Event-triggered consensus control for second-order multi-agent systems

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Abstract: In this study, an event-triggered control strategy is proposed to achieve consensus in a multi-agent system under a directed topology. The proposed control strategy utilises a piecewise continuous control law and an event-triggering function for each agent. The control law only updates at discrete event instants computed using an event-triggering function, which depends on the states of the agents at the current and outdated event instant. This control approach is first applied to a first-order system and is further extended to a second-order system. Simulation examples are presented to illustrate the efficiency of the proposed control strategy.

1 Introduction

Along with rapid development in various engineering fields, both the size and complexity of a practical system have increased substantially. One way to address this issue is to divide the entire system into several sub-systems. Each sub-system is able to individually complete its part of the work and thereby accomplish the whole objective as a system. This cooperation makes the system efficient, flexible and powerful and in some instances the ability of the system can also be enhanced significantly compared with a monolithic system. By considering each sub-system as an agent, collective behaviours such as flocking [1], swarming [2], formation control [3] and consensus reaching [4–6] have been studied. Various tools such as matrix theory, algebraic graph theory and control theory have been explored on controller design, stability criteria analysis and some other aspects [7, 8].

However, the cooperation among agents consumes large quantity of energy on continuous updating and frequent transmissions. With an increase in the complicity of task, the congestion of communication channel, the cost of incremental energy is becoming a major issue. The discrete control strategy associating with the sampled state [9–11] offers an efficient control strategy.

For sake of saving energy, scheduling the controller to update at a variable sampling rate is more efficient and flexible compared with a fixed sampling rate. Such control method is provided by the event-triggered control strategy [12]. The sampling instants are determined by a triggering function, which is defined with respect to the measurement error. When the triggering function overpasses zero, an event is released. The control actuator will keep steady during the interval in between of two instants. This event-triggered control rule can be applied to different control strategies, such as output feedback control [13], model-based control [14], quantisation control [15] and consensus control [16–20]. Centralised event-triggered consensus control in multi-agent system was first proposed in [16]. The whole system shares one event-triggering function that triggers all the agents at the same time. Meanwhile, the centralised event-triggering function depends on the states of all the agents. It is obvious that this centralised scheme is not suitable as the number of agents increases. A decentralised consensus control strategy is therefore proposed to improve the scalability of such a control strategy [21]. The decentralised strategy gives distributed event-triggering function to each agent in the system. In the literature, most of the event-triggering consensus control strategies are applied to a first-order multi-agent system [20, 21]. The second-order consensus problem, which cannot be solved by a simple extension of the first-order case has also attached some attentions. The event-triggered control strategy has been used to solve this problem in the centralised way in [22–24]. The result in literature [18] extended the centralised event-triggering function in [24] into decentralised pattern. Both of the literatures utilised a control law that designed under an exponential relation between the coefficient of the position state and the coefficient of the velocity state. This coupling requirement may constrain the application of the control law. In [19, 20], a sampling model is purposed that the agent samples both its own and its neighbours state at the agent’s event instants. Various methods are used together with the event-triggered control strategy in the consensus analysis of
multi-agent systems, such as matrix analysis and the linear matrix inequality (LMI) method [25]. Different topologies have been considered as well. The multi-agent system under undirected topology is studied in [21]. The directed topology case, which is more difficult to analysis, has attracted a lot of attentions.

In this paper, event-triggered control strategies are proposed for both first-order and second-order dynamics of a kind of multi-agent system under directed topology. The event-triggered strategy is decentralised that each agent has assigned a specific event-triggering function that utilises only neighbours’ information. The decentralised event-triggering function depends on the measurement error, that is, the difference between state at the current time and at the last event instant. An agent is triggered when its event-triggering function exceeds the designed threshold. Once the agent is triggered, its current state information will be sampled and sent to its neighbours and thereby the controller updates the control actuation. Furthermore, an improved event-triggering function is designed for each agent that only uses the information for the agent and the discrete information that sampled and sent by neighbours at their event instants. By applying the improved event-triggering function, the communication energy between agents can also be minimised.

The paper is organised as follows: Section 2 states some preliminaries that are used in this paper. Section 3 addresses the consensus control method for the first-order multi-agent system under directed topology. Besides, the event intervals between two events have been studied in this section as well. Section 4 addresses the consensus problem for the second-order system from applying the event-triggered rule. In Section 5, the improved event-triggering functions are designed for both the first-order and the second-order systems. The effectiveness of the improved event-triggering functions is proved by using Lyapunov method. The event intervals are analysed to show that infinite accumulation of events can be avoided. Simulation results are illustrated in Section 6 and finally a conclusion is presented in Section 7.

Notation. Throughout this paper, $I_N \in \mathbb{R}^{N \times N}$ is an identity matrix and $0_N \in \mathbb{R}^{N \times N}$ is a zero matrix. For a vector $x$, $x'$ denotes the transpose vector and $\|x\|$ denotes the Euclidean norm of the vector. For a matrix $A$, $A'$ denotes the transpose of matrix of $A$ whereas $\|A\|$ represents the matrix 2-norm of the matrix $A$. $N$ denote the set of non-negative integers.

2 Preliminaries

2.1 Graph theory

For a multi-agent system that consists of $N$ agents, the topology of this system can be represented by a graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of all the agents in the system and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of all edges between connected agents. A directed edge in the set $\mathcal{E}$ is represented by $(i, j)$, which implies that agent $j$ can receive information from agent $i$. Thus agent $i$ is a neighbour of agent $j$. The neighbours of agent $j$ can be denoted as a set $N_j = \{i \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. A directed path from agent $j$ to agent $k$ in $G$ is a sequence of ordered edges $(j, j+1), (j+1, j+2), \ldots, (k-1, k) \in \mathcal{E}$. In a directed topology, a directed spanning tree exists if there are directed paths from at least one agent to all the other agents. A directed topology is strongly connected if there always exists a directed path from agents to each of the other agent. An adjacency matrix $A \in \mathbb{R}^{N \times N}$ is defined to describe the topology of the whole system. $A = [a_{ij}]_{N \times N}$, where $a_{ii} = 0$, $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$. $a_{ij} = 0$ otherwise. The in-degree of the agent $i$ is defined as $d_{\text{in}}(i) = \sum_{j \in \mathcal{V}} a_{ij}$. For all of the agents in the system, the in-degree matrix is defined as $\Gamma = \text{diag}(d_{\text{in}}(i))$. The Laplacian matrix of the graph $L \in \mathbb{R}^{N \times N}$ is defined as $L = I - A$.

Lemma 1. For any $x, y \in \mathbb{R}$ and $\epsilon > 0$, one has the following properties:
1. $xy \leq \frac{1}{2}x^2 + \frac{1}{2}y^2$;
2. $(x^2 + y^2) \leq (x + y)^2$, if $xy \geq 0$.

Lemma 2 [7]: If the directed graph $G$ of a multi-agent system contains a directed spanning tree, then the eigenvalues of the Laplacian matrix $L$ satisfy $0 = \lambda_1 \leq \mathcal{R}(\lambda_2) \leq \cdots \leq \mathcal{R}(\lambda_N)$. Where $\lambda_i$ is the $i$th eigenvalues of the Laplacian matrix $L$, $\forall i = 1, \ldots, N$, respectively, and $\mathcal{R}(\lambda_i)$ denotes the real part of the $\lambda_i$.

Lemma 3 [26]: The Laplacian matrix $L$ is irreducible if and only if the topology of the directed system is strongly connected.

Lemma 4 [26]: If the Laplacian matrix $L$ is irreducible, then there exists a positive vector $\xi = [\xi_1, \xi_2, \ldots, \xi_N]'$, $\sum_{i=1}^{N} \xi_i = 1$ such that $\xi$ is the left eigenvector of $L$ associated with eigenvalue 0. Moreover, $L^T \Sigma + \Sigma L$ is semi-positive definite where $\Sigma = \text{diag}[\xi_1, \xi_2, \ldots, \xi_N]$. By taking extractions of square root of each element of $\Sigma$, a matrix is obtained as $\Lambda = \text{diag}[\rho_1, \rho_2, \ldots, \rho_N]$ such that $\rho_i = \sqrt{\xi_i}$, $i = 1, \ldots, N$.

