Consensus disturbance rejection control of directed multi-agent networks with extended state observer

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Abstract This paper investigates the consensus disturbance rejection problem among multiple high-order agents with directed graphs. Based on disturbance observers, distributed consensus disturbance rejection protocols are constructed in leaderless and leader-follower consensus setups. Different from the previous related papers, the consensus protocols in this paper are developed in a fully distributed fashion, relying on only the state information of each agent and its neighbors. Sufficient conditions are provided to guarantee that the asymptotic stability of high-order multi-agent systems can be reached with matched disturbances.

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1. Introduction

Cooperative control of multi-agent systems is an emerging and cutting edge research topic in the field of systems and control over the past decades, owing to its application in military and civilian areas such as satellite formation, telecommunication relay, and target search.1–4 All of these applications benefit from the improvements of the study on consensus control approaches. From different perspectives, many pioneers have put forward many insights into the consensus control problem and its related problems.5–8 One common demerit in the most existing works is that the controller design requires the smallest nonzero eigenvalue information of the Laplacian matrix of the communication topology.9–12 However, the agent cannot calculate and implement it without the knowledge of the entire communication topology. Therefore, although these consensus controllers can be implemented in a distributed fashion, they cannot be designed by each agent in a distributed way. In other words, they are not fully distributed. To overcome this limitation, Ref.13 proposed fully distributed protocols to deal with the consensus problem for both the leaderless case and the leader-follower case with the use of adaptive parameters to estimate the unknown network connectivity. Further extension of Ref.13 can be found in Ref.14, where researchers developed distributed adaptive consensus protocols for any communication topologies with a directed spanning tree.
Note that the adaptive consensus controllers proposed in Refs.13,14 were only feasible to the case without external disturbances attributing to the parameter drift phenomenon.15 To alleviate the effect of external disturbances, one commonly used method is to design an observation mechanism to obtain the estimate of disturbances and then generate proper compensation.16–19 By integrating the sliding-mode control and the disturbance observer mechanism, Ref.17 made a major advance in the research of the finite-time consensus problem. In Ref.18, systematic designs for disturbance-observer-based protocols were proposed and analyzed for linear systems for both the cases with and without a leader. Ref.19 extended Ref.18 to the case with event-triggered communications. However, most of the above-reported controller designs regarding the consensus disturbance rejection problem are not fully distributed and only feasible to undirected graphs. As far as we know, how to build fully distributed disturbance rejection schemes with general directed graphs is still open.

Following the above discussions, we intend to solve the disturbance rejection consensus problem with directed graphs in this paper. To address this problem, we first get the estimate of matched disturbances and model dynamics by constructing a Linear Extended State Observer (LESO) with the use of the local output information. Then, when only the LESO’s states are available, distributed adaptive disturbance rejection consensus protocols are designed to achieve consensus and reject the influence of the external disturbances under directed graphs. For both the cases without and with a virtual leader, the practical algorithms are developed to calculate the gain matrices of the control protocols and the adaptive laws. The main difficulty is to achieve a joint design of the observer gain parameter and fully distributed consensus protocols such that the performance requirement can be satisfied. In contrast to the related works, this paper has the following contributions. First, the asymptotically convergent consensus error can still be reached by multi-agent systems with matched disturbances, while the consensus error in Ref.20 can only asymptotically converge to a bounded region. Second, our consensus protocols can achieve leaderless consensus and leader-follower consensus for directed graphs with some common assumptions in the literature about network connectivity. Only undirected graphs were considered in Refs.17,19. Third, compared with Refs.18,19, all the control protocols proposed in this paper do not require any global information, which means they are fully distributed. A preliminary version of this paper was reported in Ref.21, where only undirected graphs were considered.

This paper is organized as follows. Section 2 describes the problem treated in this paper and recalls some basic knowledge of algebraic graph theory. The leaderless disturbance rejection consensus problem with strongly connected graph is addressed in Section 3. The leader-follower disturbance rejection problem with any directed graphs containing a directed spanning tree is considered in Section 4. Section 5 gives a numerical simulation to validate the feasibility and efficiency of the obtained theoretical results. A brief conclusion is drawn in Section 6.

