A sensitivity analysis of DC resistivity prospecting on finite, homogeneous blocks and columns

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ABSTRACT

Besides field applications to geophysical prospecting and subsurface hydrology, electrical resistivity tomography can be applied to finite-scale blocks in the laboratory to characterize the resistivity structure of the blocks and to monitor internal physical and chemical processes. This requires a fast and accurate calculation of the sensitivity matrix to perform a successful resistivity inversion for such blocks. However, the complex geometric shape and boundary and the finite size of the block limit the application of field-suitable sensitivity calculation methods to these blocks. As blocks and finite columns are often used in the laboratory experiments, this paper develops practical analytic expressions, based on the method of image charges, for the sensitivity matrix for these two types of homogenous bodies. The corresponding formulae for the electric potential distribution and the electrode array coefficient are also presented. As a result of the theory, the effects of placing limits on the sum index in the electric-potential calculation can be analyzed, and a comparison of the theoretical and the numerically simulated electric potential is shown. The results demonstrate the correctness of the theory and indicate that even the addition of only one set of mirror current sources greatly reduces the effects of the block boundary on the electric-potential calculation. Finally, several interesting sensitivity distributions for cross-surface arrays on blocks, and for circular and vertical arrays on columns, are given. Although the formulae developed here are only valid for homogeneous blocks and columns, and an element of relatively small volume is required to permit a good approximation to the sensitivity, the theory is useful in the verification of numerically simulated results, in sensitivity-analysis for optimum probing-scheme design, and in successful resistivity inversion calculation for finite bodies.

INTRODUCTION

With the advantages of noninvasive, multidimensional and scale-variable monitoring, high-density electrical resistivity tomography (ERT) is receiving much attention for near-surface geophysical prospecting and subsurface hydrology (e.g., Daily and Ramirez, 2000; Zhou et al., 2001, 2002, 2004; Kemna et al., 2002; Michot et al., 2003). Applications either use the earth resistivity to characterize the site and understand the subsurface geological structure and lithology (e.g., Shima, 1992, 1995; Suzuki and Ohnishi, 1995; Tsokas and Tsourlos, 1997), or utilize the temporal variations of earth resistivity to investigate underlying physical and chemical processes (e.g., Ramirez et al., 1993; White, 1994; Binley et al., 1996; Park, 1998; Yang et al., 1999; Zhou et al., 2001, 2002, 2004; Kemna et al., 2002). Because usually only earth resistance or electric potential data are available, a successful application of ERT requires a reliable inversion algorithm that can convert the measured data to a spatial distribution of resistivity.

Inversion algorithms used for ERT are usually based on a least-squares solution. Gauss-Newton and Levenberg-Marquardt methods are often used (e.g., Binley et al., 1996; Zhou et al., 1999; Kemna et al., 2002; Zhou and Greenhalgh, 2002). Although these algorithms may differ from each other in search-direction and step-length formulation, most of them require an expensive sensitivity (Jacobian) matrix calculation at each iteration. Recently, Yeh et al. (2002) developed a technique (successive linear estimator) that can be used to estimate the spatial distribution of resistivity from the measured resistivity and electric potential data, and they demonstrated its validity through numerical examples. However, the calculation of the sensitivity matrix is still not avoided by this technique.

Calculation of the sensitivity matrix depends on the presumed homogeneity of the medium of interest. For a homogeneous half-
space, the sensitivity matrix can be analytically obtained (e.g., Park and Van, 1991; Zhou and Greenhalgh, 1999, 2002). However, for a heterogeneous medium, the sensitivity matrix can be calculated only by a numerical method or by a later correction (updating) of the sensitivity matrix for a homogeneous medium. Although numerical methods can be used for almost any heterogeneous medium, they can become extremely time-consuming. For this reason, the sensitivity matrix for a heterogeneous medium is often approximated by updating the matrix derived for a homogeneous medium (e.g., Loke and Barker, 1996a; 1996b; Van, 1991; Zhou and Greenhalgh, 1999, 2002). Thus, the analytic calculation of the sensitivity matrix for a homogeneous medium is a vitally important first step in the ERT inversion.