Lemma 5 [27]: The following LMI
\[
\begin{bmatrix}
Q(x) & S(x) \\
S^T(x) & R(x)
\end{bmatrix} \geq 0
\]
is equivalent to either of the following conditions:
1. $Q(x) > 0, R(x) - S(x)'Q^{-1}(x)S(x) \geq 0$;
2. $R(x) > 0, Q(x) - S(x)'R^{-1}(x)S(x) \geq 0$.

3 Consensus control for the first-order multi-agent system

In this section, the event-triggered control strategy for the first-order multi-agent systems is discussed. At first, on the basis of generate control law, two piecewise continuous control laws together with centralised event-triggering function and decentralised event-triggering function are proposed to minimise the frequency of controller updating. The results will show that the agents are able to achieve consensus associating with appropriate designed event-triggering functions. The event intervals under both centralised and decentralised scenarios are analysed to guarantee positive lower bound of the intervals between two events.

3.1 Problem description

Consider a first-order multi-agent system associated with $N$ agents. The dynamics of the agents are described by
\[
\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad i = 1, \ldots, N
\]
where $x_i(t) \in \mathbb{R}$ is the state of the agent $i$, specifically, the position; $x_{i0}(0)$ is the initial position; $u_i(t) \in \mathbb{R}$ is the given control input.

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Lemma 4. When an event occurs as $e(t_k) = x(t_k) - x(t_{k-1}) \geq 0$, the measurement error is set to zero and it will grow until the event-triggering function overpasses zero.

Definition 1: The multi-agent system listed above is said to achieve consensus if, for any initial condition
\[
\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \quad i = 1, \ldots, N
\]
A widely used control law for this system is
\[
u_i(t) = -\sum_{j \in N_i}(x_i(t) - x_j(t))
\]
This continuous updating control law can drive the system to achieve consensus while consuming unnecessary transmission energy and occupying communication channel. An event-triggered control strategy that minimises the update frequency thereby minimising the energy consumption is proposed for the system. By using the event-triggered control strategy, the controllers of agents update at discrete event instants, which are calculated by predesigned event-triggering function.

3.2 Event-triggered consensus control of the first-order multi-agent system

At first, a centralised event-triggering control strategy for the first-order system (1) will be studied. Latter, on the basis of the centralised event-triggered strategy, a decentralised event-triggered strategy is proposed.

3.2.1 Centralised event-triggered control strategy:

From this strategy, the system only has one global event-triggering function. The event instants are indexed by, $k = 0, 1, \ldots$, such that $t_k$ denotes the $k$th event of the system. At the event instants, all the agents in the system will synchronously send their states to neighbours and update the control law with the received states. As the controllers of agents update at discrete event instants, which are calculated by predesigned event-triggering function.

By using the event-triggered strategy, the control input for agent $i$ is designed as
\[
u_i(t) = -\alpha \sum_{j \in N_i}(x_i(t_k) - x_j(t_k)), \quad t \in [t_k, t_{k+1}) \tag{2}
\]
where $\alpha$ is a positive constant. $x_i(t_k), i = 1, \ldots, N$ is the state of agent $i$ at the $k$th event instants. In order to synthesise all the agents, we use $x(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^\top$ to denote the states of agents. From the definition, $e(t)$ equals to 0 when an event occurs as $e(t_k) = x(t_k) - x(t_{k-1}) = 0$. Before stating the centralised event-triggering function, we first define the measurement error $e(t) = x(t_k) - x(t)$. The dynamics of agents can be written into
\[
\dot{x}(t) = -L(x(t) + e(t)) \tag{3}
\]
For the first-order multi-agent system (1), the centralised event-triggering function is defined as
\[
f_i(t) = \|e(t)\| - \alpha \frac{\|\Lambda Lx(t)\|}{\|\Lambda\|}, \quad 0 < \alpha < 1 \tag{4}
\]
where $\Lambda$ is the same diagonal matrix as described in Lemma 4. When $f_i(t)$ reaches zero, all the agents will be triggered. The measurement error is set to zero and it will grow until the event-triggering function overpasses zero again.

Theorem 1: Considering the first-order system (1) under directed strongly connected topology, the event-triggered control law (2) is utilised and the event-triggering function (4) is enforced to satisfy $f_i(t) \geq 0$. Then for any initial condition, the consensus of the system (1) can be reached.

Proof: To analyse the effectiveness of the proposed strategy, a candidate Lyapunov function is considered
\[
V(t) = \frac{1}{4}V(t)^\top (L^\top \Xi + \Xi L)x(t) \tag{5}
\]
where $\Xi \in \mathbb{R}^{N \times N}$ is the diagonal matrix described in Lemma 4. From Lemma 4, $L^\top \Xi + \Xi L$ is semi-positive definite and it equals to zero only when $x = a \mathbf{1}_N$ for $a$ is a constant. Consequently, $V(t) \geq 0$ and $V(t) = 0$ if and only if the system has achieved consensus. Along with the trajectories of the state as described in (3), the time derivative of the Lyapunov function is
\[
\dot{V} = x(t)^\top L^\top \Xi(-\alpha(Lx(t) + Le(t))) \\
\leq -\alpha \|\Lambda Lx\|^2 + \alpha \|\Lambda Lx\| \|\Lambda L\| \leq 0
\]
Since the event-triggering function is enforced to be greater than zero, one can obtain that $\|\Lambda Lx\| \leq \|\Lambda L\| \|x\| \leq \sigma \|\Lambda Lx\|$. Hence the time derivative of the Lyapunov function turns into
\[
\dot{V} \leq -\sigma \|\Lambda Lx\|^2 (\sigma - 1)
\]
As $0 < \sigma < 1$, one has $\dot{V} \leq 0$ and $\dot{V} = 0$ if and only if the consensus is achieved. The centralised event-triggered strategy is proved effective to solve the consensus problem of the system (1).

3.2.2 Decentralised event-triggered control strategy:

It is obvious that the decentralised event-triggering function requires the states of all the agents. This disadvantage causes unnecessary communication cost. Hence in order to improve the performance of the event-triggered strategy, a decentralised event-triggering strategy is proposed in the following. The decentralised event-triggering strategy assigns each agent a distributed event-triggering function such that only neighbours’ information are required. By applying the decentralised event-triggering function, each agent only samples its state when it triggered.

The $k$th event of agent $i$ is denoted by the $t'_{ik}$, as $t'_{ik}, t'_1, \ldots$ denotes the event sequence of the agent. Note that since agents trigger asynchronously, each agent has their own event sequence. The measurement error of agent $i$ is defined as $e_i(t) = x(t_k) - x(t_{k-1})$. It is clear that $e_i(t) = 0$ when $t = t'_{ik}$. The control input of agent $i$ is designed as
\[
u_i(t) = -\alpha \sum_{j \in N_i}(x_i(t'_k) - x_j(t'_k)), \quad t \in [t'_{ik}, t'_{i(k+1)}] \tag{6}
\]
where $t'_{ik} \triangleq \arg \min_{t \in \mathbb{R}}(t - t')$ denotes the last event instant of agent $j$. From (6), the controller of agent $i$ will update at both its own event instants $(t'_{ik}, t'_1, \ldots)$ and the
neighbours’ event instants \((t_i', t_i'', \ldots)\). As mentioned before, \(t_i'\) is the instant that agent \(i\)’s event-triggering function overpasses 0, where the function is defined as follow

\[
\hat{f}_i(t) = \|e_i(t)\| - \frac{\sigma}{\|\Lambda L\|}, \quad 0 < \sigma < 1
\]

\(\hat{e}_i\) is the \(i\)th elements of vector \(\hat{e}\) where \(\hat{e} = [\hat{e}_1, \ldots, \hat{e}_n]^\top \triangleq Lx\). According to the definition of \(\hat{e}_i\), one has \(\hat{e}_i = \sum_{j \in N_i} (x_i - x_j)\), which means that \(\hat{e}_i\) only involves local information. At the \(k\)th triggering instant of agent \(i\), the agent will sample its state, and then update its own controller by using the newly sampled state and send the state to its neighbours. The neighbours will therefore update their controllers with the received state as well, but they will not sample their states at this moment unless their event-triggering functions exceed zero.

Then combining (1) and (6) and the description of the measurement error, the dynamics of agents turns into

\[
\dot{x}_i(t) = -\alpha L(x_i(t) + e_i(t))
\]

By synthesising all the agents, we also have

\[
\dot{x}(t) = -\alpha L(x(t) + e(t))
\]

**Theorem 2:** Considering the first-order multi-agent system described by (1) under directed strongly connected topology. By using the piecewise-continuous control law (6), the system can achieve consensus, for any initial condition, if the distributed event-triggering function (7) is enforced to satisfy \(f_i(t) \leq 0\).

