliding mode control. Super-twisting control was offered as a method of enforcing sliding modes with continuous control action [right-hand side in (4)] and as a result reducing chattering effect, caused by discontinuous input in (5). However, this term is not the only source of chattering. The local gain of term $-\sqrt{x} \cdot \text{sign}(x)$ in (4) tends to infinity when approaching the origin in the state space and it results in chattering at the presence of unmodeled dynamics as well. On one hand the desire to reduce chattering effect leads to decreasing the magnitude of the discontinuous input $M_0$ and decreasing $m$. On the other hand parameter $a$ should be increased for small values of $m$ to get finite time convergence which leads to opposite effect—increasing chattering amplitude. Open problem: how to find the tradeoff between values of $M_0$ and $a$ to minimize chattering amplitude at the presence of unmodeled dynamics. Of course it would be of interest to compare this chattering effect with that.
of the traditional first order sliding mode control in the system without the second integrator \((5)\) and with the discontinuous input of the first integrator \([\dot{x} = -M_0 \text{sign}(x) + f(t)]\).

APPENDIX

Find an upper estimate of the solution \(a_{\gamma}\) to the equation

\[
\left( \frac{M}{a} \right)^{\frac{3}{2}} h(a) = \gamma^2 \quad \text{or} \quad \frac{M}{a} \left[ \left( \frac{2e}{ma} \right)^{\frac{1}{3}} + 1 \right] = \gamma.
\]

\[
\left( \frac{2e}{ma} \right)^{\frac{1}{3}} \geq \frac{1}{a} \quad \text{for} \quad a \geq \sqrt[3]{\frac{m}{2e}}
\]

\(a'_{\gamma} > a_{\gamma}\), \(a'_{\gamma}\) is solution to the equation

\[
\frac{M}{a} \left( \frac{2e}{ma} \right)^{\frac{1}{3}} = \gamma. \quad (\text{See Fig. 5})
\]

\(a'_{\gamma} = 2(M^3 e / \gamma^3 m)^{1/4}\). For convergence analysis parameter \(a\) should be selected \(> \max(\sqrt{m}/2e, 2(M^3 e / \gamma^3 m)^{1/4})\).

Fig. 5. Appendix.

REFERENCES


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