Whereas ERT is used extensively in the field; recently, applications of ERT on finite blocks in the laboratory have also been reported (e.g., Binley et al., 1996; Zhou, 2002). A main difficulty for ERT in the laboratory is the finite size of the medium under investigation. Especially when the electrode interval is comparable to the scale of the target, the medium cannot be simply approximated by an infinite one, although this approximation is usually satisfactory for ERT in the field. Boundary effects and the geometry of the block need to be incorporated into the calculation of the sensitivity matrix and the forward-problem solution. This makes the analytic calculation of the sensitivity matrix difficult for homogeneous blocks. While for a homogeneous half-space, the analytic solution of the sensitivity matrix can be found in the literature (see citation above); the corresponding analytic solution for homogeneous finite blocks is still not reported, though Gheith and Schwartz (1998) did present a formula for electrode array coefficient calculation on a homogeneous block. A sensitivity analysis of DC resistivity prospecting on homogeneous finite blocks is needed for successful application of ERT to laboratory-scale targets. A laboratory-scale sensitivity analysis could be also helpful for ERT in the field, especially in situations where the prospecting target has a special geometric shape.

In this paper, analytic expressions of the sensitivity matrix are developed for homogeneous blocks (rectangular parallelepipeds) and columns (finite, right, circular cylinders). Then, as an application of the theory, factors affecting the theoretical electric-potential calculation are investigated, and a comparison is made between the theoretical and the numerically calculated electric potential. Finally, as a guide for electrode installation and measurement-scheme optimization of ERT in the laboratory, several interesting sensitivity distributions are presented.

THEORY

The analytic expressions of electric potential, electrode array coefficient, and sensitivity for a homogeneous block

We consider the cases in which ERT is applied to a block of homogeneous resistivity. The block may be a porous medium, fractured rock, or other material, but its size is much larger than a typical pore size or fracture aperture. Assume one current electrode, with \( A = \{A_x, A_y, A_z\} \) Cartesian coordinates (using top left corner of the block as the origin, and with the \( z \) axis positive downwards), is installed on the surface or in the interior of the block. On the basis of the method of image charges (Griffiths, 1999), an infinite number of mirror current sources that have the same current as \( A \) need to be added to incorporate the block boundary conditions into the electric-potential calculation. In this paper, we assume that the electrode occupies a single point. Let \((i, j, k)\) be one of the mirror current sources, and \(M = \{M_x, M_y, M_z\}\) be the Cartesian coordinates of a potential electrode. The distances between the \((i, j, k)\) mirror current source and the potential electrode \(M\) in the \(x, y, z\) directions, respectively, are

\[
D_x^{AM}(i,H) = \left[ i + \frac{1 - (-1)^i}{2} \right] H_x + (-1)^i A_x - M_x,
\]

\[
D_y^{AM}(j,H) = \left[ j + \frac{1 - (-1)^j}{2} \right] H_y + (-1)^j A_y - M_y,
\]

\[
D_z^{AM}(k,H) = \left[ k + \frac{1 - (-1)^k}{2} \right] H_z + (-1)^k A_z - M_z,
\]

where \(H = \{H_x, H_y, H_z\}\) is the block size, and \((i, j, k)\) are integer indices varying from \(-\infty\) to \(+\infty\). Indices \(0, 0, 0\) indicate the current source itself. Figure 1 gives an example about the mirror current source positions for a current electrode in the block. Thus, the distance between mirror current source \((i, j, k)\) and the potential electrode \(M\) can be expressed by

\[
R_{ijk}^{AM}(H) = \left[ (D_x^{AM})^2 + (D_y^{AM})^2 + (D_z^{AM})^2 \right]^{1/2}.
\]

For a four-electrode array with \(A\) and \(B\) as the positive and negative current electrodes, and \(M\) and \(N\) as the potential electrodes, the electric potential at \(M\) can be approximated by

Figure 1. The mirror current source positions for a current electrode in the block. Only the positions for indices \((i, j, k)\) taking \(-1, 0, 1\) are shown. The current electrode is located at \((0.2\ m, 0.2\ m, 0.2\ m)\), and the block (solid line) has a size of \(1.0\ m \times 1.0\ m \times 1.0\ m\).
\[ P_{ijk}^M(A,B,H,\rho, I, g) = \sum_{k=-g}^{g} \sum_{j=-g}^{g} \sum_{i=-g}^{g} P_{ijk}^M, \]  
\[ P_{ijk}^M(A,B,H,\rho, I) = \frac{\rho I}{4\pi R_{ijk}^M} - \frac{\rho I}{4\pi R_{ijk}^{BM}}, \]

\( \rho \) is the resistivity of the block, \( I \) is the current intensity, and \( g \) is the limit on the sum of mirror sources from the current source \( g=0 \) in each of the six directions. When \( g \to \infty \), which means an infinite sequence of mirror sources, \( P_{ijk}^M \) approximates the exact potential value.