**Proof:** Since the distributed event-triggering function in enforced to satisfy \(f_i(t) \leq 0\), we have \(\|e_i\| \leq \frac{\sigma}{\|\Lambda L\|}\|\Lambda L\|\|\hat{e}_i\|\|\Lambda L\|\). It follows that \(\|e\| \leq \sigma \|\Lambda L\|\|\hat{e}\|\|\Lambda L\|\). We also have \(\|\Lambda L e\| \leq \|\Lambda L\|\|e\| \leq \sigma \|\Lambda L\|\). Using the same Lyapunov function that used in Theorem 1, one can obtain \(V \leq \alpha \|\Lambda L\|^2 (\sigma - 1)\). Consequently one can easily jump to the result that the control law (6) is capable to drive the system reaching consensus by using the decentralised event-triggering function (7).

### 3.3 Event interval analysis of the first-order multi-agent system

The purpose of the event-triggered strategy is energy saving, hence the accumulation of events should be avoided. The intervals between events that determined by the centralised event-triggering function is first analysed.

#### 3.3.1 Event intervals under the centralised strategy

According to the definition of the measurement error, we have \(e(t) = -\dot{x}(t) = -Lx(t), \forall t \in [t_k, t_{k+1})\) and \(e(t_k) = 0\). Besides, from the mechanism of event-triggering strategy, the next event occurs when \(e(t) = \sigma \|\Lambda L\|\|\Lambda L\|\|\hat{e}\|\|\Lambda L\|\|\hat{e}\|\|\Lambda L\|\|\hat{e}\|\|\Lambda L\|\|\hat{e}\|\|\Lambda L\|\|\hat{e}\|\|\Lambda L\|\|\hat{e}\|

\[
\|e\| \leq \|\hat{e}\| \|\Lambda L\| \leq \sqrt{\|\hat{e}\|\|\Lambda L\|}
\]

Hence the time \(t_k = \|e\|/\|\hat{e}\|\|\Lambda L\|\) reaches \(\|\hat{e}\|/\|\Lambda L\|\) longer than \(\sqrt{\|\hat{e}\|\|\Lambda L\|}\) spends. Thereby one has that \(t_e \geq \tau^*\),
where \( \tau^* \) is the time \( \|e_i/\|\Lambda \hat{x}_i \| \) grows from 0 to \( \sigma/\sqrt{N} \|\Lambda L \| \). From (8), one can obtain

\[
\tau^* = \frac{\phi(\tau^*, 0)}{\sqrt{N}\alpha \|\Lambda \| (1 + \|\Lambda L \|\phi(\tau^*, 0))}
\]

\[
= \frac{\sigma}{(\sqrt{N} + \sigma)\alpha \|\Lambda L \| \|\Lambda^{-1} \|}
\]

It leads to the that the minimal interval between two event instants of agent \( q \) is

\[
\tau_q = \frac{\sigma}{(\sqrt{N} + \sigma)\alpha \|\Lambda L \| \|\Lambda^{-1} \|}
\]

Remark 1: The thresholds of event-triggering functions can be different. As listed in (7), the event-triggering function of agent \( i \) is that \( f_i(t) \triangleq \|\rho_i \hat{x}_i \| \|\Lambda L \| \). The derivative of the Lyapunov function (5) has \( \dot{V} \leq -\alpha \|\Lambda x \| (\|\Lambda L \| - \|\Lambda L \|\|e_i \|) \). In this paper, the upper threshold of the event-triggering function of agent \( i \) is set to 0, that is, \( f_i(t) \leq 0 \), hence we have \( \dot{V} \leq \alpha \|\Lambda L \|^2(\sigma - 1) < 0 \), which guarantees the consensus achievement. Similarly, we can obtain the result if the thresholds of the event-triggerings are different from each other. Let us choose \( -\beta_i \leq 0 \), to be the threshold of the event-triggering function \( f_i(t) \), that is, \( f_i(t) \leq -\beta_i \). Then the derivative of the Lyapunov function turns into \( \dot{V} \leq \alpha \|\Lambda L \| (-\beta_{\text{min}} \|\Lambda L \| (1 - \sigma \|\Lambda L \|) < 0 \), where \( \beta_{\text{min}} = \min\{\beta_i, i = 1, \ldots, N\} \). It is clear that the system can reach consensus by utilising event-triggering functions with different non-positive thresholds. (If \( -\beta_i > 0 \), the convergence of the system may not be guaranteed.) It is worth mentioning that a negative threshold lowers the upper bound of the measurement error compare the zero threshold. Moreover, if \( -\beta_i < 0 \), from the description of the event-triggering function, the agent \( i \) will be triggered once \( \|\rho_i \hat{x}_i \| \|\Lambda L \| \) is smaller than \( \beta_i \). For the worst case, the accumulation of events may exists. Consequently the performance of the event-triggered strategy is restrained by the negative threshold. In this paper, for simplicity and for best performance of the event-triggered strategy, we set the thresholds of all the event-triggering functions \( f_i(t) \) to be zero.

4 Consensus control for the second-order multi-agent system

Since the agent dynamics of the second-order system is more complicated, the consensus controller design for the second-order multi-agent system is more difficult comparing to the first-order case.

4.1 Problem description

Considering a second-order multi-agent system under directed strongly connected topology, the dynamics of the agents are described as

\[
\begin{aligned}
\dot{x}_i(t) &= v_i(t) \\
\ddot{v}_i(t) &= u_i(t), \quad i = 1, \ldots, N
\end{aligned}
\]

(9)

where \( x_i(t) \in \mathbb{R} \), \( v_i(t) \in \mathbb{R} \) and \( u_i(t) \in \mathbb{R} \) are the position state, the velocity state and the control input of agent \( i \), respectively. All initial values of position states and velocity states belong to \( \mathbb{R} \).

Definition 2: The consensus of the second-order multi-agent system (9) is achieved if, for any initial conditions

\[
\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \quad i, j = 1, \ldots, N
\]

For the second-order multi-agent system (9), the generate control model that utilising both the position and the velocity states is

\[
\begin{aligned}
u_i(t) &= -\alpha \sum_{j \in \mathcal{N}_i} (x_i(t) - x_j(t)) \\
&\quad - \beta \sum_{j \in \mathcal{N}_i} (v_i(t) - v_j(t))
\end{aligned}
\]

where \( \alpha, \beta \) are independent positive constant coefficients. It is clear that this control law also requires continuous updating and communication.

4.2 Centralised event-triggered consensus strategy of the second-order multi-agent system

The segmental continuous control law for the second-order multi-agent system has a similar form to the first-order control law. In stead of continuous states, both the position and

\[
\begin{aligned}
v_i(t) &= \dot{x}_i(t) = \dot{v}_i(t) = u_i(t) \\
&\quad -\alpha \sum_{j \in \mathcal{N}_i} (x_i(t) - x_j(t)) \\
&\quad - \beta \sum_{j \in \mathcal{N}_i} (v_i(t) - v_j(t))
\end{aligned}
\]

where \( t \in [t_k, t_{k+1}] \) (10)

where \( t_k \) is the latest event instant of the system. \( x_i(t), v_i(t), i = 1, \ldots, N \) are the position and velocity states of agent \( i \) at the event instant, respectively. \( \alpha, \beta \) are independent positive constant coefficients.