2. Problem formulation

The multi-agent system considered in this paper consists of N identical agents, described by

$$\begin{align*}
\dot{x}^{(i)}_t & = x^{(i+1)}_t, & t = 1, 2, \ldots, n - 1 \\
\dot{x}^{(0)}_t & = f(x_t, m_t) + u_i, & i = 1, 2, \ldots, N
\end{align*}$$

(1)

where $x_t = [x^{(1)}_t, x^{(2)}_t, \ldots, x^{(n)}_t]^T \in \mathbb{R}^n$ is the state information of agent $i$. The nonlinear function $f(x_t, m_t)$ denotes the inner dynamics of agent $i$ with $m_t$ being the external disturbances. $u_i \in \mathbb{R}$ is the control signal. In the Extended State Observer (ESO) scheme, we define a new state by $\hat{x}^{(i+1)}_t = f(x_t, m_t)$. In this work, we presume that $f(x_t, m_t)$ is differentiable, then we can transform Eq. (1) to

$$\begin{align*}
\dot{x}^{(i)}_t & = \hat{x}^{(i+1)}_t, & t = 1, 2, \ldots, n - 1; & i = 1, 2, \ldots, N \\
\dot{x}^{(0)}_t & = \hat{x}^{(n+1)}_t + u_i \\
\hat{x}^{(n+1)}_t & = h(x_t, m_t)
\end{align*}$$

(2)

where $h(x_t, m_t) = f(x_t, m_t)$, and the following assumption is needed.

Assumption 1. $h(x_t, m_t)$ is globally Lipschitz with respect to $x_t$. In other words, for all $x_t, \dot{x}_t$ and $m_t$, we can always find a constant $\epsilon$ such that $h(x_t, m_t)$ satisfies the inequality $|h(x_t, m_t) - h(\dot{x}_t, m_t)| \leq \epsilon |x_t - \dot{x}_t|$. Let

$$\begin{align*}
A & = \begin{bmatrix} 0_{(n-1) \times 1} & I_{(n-1)} \\ 0 & 0_{1 \times (n-1)} \end{bmatrix} \in \mathbb{R}^{n \times n} \\
B & = [0, \ldots, 0, 1]^T \in \mathbb{R}^n
\end{align*}$$

Then, system (2) can be transformed to

$$\begin{align*}
\dot{x}_t & = Ax_t + Bu_i + f(x_t, m_t) \\
& = Ax_t + Bu_i + \hat{x}^{(n+1)}_t
\end{align*}$$

(3)

It is natural to use a directed graph $G(V, E)$ with the node set $V$ and the edge set $E$ to model the information exchange relationship between distinct agents. A node represents an agent in the network. An edge $(i, j)$ represents that agent $j$ can receive the output information of agent $i$. A directed path is a finite sequence of edges of the form $(i_1, i_2), (i_2, i_3), \ldots$. If there exists a directed path from every node to every other node, the directed graph $G$ is strongly connected. The directed graph $G$ is said to contain a directed spanning tree if there exists a node called the root that has a directed path to all the other nodes in $G$. Define the adjacency matrix $A$ with elements $a_{ij}$ such that $a_{ij} = 1$ if $(i, j) \in E$, and $a_{ij} = 0$ otherwise. Define the Laplacian matrix $L$ in the normal way as $L = \sum_i a_{ij} l_i$ and $l_i = -a_{ij}$ when $i \neq j$. Let $r = [r_1, r_2, \ldots, r_n]$ be the positive left eigenvector of $L$ associated with the zero eigenvalue, $R = \text{diag}(r_1, r_2, \ldots, r_n)$, and $\bar{L} = RL + L^TR$. Then, we have the following lemmas.

Lemma 1. 22 For a directed strongly connected graph $G$, $\bar{L}$ is a symmetric matrix. Furthermore, the smallest nonzero eigenvalue $\lambda_2(L)$ of $L$ satisfies

$$\min_{\xi > 0} \frac{\xi^T L \xi}{\xi^T \xi} > \frac{\lambda_2(L)}{2},$$

where $\xi$ is any vector with positive entries.

Lemma 2. 23 If the directed graph $G$ has a directed spanning tree, zero is a simple eigenvalue of corresponding Laplacian matrix $L$ and all nonzero eigenvalues have positive real parts.
Our purpose is to design fully distributed consensus control protocols with directed graphs and matched disturbances satisfying Assumption 1, under which the states of all the agents in Eq. (1) come into agreement, i.e.\[ \lim_{t \to +\infty} \| x_i - x_j \| = 0 \quad \forall i, j = 1, 2, \ldots, N. \]

3. Leaderless consensus

Due to the existence of matched disturbances, the following control design consists of two parts, namely the LESO design and the leaderless consensus protocol design.