Figure 2a illustrates the relationship of \( P_{ijk}^M \) and sum limit \( g \). The alternating convergence properties in the relation of \( P_{ijk}^M \) and \( g \), as described by equation 5, are clear. Thus, averaging \( P_{ijk}^M \) over \( g \) and \( g+1 \), we should get a reasonable approximation of the exact electric potential. So, at potential electrodes \( M \) and \( N \), we have

\[ \tilde{P}_i^M(A,B,H,\rho, I, g) = \frac{1}{2} \left( \sum_{k=-g}^{g} \sum_{j=-g}^{g} \sum_{i=-g}^{g} P_{ijk}^M \right) + \frac{1}{2} \sum_{k=-g}^{g} \sum_{j=-g}^{g} \sum_{i=-g}^{g} \frac{P_{ijk}^M}{g+1}, \]

and

\[ \tilde{P}_i^N(A,B,H,\rho, I, g) = \frac{1}{2} \left( \sum_{k=-g}^{g} \sum_{j=-g}^{g} \sum_{i=-g}^{g} P_{ijk}^N \right) + \frac{1}{2} \sum_{k=-g}^{g} \sum_{j=-g}^{g} \sum_{i=-g}^{g} \frac{P_{ijk}^N}{g+1}. \]

With a substitution of equation 6 into equations 7 and 8, the difference in the electric potential between electrodes \( M \) and \( N \) becomes

\[ \tilde{P}_i^M - \tilde{P}_i^N = \frac{\rho I}{8\pi} \left[ \sum_{k=-g}^{g} \sum_{j=-g}^{g} \sum_{i=-g}^{g} \frac{1}{R_{ijk}^M} - \frac{1}{R_{ijk}^{BM}} - \frac{1}{R_{ijk}^{AN}} \right] \]

\[ + \frac{1}{R_{ijk}^{BM}} \right) + \sum_{k=-g}^{g+1} \sum_{j=-g}^{g+1} \sum_{i=-g}^{g+1} \left( \frac{1}{R_{ijk}^M} \right) \]

\[ - \frac{1}{R_{ijk}^{BM}} - \frac{1}{R_{ijk}^{AN}} + \frac{1}{R_{ijk}^{BN}} \right]. \]

Thus, the apparent resistivity can be calculated with

\[ \rho_a = K_{AB}^M \frac{\tilde{P}_i^M - \tilde{P}_i^N}{I}, \]

where

\[ K_{AB}^{MN} = 8\pi \left[ \sum_{k=-g}^{g} \sum_{j=-g}^{g} \sum_{i=-g}^{g} \left( \frac{1}{R_{ijk}^M} - \frac{1}{R_{ijk}^{BM}} - \frac{1}{R_{ijk}^{AN}} \right) \right] + \frac{1}{R_{ijk}^{BM}} \]

\[ + \frac{1}{R_{ijk}^{AN}} + \frac{1}{R_{ijk}^{BN}} \right)^{-1}. \]

is the electrode array coefficient. When the averaging over sum limits \( g \) and \( g+1 \) is not applied, equation 11 reduces to that given by Gheith and Schwartz (1998).

For a given electrode array, an entry of the sensitivity matrix (Jacobian) is defined as the derivative of the measured voltage (V) with respect to a small change of resistivity in a volume element. It differs from common measures of the sensitivity of ERT, such as changes in ERT images with respect to variations in material structure, solute concentration of pore water, and water content in the porous medium. For a pole-pole array with current electrode at D and potential electrode at M in a homogeneous medium, according to Park and Van (1991), the sensitivity S can be expressed as

\[ S = \frac{\partial V}{\partial \rho} = \frac{1}{\rho} \int \nabla P \cdot \nabla P' \, dv, \]

where \( P(x, y, z) \) is the electric potential resulting from a current source I at the current electrode A, \( P'(x, y, z) \) is the electric potential caused by a unit current source at the potential electrode M, \( V \) is the voltage measured at M related to current injected at A, and \( \rho \) is the material volume where the resistivity is changed. For arbitrary four-electrode array of A-B-M-N on the surface or in the interior of a cubic block, the expressions for \( P \) and \( P' \) are

\[ a) \]
\[
\overline{P}_{i}^{(x,y,z)}(A,B,H,\rho,l,g) = \frac{1}{2} \left( \sum_{k=-g}^{g} \sum_{j=-g}^{g} \sum_{i=-g}^{g} P_{ijk}^{(x,y,z)}(A,B,H,\rho,l) + \sum_{k=-(g+1)}^{g+1} \sum_{j=-(g+1)}^{g+1} \sum_{i=-(g+1)}^{g+1} P_{ijk}^{(x,y,z)}(A,B,H,\rho,l) \right) \times P_{ijk}^{(x,y,z)}(A,B,H,\rho,l),
\]