According to the proposed control law (10), the dynamics of the agent \( i \) can be written into

\[
\begin{aligned}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= -\alpha \sum_{j \in \mathcal{N}_i} (x_i(t) - x_j(t)) \\
&\quad - \beta \sum_{j \in \mathcal{N}_i} (v_i(t) - v_j(t))
\end{aligned}
\]

where \( t \in [t_k, t_{k+1}] \). The position and the velocity measurement errors are defined separately that \( e_i(t) = [e^x_i(t), e^y_i(t), \ldots, e^n_i(t)] \triangleq (x(t) - x_i(t)), e_i(t) = [e^x_i(t), e^y_i(t), \ldots, e^n_i(t)] \triangleq (v(t) - v_i(t)) \). Then the uniform expression of the dynamics of the system (9) is obtained by using \( x(t) = [x_1(t), x_2(t), \ldots, x_N(t)] \), \( v(t) = [v_1(t), v_2(t), \ldots, v_N(t)] \)

\[
\begin{aligned}
\dot{x}_i(t) &= v(t) \\
\dot{v}(t) &= -\alpha L x(t) + e_i(t) - \beta L (v(t) + e_i(t))
\end{aligned}
\]

Given two vectors as \( y = [x^T, v^T]^T \) and \( e = [e^x_i, e^y_i]^T \), and then the dynamics of the system is as below

\[
\dot{y}(t) = \begin{bmatrix}
0 & I_N \\
-\alpha L & -\beta L
\end{bmatrix} y + \begin{bmatrix}
0 \\
-\alpha L & -\beta L
\end{bmatrix} e
\]

\[(11)\]

Similar to the definition of \( \hat{x} \), we define \( \hat{v} = [\hat{v}_1, \ldots, \hat{v}_N] \triangleq \hat{L} v, \quad \hat{y} = (I_2 \otimes L) \hat{y} \) and \( \hat{y} = (I_2 \otimes \Lambda) \hat{y} \). The centralised
It is easy to acquire that states of the agents achieve consensus. From Lemma 4, $c$ achieve consensus. Similar to Lemma 6 in [26], a positive $\alpha$ is given by

$$V = \frac{1}{2} \begin{bmatrix} rL^T \Xi L & k_i L^T \Xi \\ k_i \Xi L & \frac{1}{2} \eta (L^T \Xi + \Xi L) \end{bmatrix} y$$

where $r, k_i, \eta$ are positive constants.

**Lemma 6**: The Lyapunov function (13) is qualified for the system (9) if the following condition holds:

$$r \geq \frac{2k_i^2}{c\eta} \xi_{\text{max}}$$

where $c$ is a positive constant; $\xi_{\text{max}}$ is the maximum element of the vector $\xi$ that mentioned in Lemma 4.

**Proof**: It is easy to obtain that the Lyapunov function equals to zero if and only if the consensus is reached. Now the case that the consensus is not reached is discussed. From (13), it follows:

$$2V = rx^T L^T \Xi Lx + k_i v^T L^T \Xi v
+ k_i v^T L^T \Xi Lx + v^T \frac{1}{2} \eta (L^T \Xi + \Xi L)v$$

It is easy to acquire that $rx^T L^T \Xi Lx = r\xi^T \hat{\xi} = r \sum_{i=1}^{N} \xi_i^2 \geq 0$. It equals to 0 only when the position states of the agents achieve consensus. From Lemma 4, (1/2)$\eta (L^T \Xi + \Xi L)$ is semi-positive definite and $v^T (1/2) \eta (L^T \Xi + \Xi L)v$ equals to 0 only when the velocity states of the agents achieve consensus. Similar to Lemma 6 in [26], a positive constant $c$ is defined such that

$$c \triangleq \min_{L \neq 0, v \neq 0} \frac{v^T (L^T \Xi + \Xi L)v}{v^T v}$$

Then one can find a diagonal matrix $\Omega = \text{diag} \{\omega_1, \ldots, \omega_N\}$ associating with $L^T \Xi + \Xi L$ that $\omega_i$ is the $i$th eigenvalue of $L^T \Xi + \Xi L$, $\forall i = 1, \ldots, N$. Then there exists a unitary matrix $P = [p_1, p_2, \ldots, p_N]$ such that $L^T \Xi + \Xi L = P \Omega P^T$. It is clear that $p_i$ is the eigenvector of $L^T \Xi + \Xi L$ associated with eigenvalue $\omega_i$. Since $L^T \Xi + \Xi L$ is semi-positive definite, it has one zero eigenvalue. Without lose of generality, assume $\omega_1 = 0$. It follows that $p_1 = 1_N$. By using notation $\xi = [\xi_1, \ldots, \xi_N] \triangleq P^T v$, one has

$$c = \min_{L \neq 0, v \neq 0} \frac{v^T (L^T \Xi + \Xi L)v}{v^T v} = \min_{L \neq 0, v \neq 0} \frac{v^T P \Omega P^T v}{v^T v} = \min_{L \neq 0, \xi \neq 0} \sum_{i=1}^{N} \omega_i \xi_i^2 \geq 0$$

One have that $c = 0$ if and only if $LP\xi \neq 0, \xi^T \xi = 1$, and $\xi_1 = \cdots = \xi_N = 0$, which implies that $\xi_1 = 1$, or $-1$. Without loss of generality, use $\xi_1 = 1$ for further analysis. In that case, $LP\xi = L[p_1, p_2, \ldots, p_N][1 0 \cdots 0]^T = Lp_1 = L1_N = 0$, which violates the condition that $LP\xi \neq 0$. Therefore $c > 0$ if the consensus is not achieved yet. From the definition of $c$, it follows that $cv^T v \leq v^T (L^T \Xi + \Xi L)v$.

From above, the Lyapunov function has

$$V \geq \frac{1}{2} v^T \begin{bmatrix} rl^T \Xi L & k_i L^T \Xi \\ k_i \Xi L & \frac{1}{2} \eta c \end{bmatrix} y$$

From Lemma 5,

$$\begin{bmatrix} rl^T \Xi L & k_i L^T \Xi \\ k_i \Xi L & \frac{1}{2} \eta c \end{bmatrix}$$

is semi-positive definite if

$$\frac{1}{2} \eta c > 0 \text{ and } rl^T \Xi L - k_i L^T \Xi \left(\frac{1}{2} \eta c \right)^{-1}
\geq 0$$

One have that if $r \geq (2k_i^2/c\eta)\xi_{\text{max}}$, then the Lyapunov function (15) is positive definite. From the condition that $r \geq (2k_i^2/c\eta)\xi_{\text{max}}$, the Lyapunov function (15) is semi-positive definite and it equals to zero if $r$ and $\eta$ only if the consensus is reached. From the above discussion, the candidate Lyapunov function (13) is positive if the consensus is not reached and it equals to 0 if and only if the consensus is achieved. Hence the Lyapunov function (13) is qualified for the second-order system.

**Theorem 4**: Consider the second-order multi-agent system as described by (9) with piecewise-continuous control law (10). The system (9) is able to achieve consensus, for any initial condition, if the centralised event-triggering function (12) is enforced to satisfy $f_i(t) \leq 0$ and the following conditions are satisfied:

$$\beta k_i + \alpha \eta \geq \frac{2k_i^2}{c\eta} \xi_{\text{max}}$$

$$k_i L^T \Xi L - \frac{1}{2} (L^T \Xi + \Xi L) \geq 0$$

**Proof**: The candidate Lyapunov function (13) can be represented as

$$V = \frac{1}{2} v^T \begin{bmatrix} rl^T \Xi L & k_i L^T \Xi \\ k_i \Xi L & \frac{1}{2} \eta L^T \Xi \end{bmatrix} y$$

Hence the derivative of the Lyapunov function along the trajectory (11) is

$$\dot{V} = y^T \begin{bmatrix} rl^T \Xi L & k_i L^T \Xi \\ k_i \Xi L & \eta L^T \Xi \end{bmatrix} \dot{y}$$

$$= y^T \begin{bmatrix} rl^T \Xi L & k_i L^T \Xi \\ k_i \Xi L & \eta L^T \Xi \end{bmatrix} y$$

$$\times \left[ \begin{bmatrix} 0 & I_N \\ -\alpha L & -\beta L \end{bmatrix} y + \begin{bmatrix} 0 & 0 \\ -\alpha L & -\beta L \end{bmatrix} e \right]$$
The consensus of the second-order multi-agent system is decentralised strategy of the first-order system, the event-triggered function. As same as the triggered control strategy is proposed to solve this problem.

\[\begin{align*}
y(t) &= y^T \left[ -\alpha k L^T \Xi L \quad r L^T \Xi L - \beta k L^T \Xi L \right] y \\
&+ y^T \left[ r L^T \Xi L \quad k L^T \Xi L \quad \eta L^T \Xi L \right] \left[ \begin{array}{c}
0 \\
0 \\
-\alpha L - \beta L 
\end{array} \right] e 
\end{align*}\]

The former part of (18) has

\[\begin{align*}
y^T \left[ -\alpha k L^T \Xi L \quad r L^T \Xi L - \beta k L^T \Xi L \right] y \\
&= -\alpha k x^T L^T \Xi L x + k x^T L^T \Xi L - \beta k L^T \Xi L \\
&+ x^T \left( r - \beta k L^T \Xi L \right) L x
\end{align*}\]

From choosing \(r = \beta k - \alpha \eta\) and the condition (17), the expression (19) is smaller or equal to

\[\begin{align*}
-\alpha k x^T L^T \Xi L x - (\beta \eta - k) v^T L^T \Xi L v
\end{align*}\]