3.1. LESO design

Inspired by Ref.\(^{24}\), the following LESO is proposed for each agent:
\[
\begin{align*}
\dot{x}_i^{(k)} &= x_i^{(k+1)} + o_k z_i \left( x_i^{(k)} - x_0 \right) \\
x_i^{(k+1)} &= x_i^{(k+1)} + o_k z_i \left( x_i^{(k)} - x_0 \right) + u_k \\
\dot{z}_k &= a_0 z_i + z_k^{(k)} - x_i^{(k)} + h(x_i, m_i) \quad i = 1, 2, \ldots, n - 1
\end{align*}
\]
where \( \dot{x}_i = \left[ x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(n)} \right]^T \in \mathbb{R}^n \) is the estimate of \( x_i \), and \( o_k \) is the positive observer gain parameter to be chosen. Choose parameters as Ref.\(^{24}\), i.e., \( z_k = \frac{o_k}{o_k + 1} \) (\( k = 1, 2, \ldots, n + 1 \)). From Eq. (4), we have
\[
\dot{x}_i = \dot{x}_i^{(1)} + \Phi C^T (x_i - \dot{x}_i)
\]
where \( \Phi = \left[ a_0, o_k z_i, o_k z_i^2, \ldots, o_k^n z_i^n \right]^T \in \mathbb{R}^n \) and \( C = [1, 0, \ldots, 0]^T \in \mathbb{R}^n \). To describe the observation error, we define \( \dot{x}_i^{(k)} \) and \( e_i^{(k)} \) as
\[
\begin{align*}
\dot{x}_i^{(k)} &= x_i^{(k)} - x_i^{(k-1)} \\
e_i^{(k)} &= o_k^{(i)} z_i^{(k)} + o_k^{(i)} z_i^{(k-1)} \quad k = 1, 2, \ldots, n + 1.
\end{align*}
\]
From Eqs. (2), (4) and (6) the estimate error can be shown as
\[
e_i = o_k A e_i + B \frac{h(x_i, m_i) - h(x_i, m_i)}{o_k^{(i)}}
\]
with
\[
\begin{align*}
A &= \begin{bmatrix}
-2I & 1 & 0 & \cdots & 0 \\
-2I & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-2I & 0 & 0 & \cdots & 1 \\
-2I & 0 & 0 & \cdots & 0
\end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)} \\
B &= \begin{bmatrix}
0, \ldots, 0, 1
\end{bmatrix}^T \in \mathbb{R}^{n+1} \\
e_i &= \left[ e_i^{(1)}, e_i^{(2)}, \ldots, e_i^{(n+1)} \right]^T \in \mathbb{R}^{n+1}
\end{align*}
\]
Following the above discussions, there are some useful lemmas.

Lemma 3\(^{24}\). Assume that Assumption 1 holds, and the observer gain parameter \( o_k > 1 + \| P \| \), where \( P \) is a positive solution to the following Lyapunov equation:
\[
PA + A^T P - 2PB \tilde{B} + \tilde{P} = 0
\]
Then the observation errors \( \dot{x}_i^{(k)} \) (\( i = 1, 2, \ldots, N; \ k = 1, 2, \ldots, n \)), will asymptotically converge to the origin.

Lemma 4. Select an appropriate observer gain, that satisfies Lemma 3. Then, leaderless consensus is reached if \( \lim_{t \to +\infty} \| x_i - x_j \| = 0 \ (\forall i, j = 1, 2, \ldots, N) \).

3.2. Leaderless consensus protocols design

In this section, we assume that the communication graph \( G \) is strongly connected. Define \( \xi = \sum_{i=1}^{N} a_{ij} (x_i - x_j) \in \mathbb{R}^n \), \( \xi = [\xi_1, \xi_2, \ldots, \xi_n]^T \text{ and } \hat{x} = [x_1, x_2, \ldots, x_n] \). Then, we have
\[
\dot{\xi} = (L \otimes I_{k}) \dot{x}
\]
where \( L \) is a lower triangular matrix and \( I_{k} \) is an identity matrix of order \( k \).