Thus,
\[
\nabla \overline{P}_{i}^{(x,y,z)}(A,B,H,\rho,l,g) = \frac{1}{2} \left( \sum_{k=-g}^{g} \sum_{j=-g}^{g} \sum_{i=-g}^{g} \nabla P_{ijk}^{(x,y,z)}(A,B,H,\rho,l) + \sum_{k=-(g+1)}^{g+1} \sum_{j=-(g+1)}^{g+1} \sum_{i=-(g+1)}^{g+1} \nabla P_{ijk}^{(x,y,z)}(A,B,H,\rho,l) \right) \times \nabla P_{ijk}^{(x,y,z)}(A,B,H,\rho,l).
\]

Considering that the integrand \( \nabla \overline{P}_{i} \cdot \nabla \overline{P}_{i} \) in equation 17 is a continuous function with singularities at the locations of the current electrodes, the sensitivity relative to a unit element volume can be expressed by

\[
S_{v}^{AMNB} = \frac{1}{\rho^{2}v} \int_{v} \nabla \overline{P}_{i} \cdot \nabla \overline{P}_{i} \, dv.
\]

where \( \nabla \overline{P}_{i} \cdot \nabla \overline{P}_{i} \) is the mean value of \( \nabla \overline{P}_{i} \cdot \nabla \overline{P}_{i} \) over the element volume \( v \). Thus, when the element volume \( v \) is sufficiently small, the sensitivity at \((x,y,z)\) can be approximated by

\[
S_{v}^{AMNB(x,y,z)} = \frac{1}{\rho^{2}} (\nabla \overline{P}_{i} \cdot \nabla \overline{P}_{i}).
\]
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\[
F_z = \frac{(-1)^n + \left(\frac{1}{2} - (-1)^2\right) + k}{D_A} W - z
\]
\[
G_x = \frac{(-1)^q + \left(\frac{1}{2} - (-1)^2\right) + k}{D_B} W - z
\]
\[
G_y = \frac{(-1)^p + \left(\frac{1}{2} - (-1)^2\right) + k}{D_N} V - y
\]
\[
D_M = 4\pi \left\{ (-1)^q + \left(\frac{1}{2} - (-1)^2\right) + k \right\} \times W - z\right)^{3/2}
\]
\[
D_N = 4\pi \left\{ (-1)^p + \left(\frac{1}{2} - (-1)^2\right) + k \right\} \times W - z\right)^{3/2}
\]

Thus, for any four-electrode array on the surface or in the interior of the block, using equations 7, 8, 11, and 20, we readily obtain the electric-potential distribution, electrode array coefficient, and sensitivity distribution.

Analytic expressions of electric potential, electrode array coefficient and sensitivity for a homogeneous finite column

For a homogeneous column (right, circular cylinder) of finite length \(L\), similar formulae for the electric potential distribution, electrode array coefficient, and sensitivity distribution can be obtained following the above procedure. Assume the origin of the Cartesian coordinate is at the center of the top surface of the column and the \(z\) coordinate is positive downwards; then the counterparts of equations 1–3 become

\[
D^\text{AM}_x = A_x - M_x,
\]
\[
D^\text{AM}_y = A_y - M_y,
\]
Here, formula 33 indicates that the mirror current source positions in the \( z \) direction follow the same rule as that for the rectangular block. However, in the \( x-y \) plane, because the boundary is circular, the mirror current source positions for a current electrode \( A \) on the column become those shown in Figure 3. The mirror sources form a circle around the current electrode with the following coordinates:

\[
A_x^i(\theta_g) = 1.25r \cos \theta_g - A_x/4, \\
A_y^i(\theta_g) = 1.25r \sin \theta_g - A_y/4, \\
A_z^i(\theta_g) = A_z
\]