Meanwhile, the latter part of (18) has

\[\begin{align*}
y^T \left[ r L^T \Xi L \quad k L^T \Xi L \quad \eta L^T \Xi L \right] \left[ \begin{array}{c}
0 \\
0 \\
-\alpha L - \beta L 
\end{array} \right] e \\
&= y^T \left[ -\alpha k L^T \Xi L - \beta k L^T \Xi L \right] e \\
&= (I_2 \otimes \Delta L) e \left[ -k_1 \alpha I_N \quad \beta I_N \right] \left( I_2 \otimes \Delta L \right) e
\end{align*}\]

Therefore the derivative of the Lyapunov function has

\[\dot{V} \leq \tilde{y}^T \left[ -\alpha k - \beta k \right] \left( I_2 \otimes \tilde{y} \right) \left[ I_2 \otimes \tilde{y} \right] \left[ -\alpha k - \beta k \right] \left( I_2 \otimes \tilde{y} \right) \left[ I_2 \otimes \tilde{y} \right] \left[ -\alpha k - \beta k \right] \left( I_2 \otimes \tilde{y} \right) \left[ I_2 \otimes \tilde{y} \right]
\]

Since the centralised event-triggering function (12) is enforced to satisfy \(f_i(t) < 0\), one can obtain that \(\dot{V}(t) < 0\). The consensus of the second-order multi-agent system (9) is achieved.

The centralised event-triggered strategy also requires global states. In the following, a decentralised event-triggered control strategy is proposed to solve this problem.

### 4.3 Decentralised event-triggered consensus control law of the second-order multi-agent system

The decentralised event-triggered strategy assigns each agent a distributed event-triggering function. As same as the decentralised strategy of the first-order system, the event instants of agent \(i\) are determined only by its own distributed event-triggering function. We still use \(t_i \) to denote the \(k\)th event instant of agent \(i\). The position measurement error and the velocity measurement are defined as \(e_i(t) = x_i(t) - x_i(t_i)\) and \(e_i(t) = v_i(t) - v_i(t_i)\), \(t \in [t_i, t_{i+1})\), respectively.

The decentralised control input of agent \(i\) is designed as

\[u_i(t) = -\alpha \sum_{j \neq i} (x_i(t_j) - x_j(t_j)) - \beta \sum_{j \neq i} (v_i(t_j) - v_j(t_j))\]

where, \(t \in [t_i, t_{i+1})\), \(t_i\) and \(t_{i+1}\) are the latest event instants of agent \(i\) and agent \(j\), \(x_i(t_i), v_i(t_i), x_j(t_i)\) and \(v_j(t_i)\) are the position and velocity states of agent \(i\) and agent \(j\) at their latest event instant respectively. \(\alpha, \beta\) are independent positive constant coefficients. According to the proposed control law (22) and the expression of measurement errors, the dynamics of the agent \(i\) can be written into

\[\begin{align*}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= -\alpha \sum_{j \neq i} (x_i(t_j) - x_j(t_j)) - \beta \sum_{j \neq i} (v_i(t_j) - v_j(t_j))
\end{align*}\]

One can obtain that the system dynamics can be represented as same as stated in (11). The distributed event-triggering function \(f_i(t)\) corresponding to agent \(i\) is

\[f_i(t) = \frac{\|\Delta L\|^2 (k_1 + \eta)}{2k} \left[ (\alpha \|e_i\|^2 + \beta \|e'_i\|^2) \right] - \sigma \left[ ak_1 - \frac{(\alpha + \beta)k_1 k_2}{2} \right] \|\tilde{x}\|^2 - \sigma \left[ \beta \eta - k_2 - \frac{(\alpha + \beta)\eta}{2} \right] \|\tilde{v}\|^2
\]

where \(0 < \sigma < 1, k, k_1, k_2, \eta\) are positive constants. \(\tilde{x}\) and \(\tilde{v}\) are the \(i\)th element of vector \(\tilde{x}\) and \(\tilde{v}\), respectively, where \(\tilde{x} \triangleq \Delta L x, \tilde{v} \triangleq \Delta L v\). Similar to the decentralised case of the first-order system, the element \(\tilde{x}\) and \(\tilde{v}\) only involves local information.

**Theorem 5:** Consider the second-order multi-agent system as described by (9) with piecewise-continuous control law (22). The system (9) is able to achieve consensus, for any initial condition, if the event-triggering function (23) is enforced to satisfy \(f_i(t) \leq 0\) while (16) and (17) and the following conditions are satisfied:

\[ak_1 - \frac{(\alpha + \beta)k_1 k_2}{2} > 0\]

\[\beta \eta - k_2 - \frac{(\alpha + \beta)\eta}{2} > 0\]
Proof: The candidate Lyapunov function (13) is considered. From (18) and (20), the time derivative of the Lyapunov function along the trajectory of (11) turns into

\[ \dot{V} \leq -ak_1 \dot{x} L \| \dot{x} \| + (\alpha + \beta) \| e \| \| \dot{e} \| + (\beta \eta - k_2) \| \dot{\psi} \| \]

One can write the latter part of (26) by using Lemma 1

\[ y^\top \left[ rL^\top \Xi L k_1 L^\top \Xi \Xi \Xi + (\alpha + \beta) \| e \| \| e \| \| \dot{e} \| + (\beta \eta - k_2) \| \dot{\psi} \| \right] e \]

where \( k \) is a positive constant. Therefore the derivative of the Lyapunov function has

\[ \dot{V} \leq \frac{(k_1 + \eta) \| A \| L^2}{2k} (\alpha \| e \| + \beta \| e \|)^2 \]

If the event-triggering function (23) can be enforced to be no greater than zero, then one has

\[ \| A \| L^2 (k_1 + \eta) (\alpha \| e \| + \beta \| e \|)^2 \leq \sigma (ak_1 - \frac{(\alpha + \beta) k k_1}{2}) \| \dot{\psi} \| \]

Hence the derivative of the Lyapunov function is negative unless the consensus of the agents is achieved. It comes to the conclusion that the agents in the second-order system can achieve consensus with the control law (22) and the event-triggering function (23).  

4.4 Event interval analysis of the second-order multi-agent system

For the second-order case, we use the similar method that used in the first-order scenario to analyse the event intervals. In order to give the event interval of the decentralised control approach, the event interval of the centralised strategy is discussed at first.

4.4.1 Event intervals under the centralised strategy: Since all the agent trigger at the same event instant, at event instant \( t_0 \), one has \( e(t_0) = 0 \). Meanwhile, the second-order system will not be triggered until the centralised event-triggering function reaches zero, which means that the following equation holds at the event instant:

\[ \| e \| \| \dot{\psi} \| = \frac{\sigma \alpha_{\min}}{\| A \| L \sqrt{(k_1^2 + \eta^2)} (\alpha^2 + \beta^2)} \]

Hence the event interval between two events is the period that \( \| e \| / \| \dot{\psi} \| \) changes from 0 to

\[ \frac{\sigma \alpha_{\min}}{\| A \| L \sqrt{(k_1^2 + \eta^2)} (\alpha^2 + \beta^2)} \]

The derivative of \( \| e \| / \| \dot{\psi} \| \) has

\[ \frac{d}{dt} \| e \| / \| \dot{\psi} \| \leq \frac{\| \dot{e} \| / \| \dot{\psi} \|}{\| \dot{\psi} \|} \]

One can obtain that

\[ \| \dot{\psi} \| = \left[ \begin{array}{cc} \Lambda & 0 \\ 0 & \Lambda \end{array} \right] \left[ \begin{array}{c} -\alpha LL(x + e_x) - \beta LL(v + e_v) \\ 0 \end{array} \right] \]

\[ \leq \left[ \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right] \left[ \begin{array}{c} \| e_x \| + \| e_v \| \\ \| \dot{e}_x \| + \| \dot{e}_v \| \end{array} \right] \]

\[ \leq \left( 1 + \sqrt{\alpha^2 + \beta^2} \| A \| L \right) \| \dot{\psi} \| \]

Similarly, one has

\[ \| \dot{e} \| = \| \dot{\psi} \| = \left[ \begin{array}{c} \alpha L(x + e_x) + \beta L(v + e_v) \\ \alpha L(x + e_x) + \beta L(v + e_v) \end{array} \right] \]