For the strongly connected graph \( G \), it is not difficult to verify that the leaderless consensus problem is addressed if \( \xi \) asymptotically converges to the origin in light of Lemma 1 and Lemma 4. By using the estimates of neighboring agents, we develop the following fully distributed disturbance rejection consensus protocol for each agent:
\[
\begin{align*}
\dot{u}_i &= (\theta_i + \rho_i) K \dot{x}_i - x_i^{(1)} \\
\dot{\theta}_i &= \xi_i^T \Gamma \xi_i \quad i = 1, 2, \ldots, N
\end{align*}
\]
where \( K \) and \( \Gamma \) are the feedback gain matrices, \( \theta_i \) is the adaptive gain of the \( i \)-th agent with \( \dot{\theta}_i(0) \geq 0 \), and \( \rho_i \) are the smooth functions. Substituting Eq. (10) into Eq. (5) yields
\[
\begin{align*}
\dot{\xi} &= I_{k} \otimes L + L [\theta + \rho \otimes BK] \xi + (L \otimes \Phi C^T)(x - \hat{x}) \quad (11)
\end{align*}
\]
where \( \theta = \text{diag}(\theta_1, \theta_2, \ldots, \theta_N), \rho = \text{diag}(\rho_1, \rho_2, \ldots, \rho_N), x = [x_1^T, x_2^T, \ldots, x_N^T]^T \), and \( \hat{x} = [\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N]^T \).

In what follows, a theorem is presented to provide a practical method to design protocol Eq. (10).

Theorem 1. Suppose that communication graph \( G \) is strongly connected, and \( \dot{P} \) is a positive solution to the following algebraic Riccati equation:
\[
PA + A^T P - 2PB \tilde{B} + \tilde{P} = 0
\]
Then the leaderless consensus problem can be addressed under the disturbance rejection consensus protocol Eq. (10) with
\[
K = -B^T P, \Gamma = P B B^T P, \text{ and } \rho_i = \xi_i^T P \xi_i (i = 1, 2, \ldots, N).
\]