(34)  
(35)  
(36)

where \( r \) is the column radius, and \( \theta_g \) is the polar angle of the \( g \)th mirror current source. The mirror sources are uniformly distributed over the range \((0, 2\pi)\), forming a continuous “mirror-current-source circle” around the original current source. To maintain the balance of current intensity, the total current intensity derived from the individual mirror current sources must equal that of the original current source. The circular structure of the mirror current sources and the balance of the current intensity imply that the electric-potential gradient at the column boundary is close to zero. Thus, the boundary condition is enforced. For a pair of current electrodes \( A \) and \( B \) at the column circular boundary surface, the total electric potential at \( M \) can be approximated as

\[
P^M_{t}(A, B, L, \rho, I, g, T) = \sum_{k = -g}^{g} \left[ \frac{\rho I}{2\pi R_{i}^{AM}} - \frac{\rho I}{2\pi R_{i}^{BM}} \right] + \frac{1}{T} \sum_{i=1}^{T} \left( \frac{\rho I}{2\pi R_{i}^{AM}} - \frac{\rho I}{2\pi R_{i}^{BM}} \right)
\]

(37)

where \( R_{i}^{AM}, R_{i}^{BM}, R_{i}^{AM} \) and \( R_{i}^{BM} \) are the distances between mirror current sources and position \( M \). While \( R_{i}^{AM} \) is the distance for the \( 4 \)th mirror current source of the \( 4 \)th mirror current source at the circle, \( R_{i}^{MN} \) is that for the \( 4 \)th mirror current source of the current electrode \( A \). The total number of the mirror sources on the circle is \( T \). The total electric potential at \( N \) is

\[
P^N_{t}(A, B, L, \rho, I, g, T) = \sum_{k = -g}^{g} \left[ \frac{\rho I}{2\pi R_{i}^{AN}} - \frac{\rho I}{2\pi R_{i}^{BN}} \right] + \frac{1}{T} \sum_{i=1}^{T} \left( \frac{\rho I}{2\pi R_{i}^{AN}} - \frac{\rho I}{2\pi R_{i}^{BN}} \right)
\]

(38)

Thus, based on equations 37 and 38, the electrode array coefficient for an \( A-B-M-N \) four-electrode array on a column becomes

\[
K_{MN}^{AB} = 2\pi \left[ \sum_{k = -g}^{g} \left( \frac{1}{R_{k}^{AM}} - \frac{1}{R_{k}^{BM}} \right) - \frac{1}{R_{k}^{AN}} + \frac{1}{R_{k}^{BN}} \right]^{-1}
\]

(39)

The sensitivity for finite-length column is approximated as

\[
S(x, y, z) = \sum_{i = x, y, z} \left[ \left( \frac{g}{T} \sum_{k = -g}^{g} F_i + \frac{1}{T} \sum_{k = -g}^{g} \sum_{i=1}^{T} H_i \right) \times \left( \frac{T}{K} \sum_{k = -g}^{g} \sum_{i=1}^{T} Q_i \right) \right]
\]

(40)

where the \( F, H, G \) and \( Q \) are given in Appendix A. Here we assume that the electrodes \( A, B, M, \) and \( N \) have the same Cartesian coordinates as in equation 20. Note that formulas 31–40 are valid whether the electrodes are placed on the circular boundary surface or in the interior of the column.

**RESULTS AND DISCUSSION**

The effects of sum limit \( g \) and polar-angle increment \( \Delta \theta \) on the electric potential calculation

To obtain an accurate electric potential, the sum in formulae 5 and 37 should be conducted for a sum limit of infinity. However, in a practical calculation, this infinity must be replaced by a finite number \( g \). Similarly, a finite number \( T \) of discrete mirror current sources
must be used to approximate the continuous mirror current source distribution surrounding the column block. The use of finite sum limits $g$ and $T$ results in an error in the electric potential that changes with the sum limit that is imposed.

Figure 2 shows the variations of electric potential with the sum limit $g$ (Figure 2a) and the polar angle increment $\Delta \theta (= 2\pi/T)$ (Figure 2b) for two measurement arrays. For the block, a single mirror current source sequence (i.e., $g = 1$) improves the results and greatly attenuates the boundary effects on the potential calculation. When $g > 4$, the calculated electric potential has effectively converged. For the column, the use of a smaller polar angle increment affords more accuracy of the electric potential (less fluctuation in Figure 2b). However, a smaller increment increases the computation cost. In the following analysis, we use $\Delta \theta = 0.05$ radians and $g = 5$ for the electric potential and sensitivity calculation.