By combing the bound of \( \| e \| / \| \dot{\psi} \| \) and \( \| \dot{e} \| / \| \dot{\psi} \| \) turns into

\[ \| e \| / \| \dot{\psi} \| \leq \left( 1 + \sqrt{\alpha^2 + \beta^2} \right) \| \dot{\psi} \| + \| A \| L \| e \| \]

where \( \gamma = \max \{1, 1/\| A \| L \} \). In the following analysis, we use \( \Gamma \) to denote \( \| A \|^{-1} \left( 1 + \sqrt{\alpha^2 + \beta^2} \right) \) for brevity. By using \( \psi \) to denote \( \| e \| / \| \dot{\psi} \| \), one has \( \psi \leq (1 + \| A \| L) \psi \Gamma \). Consequently \( \psi \) satisfies the bound \( \psi \leq \hat{\psi}(t, \theta_0, \vartheta) \), where \( \hat{\psi}(t, \theta_0) \) is the solution of \( \dot{\hat{\psi}} = (1 + \| A \| L)^T \), \( \hat{\psi}(0, \theta_0, \vartheta) = \vartheta \).
Hence the event interval of the centralised control strategy is positive lower bounded by
\[
\sigma \alpha_{\text{min}} \frac{\|\mathcal{L}\| \sqrt{(k_1^2 + \eta^2)} (\alpha^2 + \beta^2)}{2k}
\]
Calculating from
\[
\frac{d\vartheta}{(1 + \|\mathcal{L}\|\vartheta)^2} = dt
\]
the event interval of the centralised control strategy is lower bounded by the time \(\bar{\tau}\) satisfying
\[
\frac{\vartheta (\bar{\tau}, 0)}{1 + \|\mathcal{L}\|\vartheta (\bar{\tau}, 0)} = \bar{\tau} \Gamma
\]
Hence, the event interval is positive lower bounded by
\[
\bar{\tau} = \frac{\sigma \alpha_{\text{min}}}{\Gamma \|\mathcal{L}\| \sqrt{(k_1^2 + \eta^2)} (\alpha^2 + \beta^2) + \sigma \alpha_{\text{min}}}
\]
On the basis of the above analysis, the following theorem is stated.

**Theorem 6:** Consider the multi-agent system as described in (9) under directed strongly connected topology, with the centralised event-triggered control law (10) and the control law is triggered when the event-triggering function (12) reaches zero. Then for any initial condition, if conditions (16) and (17) hold, the intervals between events are positive lower bounded.

### 4.4.2 Event intervals under the decentralised strategy:

From the scheme of the decentralised event-triggered strategy, one can obtain that the interval of decentralised event-triggering function (23) is the time
\[
\frac{\|\mathcal{L}\|^2 (k_1 + \eta) \left(\alpha \|e_i'\|^2 + \beta \|e_i'^{\prime}\|^2\right)}{2k}
\]
sends for it growing from 0 to
\[
\sigma \left(\alpha k_1 - \frac{(\alpha + \beta)k_1k}{2}\right) \|\bar{v}\|^2 + \sigma \left(\beta \eta - k_2 - \frac{(\alpha + \beta)\eta k}{2}\right) \|\bar{v}\|^2
\]
Here we use \(\tau_i\) to denote this period. Then \(\tau_i\) is longer or equal to the time that
\[
\frac{\|\mathcal{L}\| (k_1 + \eta) \zeta_{\text{max}} (\|e_i'\|^2 + \|e_i'^{\prime}\|^2)}{2k}
\]
grows from 0 to \(\sigma \zeta_{\text{min}} (\|\bar{v}\|^2 + \|\bar{v}\|^2)\), where
\[
\zeta_{\text{min}} \triangleq \min \left\{ \alpha k_1 - \frac{(\alpha + \beta)k_1k}{2}, \beta \eta - k_2 - \frac{(\alpha + \beta)\eta k}{2} \right\},
\]
\[
\zeta_{\text{max}} \triangleq \max \{\alpha, \beta\}
\]
Suppose that it takes \(\tilde{\tau}\) for
\[
\frac{\|e_i'\|^2 + \|e_i'^{\prime}\|^2}{\|\bar{v}\|^2 + \|\bar{v}\|^2} \quad \text{to reach} \quad \frac{2\sigma k \zeta_{\text{min}}}{(k_1 + \eta) \zeta_{\text{max}} \|\mathcal{L}\|}
\]
Then the \(k + 1\)th event of agent \(i\) occurs after the time \(t_i + \tilde{\tau}\).

**Theorem 7:** Consider the multi-agent system as described in (9) under connected topology with the decentralised event-triggered control law (22) and the control law is triggered when the event-triggering function (23) reaches zero. Then for any initial condition, if conditions (16), (17), (24), (25) hold, there exists at least one agent \(q \in \mathcal{N}\), which has strictly positive event interval \((t_{i_1} + 1 - t_i)\) lower bounded by \(\tau_q > 0\).

**Proof:** Denote \(q = \arg \max_{i \in \mathcal{V}} \|\bar{v}_i\|^2 + \|\bar{v}_i\|^2\). It follows that
\[
\frac{\|e_i'\|^2 + \|e_i'^{\prime}\|^2}{\|\bar{v}\|^2 + \|\bar{v}\|^2} \leq \frac{N (\|e_i'\|^2 + \|e_i'^{\prime}\|^2)}{\|\bar{v}\|^2}
\]
From the idea that presented in the first-order scenario, \(\tilde{\tau}_q\) is longer than \(\tilde{\tau}_q\) that satisfies
\[
\frac{\vartheta (\tilde{\tau}_q, 0)}{1 + \|\mathcal{L}\|\vartheta (\tilde{\tau}_q, 0)} = \tilde{\tau}_q \Gamma
\]
where
\[
\vartheta (\tilde{\tau}_q, 0) = \frac{2\sigma k \zeta_{\text{min}}}{\sqrt{N} (k_1 + \eta) \zeta_{\text{max}} \|\mathcal{L}\|}
\]
It follows:
\[
\tilde{\tau}_q = \frac{2\sigma k \zeta_{\text{min}}}{\|\mathcal{L}\| \sqrt{N} (k_1 + \eta) \zeta_{\text{max}} + 2\sigma k \zeta_{\text{min}}}
\]
Hence one has \(\tau_q \geq \tilde{\tau}_q \geq \tilde{\tau}_q > 0\). The proof is complete. \(\square\)

After that, by using sampled neighbours’ information instead of continuous information, improved event-triggering functions are proposed to further save communication energy.

## 5 Improved event-triggered strategy

In the former sections, the proposed event-triggering function requires continuous information exchange between connected agents in the calculation of the event instances. Hence, in order to minimise the communication energy, an improved event-triggered strategy is developed. This improved event-triggered strategy only uses the discrete states that sampled and sent by neighbours at their event instants.

### 5.1 Consensus control from using the improved event-triggered strategy

The improved event-triggered strategy is applied to the first-order multi-agent system first and then extend to the second-order multi-agent system. Two event-triggering functions are designed without continuous communication between connected agents. Before stating the designed event-triggering functions, we first give some notations for simplicity: \(x(t_i) \triangleq [x_1(t_i), \ldots, x_N(t_i)]^{\top}\). \(t_i\) is the latest event time of agent \(i\) before or at the time \(t_i\). \(\forall i = 1, \ldots, N\). Similarly, we denote \(v(t_i) = [v_1(t_i), \ldots, v_N(t_i)]^{\top}\). Besides, we use \(L_i\) to represent the ith raw of the Laplacian matrix \(L\).