Proof 1. Denote
\[
\begin{align*}
V_{11} &= \frac{1}{2} \xi^T [(\theta + \rho) R \otimes P] \xi \quad (13) \\
V_{12} &= \frac{o_k}{o_k + 1} \sum_{i=1}^{N} \xi_i^T P \tilde{e}_i \\
V_{13} &= \frac{1}{\rho} \sum_{i=1}^{N} \rho_i (\theta_i - \vartheta_i)^2
\end{align*}
\]
where \( \vartheta_i \) is a positive constant to be determined later. Consider the Lyapunov function candidate:
\[
V_i = V_{11} + V_{12} + V_{13}
\]
It is easy to see that \( V_i \geq 0 \) and
Taking the derivative of $V_{11}$ along with Eq. (11) gives
\begin{equation}
V'_{11} = \xi^T[(\dot{\theta} + \rho) R \otimes P_{R}e_{1}^T] + \xi^T[(\dot{\theta} + \rho) R \otimes \dot{P}] \xi = \xi^T[(\dot{\theta} + \rho) R \otimes P_{R}e_{1}^T] + \xi^T[(\dot{\theta} + \rho) R \otimes \dot{P}] \xi
\end{equation}
where $\dot{\theta} = \text{diag}(\xi_{1}^{\Gamma}, \xi_{2}^{\Gamma}, \ldots, \xi_{n}^{\Gamma})$ and $\dot{\rho} = \text{diag}(2\xi_{1}^{P_{1}e_{1}}, 2\xi_{1}^{P_{2}e_{1}}, \ldots, 2\xi_{n}^{P_{n}e_{1}})$. Let $\sigma = [(\theta + \rho) \otimes L] \xi$. Noting that $r^T L = 0$, we can get that
\begin{equation}
\sigma^T [(\theta + \rho)^{-1}r \otimes 1] = \xi^T(r \otimes 1) = \xi^T(L^r \otimes 1) = 0
\end{equation}
Invoking Lemma 1, we have
\begin{equation}
\sigma^T(L \otimes L) \sigma > \frac{\lambda}{N} \sigma^T = \frac{\lambda}{N} \xi^T[(\theta + \rho)^{-2} \otimes L] \xi
\end{equation}
Note that
\begin{equation}
\xi^T(\dot{R} \otimes P^{-1}) \xi = \sum_{i=1}^{N} r_i \rho_i \hat{c}_i = \xi^T(\rho R \otimes \Gamma) \xi
\end{equation}
and
\begin{equation}
2\xi^T[(\theta + \rho) R \otimes \Phi \Phi^T](x - \hat{x}) 
\leq \xi^T[(\theta + \rho)^{-2} \otimes L] \xi + \|RL \otimes \Phi \Phi^T\|^2 \sum_{i=1}^{N} \|x_i - \hat{x}_i\|^2
\end{equation}
where we have used Young's inequality Eq. (25) and matrix norm properties to get Eq. (20). Substituting Eqs. (18)–(20) into Eq. (16), we can obtain
\begin{equation}
V'_{11} \leq \xi^T[(\theta + \rho) R \otimes (PA + A^T P) \xi] - \frac{\lambda}{N} \xi^T[(\theta + \rho)^{-2} \otimes L] \xi + 2\xi^T[(\theta + \rho) R \otimes \Phi \Phi^T](x - \hat{x}) + \xi^T(\rho R \otimes \Gamma) \xi
\end{equation}
Using Eq. (7), the time derivative of $V_{12}$ is given by
\begin{equation}
V'_{12} = \omega_{0}^2 \|RL \otimes \Phi \Phi^T\|^2
\end{equation}
\begin{equation}
\times \sum_{j=1}^{N} \left[ \omega_{0}^2(\dot{A}P + \dot{PA}^T)e_{j} + 2e_{j}^T \dot{P}B \frac{h(x_{j}, \sigma_{j}) - h(\hat{x}_{j}, \sigma_{j})}{\omega_{0}^2} - \frac{h(x_{j}, \sigma_{j})}{\omega_{0}^2} \right]
\end{equation}
Note that
\begin{equation}
2e_{j}^T \dot{P}B \frac{h(x_{j}, \sigma_{j}) - h(\hat{x}_{j}, \sigma_{j})}{\omega_{0}^2} \leq 2e_{j}^T \dot{P}Bc_{j} \|x_{j} - \hat{x}_{j}\|/\omega_{0}^2
\end{equation}
and
\begin{equation}
\|x_{j} - \hat{x}_{j}\| \leq \sqrt{\epsilon_{j}^{(12)} + \omega_{0}^2 \epsilon_{j}^{(22)} + \cdots + \omega_{0}^2 \epsilon_{j}^{(n2)}} \leq \omega_{0}^2 \|e_{j}\|
\end{equation}
where we have used the fact that $\omega_{0} \geq 1$. Therefore, we obtain
\begin{equation}
2e_{j}^T \dot{P}B^{T} h(x_{j}, \sigma_{j}) - h(\hat{x}_{j}, \sigma_{j})/\omega_{0}^2 \leq \left(1 + \|\dot{P}Bc_{j}\|^2\right)\|e_{j}\|^2
\end{equation}
Substituting Eqs. (8) and (25) into Eq. (22), we have
\begin{equation}
V'_{12} \leq \left(\omega_{0}^2 \|RL \otimes \Phi \Phi^T\|^2\right) \sum_{j=1}^{N} \|e_{j}\|^2
\end{equation}
From Eq. (10), we have
\begin{equation}
V'_{13} = \sum_{i=1}^{N} r_{i}(\theta_i - \hat{\theta}_i)^2 \xi_{i}^{\Gamma} \xi_{i}
\end{equation}
Substituting Eqs. (21), (26) and (27) into Eq. (15) yields
\begin{equation}
V'_{1} \leq \xi^T[(\theta + \rho)R \otimes \dot{PA} + A^T P \otimes \Gamma] \xi - \xi^T\left(\frac{\lambda}{N} (\theta + \rho)^{-2} \otimes L\right) \xi + \|RL \otimes \Phi \Phi^T\|^2 \sum_{i=1}^{N} \|x_i - \hat{x}_i\|^2 + \xi^T(\theta + \rho)^2 \otimes L \xi
\end{equation}
Invoking Young's inequality again, we have
\begin{equation}
-\xi^T\left(\frac{\lambda}{N} (\theta + \rho)^{-2} \otimes L\right) \xi \leq -\xi^T\left(\frac{\lambda}{N} (\theta + \rho)^{-2} \otimes \Gamma\right) \xi \leq -2\xi^T\left(\frac{\lambda}{N} \theta + \rho)^{-2} \otimes \Gamma\right) \xi
\end{equation}
Choosing $\theta^\ast \geq \frac{\max_{i=12} \abs{\lambda}}{\lambda_{N}}$, we have
\begin{equation}
-\xi^T\left(\frac{\lambda}{N} (\theta + \rho)^{-2} \otimes \Gamma\right) \xi \leq -\xi^T\left(\frac{\lambda}{N} (\theta + \rho)^{-2} \otimes \Gamma\right) \xi
\end{equation}
Substituting Eqs. (24) and (30) into Eq. (28), we get
\begin{equation}
V'_{1} \leq \xi^T[(\theta + \rho)R \otimes (PA + A^T P - 2\Gamma) \xi] + \xi^T[(\theta + \rho)^2 \otimes \Gamma] \xi
\end{equation}
where we have used Eq. (12) to get the last inequality. Since $V_{1} \leq 0$, $V_{1}$ is bounded and thereby $\theta$ are bounded. By noting that $\theta_{j}$ are monotonically increasing, it follows that each cou-
pling weight \( \vartheta_i \) converges to some steady-state value. Note that \( \dot{V}_1 \equiv 0 \) means that \( \sigma \equiv 0 \) and thereby \( \xi \equiv 0 \). Invoking LaSalle’s Invariance principal,\(^{25}\) we can conclude that the consensus error \( \xi \) asymptotically converges to zero, which implies that the leaderless consensus problem is solved.