Applications of the theory in the verification of numerically simulated results

As an application of the theory, electric potential values computed using equations 7 and 37 were compared with those obtained by numerical simulation. The simulation used a three-dimensional finite element method. The forward matrix equation is solved by a conjugate gradient iteration method. The code originates from the early work of Zhou (1999) and Zhou et al. (1999). For the block, we assume a size of $0.4 \times 0.4 \times 0.2$ m. The block is divided into 20,000 tetrahedra with 4851 nodes. For the column, the diameter is 0.06 m, and the length is 0.335 m. It is discretized into $4.553 \times 10^4$ tetrahedra and 8840 nodes. To incorporate the block boundary into the numerical simulation, zero-flux boundary conditions were used.

Figure 4 shows a comparison of the electric potentials obtained by equations 7 and 37 with those obtained by numerical simulation for four different measurement arrays. It is clear that for the block (Figure 4a and b), the simulated potential coincides very well with the theoretical analysis. However, for the column (Figure 4c and d), although the simulated potential agrees in large part with the theoretical value, the two electric potentials deviate somewhat at large values. This is especially true when the electrode array is aligned with the column axis (Figure 4d). One reason for the poor performance of the column calculations is the size difference between the cubic and the column blocks. For the same input current, the smaller the block, the higher the electric potential. If no further element subdivision is conducted in the region near the current sources, deviation in the simulated electric potential would be larger the smaller the block. Another reason may be the finite size of the blocks. For a finite block, a zero-electric-potential reference point actually does not exist. Although the simulated electric-potential difference between two points would not change with the calculation conditions, theoretically the simulated electric potential can be any value. These factors cause the simulated results to depend somewhat on the initial electric-potential distribution and iteration resolution in the numerical calculation. Thus, the theory developed here is useful in the verification of the simulated results.

Sensitivity distributions for cross-surface electrode arrays on a homogeneous block

Although an electrode array similar to one used in the field can be applied to a single surface of a finite block, an advantage of laboratory ERT measurements on a finite box is that the electrodes can be directly installed on different sides of the block. For a given electrode interval, cross-surface measurements force electric current flow through the block interior and provide reliable information about internal resistivity structure.

For the block, a four-electrode array can be applied with the electrodes all on the same surface, or split among two, three, or four surfaces. For each of these electrode arrays, the corresponding three-dimensional sensitivity distribution can be obtained using equation 20. As an example, Figure 5 presents some of the two-dimensional ($x$-$y$ plane) sensitivity distributions for selected cross-surface electrode arrays on a $300 \Omega$m homogeneous cube. The block size is $1.0 \times 1.0 \times 1.0$ m, and the input current is 10 mA. All of the electrodes are placed on the block surface at $0.5$ m in the $z$-axis direction as shown. For comparison, the sensitivity distribution with all of the four electrodes placed on the same surface is also provided (Figure 5i).

From the figure, we see that different electrode arrays have quite different distribution patterns of the sensitivity. For the array with the electrodes all set at the same surface (Figure 5i), the high-sensitivity (absolute value) areas are mainly concentrated in the region near the surface. As the possible maximum electrode interval is restricted by the side length of the block, the depth that can be measured by this array is limited. In contrast, when the electrodes are
placed on more than one surface, although the high-sensitivity areas of the cross-surface arrays are still concentrated around the electrodes, the sensitivity at the interior of the block is enhanced. For example, the sensitivities at the electrodes in Figure 5a-h are much smaller than those in Figure 5i. The area inside the block with sensitivity higher than a given value (e.g., 0.1) is also generally larger than that in Figure 5i. The main reason for the enhanced interior sensitivity is that the cross-surface arrays have larger electrode intervals. It is possible to optimize the electrode positions in the cross-surface array to make the measurement sensitive to areas deeper within the block. The cross-surface electrode array is generally an effective probe of the interior resistivity of a block.

Sensitivity distributions for electrode arrays on a finite, homogeneous column

Figure 6 shows the sensitivity distributions for four different electrode configurations. The distributions are shown in cross section at the middle of the column. Locations of the A-B-M-N electrodes are noted by the labels on each plot. Because the sensitivity distribution pattern does not change with column radius (the corresponding figure is not shown); we use the polar angle instead of the distance between electrodes to represent the electrode interval. As expected, when the potential electrodes are placed at the smaller polar angle between the current electrodes, as in the 15-, 30-, and 60-degree cases, the measurement reflects deeper properties of the block with increasing electrode separation. However, if the potential electrodes are placed in the larger polar angle outside of the current electrodes, as in the 90-degree case, the sensitivity distribution exhibits a complicated pattern and it is difficult to determine the region that the measurement is probing.