### 5.1.1 First-order multi-agent systems:

For the first-order multi-agent system as described in (1), the same event-triggered control law (6) is utilised for the consensus
reaching. For agent $i$, the improved event-triggering function is designed as
\[
\tilde{f}_i(t) = \|e_i(t)\| - \sqrt{\frac{a}{1 + a + ab}} \frac{1}{\|L\|} \|L\hat{x}(t_i)\| \tag{28}
\]
where $a, b$ are positive constants. Note that $L\hat{x}(t_i) = \sum_{j\in N_i} (x_j(t_i) - x_i(t_i))$. From the description, instead of continuous information sent by neighbours, the event-triggering function of agent $i$ only use the sampled information that sampled and sent by neighbours at their event instants $t_i^k$. When $\tilde{f}_i(t)$ violates $\tilde{f}_i(t) < 0$, the agent will be triggered. Similar to the former mentioned strategy, the agent will sample its current state at the event instant, the sampled state will be sent to neighbours and be used to update the agent’s control actuator. From the event-triggering function (28), one can obtain that
\[
\|L\epsilon\| \leq \|L\|\|\epsilon\| \leq \sqrt{\frac{a}{1 + a + ab}} \|L\epsilon(t_i)\|)
\]
From Lemma 1, it follows that:
\[
e^\top L^\top L \epsilon \leq \frac{a}{1 + a + ab} ((x + \epsilon)^\top L L(x + \epsilon))
\[
= \frac{a}{1 + a + ab} (x^\top L^\top L x + \epsilon^\top L^\top L \epsilon + 2x^\top L^\top L \epsilon)
\[
\leq \frac{a}{1 + a + ab} \left(1 + \frac{1}{b}\right) x^\top L^\top L x
\[
+ \frac{a(1 + b)}{1 + a + ab} e^\top L^\top L \epsilon
\]
\[
\Rightarrow e^\top L^\top L \epsilon \leq \left(a + \frac{a}{b}\right) x^\top L^\top L x
\]

**Theorem 8:** Consider the first-order multi-agent system described by (1) under directed strongly connected topology. By using the piecewise-continuous control law (6), the system can achieve consensus, for any initial condition, if the distributed event-triggering function (28) is enforced to satisfy $\tilde{f}_i(t) < 0$ and if the following condition is satisfied:
\[
-\tilde{x}_i + \frac{\xi^2}{2d} + \frac{d(a + a/b)}{2} > 0 \tag{29}
\]
where $d$ is a positive constant, $a$ and $b$ are positive constants that first mentioned in (28).

**Proof:** Here the same Lyapunov function as listed in (5) is considered. The time derivative of the Lyapunov function along the trajectories of the state as described in (3) is
\[
\dot{V} = -ax^\top L^\top L \xi - ax^\top L^\top L \xi
\]
By using Lemma 1, we have $x^\top L^\top L \xi \leq (1/2d)x^\top L^\top L \xi^2 + (\|de^\top L^\top L \xi\|/2)$. Hence the time derivative of the Lyapunov function turns into
\[
\dot{V} \leq -ax^\top L^\top L \xi x + \frac{\alpha}{2d} x^\top L^\top L \xi^2
\[
\leq -a \sum_{i=1}^{N} \left(-e^\top_i + \frac{\xi^2}{2d} + \frac{d(a + a/b)}{2}\right) \|\tilde{x}_i\|^2
\]
The condition (29) guarantees $\dot{V}(t) \leq 0$ unless $x_i = \cdots = x_N$. From the definition of the consensus, it is easy to obtain that the consensus of the system (1) can be achieved by using the improved event-triggering function (28).

**5.1.2 Second-order multi-agent systems:** The event-triggering function (23) for the second-order multi-agent system (9) also requires continuous communication between neighbours. Hence similar to the first-order scenario, the improved event-triggering function is also designed for the second-order system to save communication energy. The improved event-triggering function for agent $i$ is
\[
f_i(t) = \|e_i(t)\| - \mu_1 \frac{1}{\|L\|} \|L\tilde{x}(t_i)\| \tag{30}
\]
\[
f_i(t) = \|e_i(t)\| - \mu_2 \frac{1}{\|L\|} \|L\tilde{v}(t_i)\| \tag{31}
\]
where
\[
\mu_1 = \sqrt{\frac{a_1}{1 + a_1 + a_1b_1}}, \quad \mu_2 = \sqrt{\frac{a_2}{1 + a_2 + a_2b_2}}
\]
The $k + 1$th event instants of agent $i$ is the moment that either (30) or (31) is positive, that is,
\[
t^k_{i+1} = \inf\{t > t^k_i : f_i(t) > 0, \text{ or } f_i(t) > 0\} \tag{32}
\]

**Theorem 9:** Consider the second-order multi-agent system as described by (9) with piecewise-continuous control law (22) and the event instant is determined by (32). The system (9) is able to achieve consensus, for any initial condition, if (16) and (17) and the following conditions are satisfied:
\[
-\alpha k_1 + \frac{(\alpha + \beta)k_1}{2} + \frac{a a_1 (k_1 + \eta) (b_1 + 1)}{2k_b} < 0 \tag{33}
\]
\[
-\beta \eta + k_2 + \frac{(\alpha + \beta)\eta k_2}{2} + \frac{\beta a_2 (k_1 + \eta) (b_2 + 1)}{2k_b} < 0 \tag{34}
\]

**Proof:** The same candidate Lyapunov function (23) is considered. According to the previous discussion, the time derivative of the Lyapunov function can be written
\[
\dot{V} \leq -ak_1 x^\top L^\top L \xi x - (\beta \eta - k_2) v^\top L^\top L \xi v
\]
\[
+ \alpha k_1 \|x^\top L^\top L \xi\| + \beta k_1 \|x^\top L^\top L \xi\| + \alpha \eta \|v^\top L^\top L \xi\| + \beta \eta \|v^\top L^\top L \xi\|
\]
\[
\leq -ak_1 x^\top L^\top L \xi x - (\beta \eta - k_2) v^\top L^\top L \xi v
\]
\[
+ \frac{(\alpha + \beta)k_1 k_2}{2} \|x^\top L^\top L \xi\| + \frac{\alpha (k_1 + \eta)}{2k_b} \|e_i^\top L^\top L \xi\|
\]
\[
+ \frac{(\alpha + \beta)\eta k_2}{2} \|\tilde{v}\| + \frac{\beta (k_1 + \eta)}{2k_b} \|e_i^\top L^\top L \xi\| \tag{35}
\]
From the scheme of the improved event-triggered strategy, the event-triggering functions (30) and (31) enforce that
\[
e_i^\top L^\top L \xi \leq \left(a_1 + \frac{a_1}{b_1}\right) x^\top L^\top L \xi
\]
\[
e_i^\top L^\top L \xi \leq \left(a_2 + \frac{a_2}{b_2}\right) v^\top L^\top L \xi
\]
Then, the time derivative of the Lyapunov function (35) turns into
\[
\dot{V} \leq \left(-\alpha k_1 + \frac{\alpha + \beta}{2}\right) \hat{x}^\top \hat{x} + \frac{\alpha_n (k_1 + \eta)}{2} \left(1 + \frac{1}{b_1}\right) \hat{\xi}^\top \hat{\xi} + \left(-\beta \eta + k_2 + \frac{\alpha + \beta}{2}\right) \hat{v}^\top \hat{v} + \frac{\beta_n (k_1 + \eta)}{2} \left(1 + \frac{1}{b_2}\right) \hat{\nu}^\top \hat{\nu}
\]

Since condition (33) and (34) hold, it is easily acquired that the consensus of the system (9) can be achieved. □

5.2 Interval analysis

5.2.1 Interval analysis for the first-order system: Following the previous event interval analysis, the interval between two events is the time that \(\|e_i(t_j)\|\) grows from 0 (\(e_i(t)\) equals to zero at the event instant) to the moment the value of the event-triggering function \(J(t_i)\) violates \(J(t) < 0\). Hence we state the following theorem.

**Theorem 10:** Consider the first-order multi-agent system described by (1) under directed strongly connected topology. By using the piecewise-continuous control law (6) to control the system and utilising the event-triggering function (28) to calculate the event instants, the time intervals between two events are positive.

**Proof:** We first consider the case that there is no neighbours’ events during the interval of two consecutive events of \(i\). Assume that the \(k + 1\)th event of agent \(i\) occurs at the time \(t_k + \tau\). It follows that \(\|e_i(t_k)\| = 0\), and
\[
\|e_i(t_k + \tau)\| = \sqrt{\frac{\alpha}{1 + \alpha + ab}} \frac{1}{\|L\\| \|L\hat{x}(t_k)\|}
\]

From the trajectory of \(e_i(t)\), one has
\[
\|e_i(t_k + \tau)\| = \left\| \int_{t_k}^{t_k + \tau} \dot{e}_i(t) \, dt \right\| = \left\| \int_{t_k}^{t_k + \tau} \dot{x}_i(t) \, dt \right\| = \alpha \|L\hat{x}(t_k)\| \tau = \sqrt{\frac{\alpha}{1 + \alpha + ab}} \frac{1}{\|L\\| \|L\hat{x}(t_k)\|}
\]

Hence one can obtain
\[
\tau = \sqrt{\frac{\alpha}{1 + \alpha + ab}} \frac{1}{\alpha \|L\\| \|L\hat{x}(t_k)\|} > 0
\]

If a neighbour triggers during the interval between two consecutive events of agent \(i\), that is, the neighbour triggers at time \(t_k + \tau\) \(\leq t_k + \tau\). Then the interval is greater than \(\tau\). From above, one can obtain that the intervals between events that generated by the improved event-triggering function are positive.