**Remark 1.** A practical method is proposed in Theorem 1 to construct a fully distributed controller that addresses the disturbance rejection consensus problem for more general situations than those studied in Refs.\(^{18,19}\). More specific, in Ref.\(^{18}\), the network connectivity was needed in the disturbance observer design to achieve leaderless consensus for a directed graph, while the consensus disturbance rejection scheme developed in this paper is dependent on only local information in both the LERO design and the consensus protocol design. Compared with Refs.\(^{18,19}\), our assumption on the external disturbance is more general.

**Remark 2.** The design approach of adaptive consensus protocol Eq. (10) is inspired by Ref.\(^{26}\). In comparison to Ref.\(^{20}\), our consensus protocols are developed by making use of the state estimates of the agent and its neighbors, instead of their real state information, which can be used in some circumstances where the agents only have access to position information. Moreover, the consensus error can only be guaranteed to converge into a bounded region in Ref.\(^{26}\) with external disturbances, while the consensus protocols developed in this paper can eliminate the impact of matched disturbances by the control input directly and guarantee the asymptotical convergence of the consensus error.

### 4. Leader-follower consensus

The reached consensus value in the earlier section relies on all the agents involved in the network and cannot be controlled. In this section, we intend to treat another form of consensus control where there is a virtual leader and other agents are controlled to follow this virtual leader. Consider a set of \( N \) agents moving with a virtual leader described by

\[
\dot{x}_0 = Ax_0
\]  

where \( x_0 = [x^{(1)}_0, x^{(2)}_0, \ldots, x^{(N)}_0]^T \in \mathbb{R}^N \). And we use the same LERO Eq. (4) to estimate the state information and the matched disturbance information of the followers. The communication topology \( \mathcal{G} \) among the \( N+1 \) agents is supposed to have a directed spanning tree where the virtual leader is the root node. Then, the corresponding Laplacian matrix \( L \) can be partitioned as

\[
\begin{bmatrix}
0 & \mathbf{0}_{1 \times N} \\
L_2 & L_1
\end{bmatrix}
\]

According to the definition of M-matrix and Lemma 2, \( L_1 \) is a nonsingular M-matrix. Then, the following lemmas will be useful in the proof of Theorem 2.

**Lemma 5**\(^{26}\). There exists a positive diagonal matrix \( \mathbf{G} = \text{diag}(g_1, g_2, \ldots, g_N) \) such that nonsingular M-matrix \( L_1 \) satisfies \( \mathbf{G}L_1 + L_1^T \mathbf{G} \succ 0 \).

**Lemma 6.** With LERO Eq. (4), the leader-follower consensus is reached if \( \lim_{t \to \infty} \| x_i - x_0 \| = 0 \) (\( i = 1, 2, \ldots, N \)).