Another important electrode array for the column is one that is aligned with the longitudinal axis of the column. If the electrode interval used is relatively small compared with the column length, it is possible to image the resistivity distribution along the column axis. Figure 7 shows sensitivity distributions of such vertical electrode arrays in x-z cross sections (y = 0.0 m) for different column lengths, radii, and electrode intervals. (Note the images have different scales in the x and y directions.) These column sizes are typical of those used in laboratory experiments. A distinct feature of the sensitivity distribution is the high-sensitivity area between potential electrodes M and N that extends laterally across the entire column. This high sensitivity is the result of a relatively large ratio of electrode interval to column diameter. For Figure 7a-d, this ratio is 1.0, 0.5, 0.4, and 0.2, respectively. Intervals between current electrodes in Figure 7a-d are larger than the column diameter. In such cases, the measured data provides spatially integrated information about the entire column between the electrodes. Resolution of the ERT in the horizontal cross section direction, however, is reduced. Therefore, to image the resistivity distribution in longitudinal cross section, an electrode interval that is not larger than the column diameter is preferred. Because the column diameter used in the laboratory is usually much smaller than the column length, when the measurements are carried out for resistivity imaging in the longitudinal cross section, a careful selection of the electrode interval is needed.

A comparison between different column radii indicates that the sensitivities in columns of smaller radius have much higher values than those of larger radius. For columns with the same radius, a reduction in column length also causes the sensitivity to increase.
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Figure 7. The sensitivity distributions in x-z cross-sections (y = 0.0 m) for vertical electrode arrays on different-scale columns. (a) Diameter 2r = 0.2 m, Length L = 1.0 m; (b) 2r = 0.2 m, L = 0.5 m; (c) 2r = 0.05 m, L = 0.1 m; and (d) 2r = 0.05 m, L = 0.05 m. The sensitivity unit is mV/(Ωm, m³), and the logarithm of the absolute value of the sensitivity is used.

However, if there is no substantial change in the electrode array, variations both in the column radius and in the length do not greatly affect the distribution patterns of the sensitivity.

CONCLUSIONS

Based on the current-source mirror method, in this study we developed analytic expressions for electric potential and sensitivity for DC resistivity prospecting on a homogeneous, finite block. We further obtained analytic expressions of electric potential, electrode array coefficient, and the sensitivity matrix for a homogeneous column of finite radius. An analysis of the effects of limiting the sum in the electric-potential calculation indicated that a nonzero sum index greatly reduced the boundary effects. For the column, use of a polar angle increment smaller than 0.05 radians provided adequate resolution for most applications. We calculated the sensitivity distributions for some special electrode arrays on blocks and columns.

When applying the results to practice, however, care should be taken since the analytic expressions are valid only for a homogeneous medium. Although results obtained from the formulae here can be used as references for a heterogeneous medium, a full understanding of the electric potential and sensitivity distribution in a heterogeneous block requires information from other methods such as numerical simulation. In the development of analytic expressions for the sensitivity, we have also assumed that the volume of the integration element is sufficiently small that the mean value of the integrand can be approximated by the integrand itself. This approximation means that the sensitivity obtained here should not be used as representative for an element of larger volume, especially when the element over which the sensitivity is to be calculated is located close to the current source. However, these limitations do not undermine the importance of the theory for verification of numerically simulated results, apparent resistivity calculation, sensitivity analysis for optimum prospecting scheme design, and successful resistivity inversion for finite blocks.