5.2.2 Interval analysis for the second-order system: Similar to the previous, we first consider the case that there is no triggering between two consecutive events of agent \(i\). If the event-triggering function (30) of agent \(i\) reaches zero at time \(t_k + \tau\), then one has
\[
\|e_i(t_k + \tau)\| = \left\| \int_{t_k}^{t_k + \tau} \dot{e}_i(s) \, ds \right\| = \left\| \int_{t_k}^{t_k + \tau} \dot{v}_i(s) \, ds \right\| = \left\| \int_{t_k}^{t_k + \tau} (\alpha L\hat{x}(t_k) + \beta L\hat{v}(t_k)) \, ds \right\| 
\]

\[
\leq \|\alpha L\hat{x}(t_k) + \beta L\hat{v}(t_k)\| \tau
\]

From the event strategy scheme, one has
\[
\tau = \frac{\mu_2 \|\rho L\hat{v}(t_k)\|}{\|L\| \|\alpha L\hat{x}(t_k) + \beta L\hat{v}(t_k)\|}
\]

Meanwhile, if the event-triggering function (31) reaches zero at time \(t_k + \tau\), then one can obtain
\[
\|e_i(t_k + \tau)\| = \left\| \int_{t_k}^{t_k + \tau} \dot{e}_i(t) \, dt \right\| = \left\| \int_{t_k}^{t_k + \tau} \dot{v}_i(t) \, dt \right\| = \left\| \int_{t_k}^{t_k + \tau} v_i(s) \, ds \, dt \right\| = \left\| \int_{t_k}^{t_k + \tau} (\alpha L\hat{x}(t_k) + \beta L\hat{v}(t_k)) \, ds \, dt \right\|
\]

From the event-triggering scheme, we obtain \(\|e_i(t_k + \tau)\| = \mu_2 (1/\|L\|) \|\rho L\hat{v}(t_k)\|\). Hence, it leads to
\[
\left(t_k + \tau\right)^2 - \left(t_k\right)^2 = \frac{2\mu_1 \|\rho L\hat{x}(t_k)\|}{\|L\| \|\alpha L\hat{x}(t_k) + \beta L\hat{v}(t_k)\|}
\]

It is easy to obtain that \(\tau > 0\) if \(L\hat{x}(t_k)\) and \(L\hat{v}(t_k)\) is non-zero at instant \(t_k\). Since agent \(i\) will be triggered when either (30) or (31) overpasses zero, the event interval can be described as \(t_k < t_k + \tau\). Now we consider the case that a neighbour triggers during the two consecutive events of agent \(i\). Assume agent \(j\), \(j \not\in N_i\) triggers at time \(t_k + \tau < t_k + \tau\). Then the interval \(t_k + \tau - t_k\) is greater than \(\tau\). By concluding the above discussion, the interval \(t_k + \tau - t_k\) is positive.

**Theorem 11:** Consider the multi-agent system as described in (9) under connected topology with the improved event-triggered control law determined by (32). Then for any initial condition, if conditions (16), (17), (33) and (34) hold, there is no infinite accumulation of events.

6 Illustrative example

In this section, two examples are given under both the first-order and the second-order multi-agent system to show the effectiveness of the proposed event-triggered control strategy.

6.1 Example for the first-order system

Consider a first-order multi-agent system as described by (1) under directed topology. The states of the agents are denoted
The consensus of the system is achieved by using the event-triggered control strategy together with the centralised event-triggering function (4). Then the decentralised event-triggering strategy is applied to the system. The agent motion is shown in the first figure in Fig. 2. One can observe that the convergence of the agents is also realised. The asynchronous event sequences are illustrated in the second figure in Fig. 2. It is shown that each agent has its own event instants sequence in a discrete pattern. The third figure in Fig. 2 shows the fluttering of the norm of the measurement error of agent 2. It indicates that when agent 2 is triggered at an event instant, the norm of the measurement error is set to 0 due to the update of the state. After that, the norm of the measurement error grows along with time until another event occurs. The simulations for the improved event-triggered strategy are given as well. The coefficients are chosen as \( \alpha = 0.6, a = 0.15, b = 4, d = 0.24 \) to ensure the condition (29). The motion of the agents, the event sequences and the norm of the measurement error of agent 2 are shown in Fig. 3. It can be observed that all three event-triggered strategy can solve the consensus problem of the first-order system (1). Since the improved event-triggering function is more related to the outdated convergence of the agents, rather than the current convergence, more events occur while using the improved event-triggering function.

### 6.2 Example for the second-order system

#### 6.2.1 Example applying the decentralised event-triggering function

A second-order multi-agent system is considered under the same directed topology with the first-order system. Together with piecewise continuous control law (22) and event-triggering function (23). From the Laplacian matrix, one can obtain that \( \bar{\xi} = [0.0263, 0.2895, 0.0526, 0.1053, 0.3158, 0.2105] \). We choose coefficients \( \alpha = 7, \beta = 7, k_1 = 0.4, \eta = 3.4, r = k_i \beta + a \eta = 26.6 \) and \( k_i = 3.5 \) to make sure conditions (16), (17), (24), (25) hold. Fig. 4 shows that the position states and the velocity states of all the agents achieved consensus without obvious fluctuation. The third figure in Fig. 4 shows the norm of position measurement error of agent 1. These figures illustrate that although the event interval is shorter than the first-order case, the event-triggered control strategy is able to drive the second-order system achieve consensus.

#### 6.2.2 Example applying the improved event-triggering function

The simulation for the improved event-triggering function is also illustrated. The Laplacian matrix is given by

\[
L = \begin{bmatrix}
2 & -1 & -1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 3 & -1 & 0 & -1 \\
-1 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 \\
-1 & -1 & 0 & 0 & -1 & 2
\end{bmatrix}
\]

The initial position states are set as \( x_0 = [-2.22, 4.2, -1.3, 1.4, -1.5, 2.6] \) and we choose the control law parameter \( \alpha = 0.6 \). We first apply the centralised event-triggered strategy to the system. The first figure in Fig. 1 shows the dynamics of agents under control of the centralised event-triggering strategy. The second figure in Fig. 1 shows the event sequences of the first-order system. The third figure in Fig. 1 illustrates the fluttering of the measurement error \( \|e\| \).

Fig. 1 shows that the consensus of the system is achieved.
the first-order and the second-order cases. Distributed piece-wise continuous control laws along with event-triggering functions are proposed to drive the system to achieve consensus. When the value of the event-triggering function exceeds zero, the controller of the agent will be updated with the agent’s state at the this instant. Applying this method, the consensus of the multi-agent systems can be achieved while communication energy can be saved as the agents send their state information only at infrequent event instants. Furthermore, two improved event-triggering functions, which can avoid continuous communication with neighbours, are proposed for both the first-order and the second-order multi-agent systems. Examples are provided to illustrate the effectiveness of the proposed control approaches. Future work include nonlinear system model, time-delay in information transmission and other cases.

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9 References


7 Conclusion

In this paper, the consensus achieving problem for multi-agent systems under directed topology is addressed for both

\[
L = \begin{bmatrix}
3 & -1 & 0 & -1 & -1 & 0 \\
-1 & 2 & 0 & 0 & 0 & 1 \\
0 & 2 & 2 & 0 & 0 & 1 \\
-1 & -1 & 2 & 0 & 0 & 1 \\
0 & 0 & -1 & 2 & 0 & 1 \\
0 & 0 & 0 & -1 & 1 & 0 \\
\end{bmatrix}
\]

We choose coefficients \( \alpha = 6.6, \beta = 1.9, \ k_1 = 0.26, \eta = 0.11, r = k_1 \beta + \alpha \eta = 3.82, \ k_2 = 0.3 \) and \( a_1 = 0.13, a_2 = 0.05, b_1 = 1.1, b_2 = 1.1 \) to guarantee the conditions (16), (17), (33), (34) hold. The simulation is shown in Fig. 5. It is observed that the improved event-triggering function is efficacious for the second-order system consensus achievement. Meanwhile, from the norm of the velocity measurement error of agent 3, one can observe that the event intervals are shorter that of the decentralised event-triggering strategy.