**Proof 2.** This lemma results from Lemma 3 immediately. Therefore, the proof is omitted here for the sake of brevity.

Define \( \zeta_i = \sum_{j=0}^N a_{ij} (x_i - x_j) \in \mathbb{R}^n, \zeta = [\zeta_1^T, \zeta_2^T, \ldots, \zeta_N^T]^T \in \mathbb{R}^{Nn} \). Then, we have

\[
\zeta = (L_1 \otimes L_2) (x - \mathbf{1} \otimes x_0)
\]  

According to Lemma 2 and Lemma 6, it is not hard to verify that the leader-follower consensus is achieved if \( \zeta \) asymptotically converges to the origin. By making use of the relative observation state information of neighboring agents, a distributed disturbance rejection consensus protocol is proposed for each follower as

\[
\left\{ \begin{aligned}
\dot{y}_1 &= (\overline{y}_1 + \mathbf{p}_1) K \overline{\zeta}_1^T - x_i^{(n+1)} \\
\dot{\overline{y}}_1 &= \overline{y}_1 \Gamma \overline{\zeta}_1^T & i = 1, 2, \ldots, N
\end{aligned} \right.
\]  

where \( \overline{y}_i \) is the adaptive gain with \( \overline{y}_i(0) \succ 0, \mathbf{p}_i \) is a smooth function. Combining Eqs. (35) and (5) together yields

\[
\dot{\overline{\zeta}} = (L_1 \otimes A + L_1(\overline{\mathbf{p}} + \mathbf{p}) \otimes B \mathbf{K}) \overline{\zeta} + (L_1 \otimes \Phi C^T)(x - \bar{x})
\]

where \( \overline{\mathbf{p}} = \text{diag}(\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_N) \) and \( \Phi = \text{diag}(p_1, p_2, \ldots, p_N) \).

**Theorem 2.** Suppose that Assumption 1 holds and \( \mathcal{G} \) has a directed spanning tree with the virtual leader as the root. With LERO Eq. (4), the leader-follower consensus can be achieved under the distributed disturbance rejection consensus protocol Eq. (36) with \( \overline{y}_i = \overline{y}_i^T \mathbf{P}_e^T (i = 1, 2, \ldots, N), K, \Gamma, \) and \( \mathbf{P} \) designed as in Theorem 1.

**Proof 3.** Denote

\[
\left\{ \begin{aligned}
V_1 &= \frac{1}{2} \overline{\mathbf{y}}_1^T \left[ 2 \overline{\mathbf{p}} + \mathbf{p} \right] G \otimes \mathbf{P} \overline{\mathbf{y}}_1 \\
V_2 &= \frac{\sigma^2 \gamma^2}{\alpha_{n+1} - \sigma^2 \gamma^2} \sum_{i=1}^N e_i^T \mathbf{P} e_i \\
V_3 &= \frac{1}{2} \sum_{i=1}^N (e_i - \overline{e}_i)^2
\end{aligned} \right.
\]

where \( \sigma^2 \gamma^2 \) is a positive constant to be determined later. \( G = \text{diag}(g_1, g_2, \ldots, g_N) \) is a positive definite matrix such that \( L_1 G + L_1^T G \succ 0 \). Construct the following Lyapunov function:

\[
V_2 = V_{21} + V_{22} + V_{23}
\]

It is not hard to see that \( V_2 \succeq 0 \) and

\[
\dot{V}_2 = \dot{V}_{21} + \dot{V}_{22} + \dot{V}_{23}
\]

Taking the derivative of \( V_1 \) with Eq. (38) gives
Consensus disturbance rejection control

\[ \dot{V}_{21} = \xi^{T} \left( (2\bar{\sigma} + \bar{p})G \otimes P \right) \xi + \xi^{T} \left( (\bar{\sigma} + \bar{p})G \otimes P \right) \xi \]

\[ = 2\xi^{T} \left( (\bar{\sigma} + \bar{p})G \otimes P \right) \xi + \xi^{T} \left( G\dot{\sigma} \otimes P \right) \xi \]

\[ = \bar{\xi}^{T} \left( (\bar{\sigma} + \bar{p})G \otimes (PA + A^{T}P) \right) - \left( (\bar{\sigma} + \lambda_{\min}(\bar{\sigma} + \bar{p})^{2}) \otimes I \right) \xi \]

\[ + 2\xi^{T} \left[ (\bar{\sigma} + \bar{p})GL_{1} \otimes \Phi \Phi^{T} \right] (x - \dot{x}) + \xi^{T} \left( G\dot{\sigma} \otimes P \right) \xi \]

\[ \leq \bar{\xi}^{T} \left( (\bar{\sigma} + \bar{p})G \otimes (PA + A^{T}P) - \left( (\bar{\sigma} + \lambda_{\min}(\bar{\sigma} + \bar{p})^{2}) \otimes I \right) \xi \right) \]

\[ + 2\xi^{T} \left[ (\bar{\sigma} + \bar{p})GL_{1} \otimes \Phi \Phi^{T} \right] (x - \dot{x}) + \xi^{T} \left( G\dot{\sigma} \otimes P \right) \xi \]