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APPENDIX A

THE EXPRESSIONS FOR F, H, G, AND Q IN EQUATION 40

\[ F_A = \frac{l - x}{D_A} - \frac{o - x}{D_B}, \]  
\[ F_y = \frac{m - y}{D_A} - \frac{p - y}{D_B}, \]  
\[ F_z = \frac{(-1)^5n + \left\{ \frac{1}{2}[1 - (-1)^k] + k \right\}L - z}{D_A} \]  
\[ - \frac{(-1)^5q + \left\{ \frac{1}{2}[1 - (-1)^k] + k \right\}L - z}{D_B}, \]  
\[ H_\theta = \frac{1.25r \cos \theta_i - l/4 - x}{D_H} \]  
\[ - \frac{1.25r \cos \theta_i - o/4 - x}{D_T}, \]  
\[ H_{\phi} = \frac{1.25r \sin \theta_i - m/4 - y}{D_H} \]  
\[ - \frac{1.25r \sin \theta_i - p/4 - y}{D_T}, \]
\[ H_z = \frac{(-1)^6 n + \left[ \frac{1}{2}(1 - (-1)^6) + k \right] L - z}{D_H} \]
\[ G_x = \frac{a - x}{D_M} - \frac{d - x}{D_N}, \quad (A-6) \]
\[ G_y = \frac{b - y}{D_M} - \frac{e - y}{D_N}, \quad (A-7) \]
\[ G_z = \frac{(-1)^6 c + \left[ \frac{1}{2}(1 - (-1)^6) + k \right] L - z}{D_M} \]
\[ \frac{(-1)^6 f + \left[ \frac{1}{2}(1 - (-1)^6) + k \right] L - z}{D_N}, \quad (A-8) \]
\[ Q_x = \frac{1.25r \cos \theta_i - a/4 - x}{D_a} \]
\[ - \frac{1.25r \cos \theta_i - d/4 - x}{D_B}, \quad (A-9) \]
\[ Q_y = \frac{1.25r \sin \theta_i - b/4 - y}{D_a} \]
\[ - \frac{1.25r \sin \theta_i - e/4 - y}{D_B}, \quad (A-10) \]
\[ Q_z = \frac{(-1)^6 c + \left[ \frac{1}{2}(1 - (-1)^6) + k \right] L - z}{D_a} \]
\[ \frac{(-1)^6 f + \left[ \frac{1}{2}(1 - (-1)^6) + k \right] L - z}{D_B}, \quad (A-11) \]
\[ D_H = 2\pi \left\{ (-1)^6 n + \left[ \frac{1}{2}(1 - (-1)^6) + k \right] \right\} \times L - z \]
\[ + (a - x)^2 + (p - y)^2 \right)^{3/2}, \quad (A-12) \]
\[ D_H = 2\pi \left\{ (-1)^6 n + \left[ \frac{1}{2}(1 - (-1)^6) + k \right] \right\} \times L - z \]
\[ + (1.25r \cos \theta_i - l/4 - x)^2 \]
\[ + (1.25r \sin \theta_i - m/4 - y)^2 \right)^{3/2}, \quad (A-13) \]
\[ D_M = 2\pi \left\{ (-1)^6 c + \left[ \frac{1}{2}(1 - (-1)^6) + k \right] \right\} \times L - z \]
\[ + (a - x)^2 + (b - y)^2 \right)^{3/2}, \quad (A-14) \]
\[ D_M = 2\pi \left\{ (-1)^6 c + \left[ \frac{1}{2}(1 - (-1)^6) + k \right] \right\} \times L - z \]
\[ + (1.25r \cos \theta_i - a/4 - x)^2 \]
\[ + (1.25r \sin \theta_i - b/4 - y)^2 \right)^{3/2}, \quad (A-15) \]
\[ D_M = 2\pi \left\{ (-1)^6 f + \left[ \frac{1}{2}(1 - (-1)^6) + k \right] \right\} \times L - z \]
\[ + (d - x)^2 + (e - y)^2 \right)^{3/2}, \quad (A-16) \]
\[ D_M = 2\pi \left\{ (-1)^6 f + \left[ \frac{1}{2}(1 - (-1)^6) + k \right] \right\} \times L - z \]
\[ + (1.25r \cos \theta_i - d/4 - x)^2 \]
\[ + (1.25r \sin \theta_i - e/4 - y)^2 \right)^{3/2}, \quad (A-17) \]
\[ D_N = 2\pi \left\{ (-1)^6 f + \left[ \frac{1}{2}(1 - (-1)^6) + k \right] \right\} \times L - z \]
\[ + (d - x)^2 + (e - y)^2 \right)^{3/2}, \quad (A-18) \]
\[ D_N = 2\pi \left\{ (-1)^6 f + \left[ \frac{1}{2}(1 - (-1)^6) + k \right] \right\} \times L - z \]
\[ + (1.25r \cos \theta_i - a/4 - x)^2 \]
\[ + (1.25r \sin \theta_i - b/4 - y)^2 \right)^{3/2}, \quad (A-19) \]
\[ D_O = 2\pi \left\{ (-1)^6 n + \left[ \frac{1}{2}(1 - (-1)^6) + k \right] \right\} \times L - z \]
\[ + (l - x)^2 + (m - y)^2 \right)^{3/2}, \quad (A-20) \]
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