(40)

where \( \lambda_{\min} \) represents the smallest positive eigenvalue of \( GL_{1} + L_{1}G \). Similar to the proof given in Theorem 1, we can get

\[ \dot{V}_{2} \leq \bar{\xi}^{T} \left( (\bar{\sigma} + \bar{p})G \otimes (PA + A^{T}P + I) \right) \xi \]

\[ - \bar{\xi}^{T} \left[ \left( \lambda_{\min}(\bar{\sigma} + \bar{p})^{2} + \bar{\sigma}G \right) \otimes I \right] \xi \]

\[ + \bar{\xi}^{T} \left( (\bar{\sigma} + \bar{p})^{2} \otimes L \right) \xi \]

(41)

Similar to Eq. (29), we have

\[ - \bar{\xi}^{T} \left( \left( \lambda_{\min}(\bar{\sigma} + \bar{p})^{2} + \bar{\sigma}G \right) \otimes I \right) \xi \]

\[ \leq -2\bar{\xi}^{T} \left( \sqrt{\lambda_{\min}(\bar{\sigma} + \bar{p})^{2} + \bar{\sigma}G} \right) \xi \]

(42)

Choosing \( \bar{\sigma} \geq \frac{\lambda_{\min}(\bar{\sigma} + \bar{p})^{2} + \bar{\sigma}G}{4\lambda_{\min}} \), we have

\[ - \bar{\xi}^{T} \left( \left( \lambda_{\min}(\bar{\sigma} + \bar{p})^{2} + \bar{\sigma}G \right) \otimes I \right) \xi \leq -3\bar{\xi}^{T} \left( (\bar{\sigma} + \bar{p})G \otimes I \right) \xi \]

(43)

Substituting Eq. (43) into Eq. (41), we get

\[ \dot{V}_{2} \leq \bar{\xi}^{T} \left( (\bar{\sigma} + \bar{p})G \otimes (PA + A^{T}P - 2I) \right) \xi + \bar{\xi}^{T} \left( (\bar{\sigma} + \bar{p})^{2} \otimes L \right) \xi \]

\[ \leq \bar{\xi}^{T} \left( \left( \lambda_{\min}(\bar{\sigma} + \bar{p})^{2} + \bar{\sigma}G \right) \otimes (PA + A^{T}P - 2I) \right) \xi + \bar{\xi}^{T} \left( (\bar{\sigma} + \bar{p})^{2} \otimes L \right) \xi \]

\[ = \bar{\xi}^{T} \left( (\sigma + \rho)^{2} \otimes (PA + A^{T}P - 2I) \right) \xi + \bar{\xi}^{T} \left( (\sigma + \rho)^{2} \otimes L \right) \xi \]

(44)

The remainder of the proof is similar to that of Theorem 1 and the details are omitted here for brevity.

![Communication graph](image1)

**Fig. 1** Communication graph.

(a) Consensus errors of the first state  
(b) Consensus errors of the second state  
(c) Consensus errors of the third state  

![Consensus errors](image2)

**Fig. 2** Consensus errors.
Remark 3. The virtual leader, who does not take the information of other agents, can be regarded as the reference model in the traditional model reference adaptive control design. This kind of set-up with the leader having the same dynamics as followers is commonly used in spacecraft and robot formation control.\textsuperscript{27}

Remark 4. The disturbance observer based consensus protocols proposed in this paper are continuous, which effectively avoid the chattering phenomenon caused by the sliding mode controller in Ref.\textsuperscript{17}. In addition, both the leaderless strongly-connected directed graphs and any leader-follower graphs with a directed spanning tree are investigated in this paper, while only leader-follower graphs, where the subgraphs among the followers were undirected, were considered in Ref.\textsuperscript{17}.

5. Simulation example

To validate the feasibility of the theoretical results, a numerical simulation is performed on a leaderless network of four third-order agents. The agent dynamics are described by Eq. (3) with

\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad i = 1, 2, \ldots, 4
\]

The disturbances associated with the agents are assumed to be \(f(x_i, m_i) = \sin(0.05x_i^{(1)}) + 1.5\sin(0.05x_i^{(2)}) + m_i\) and \(m_i = \sin(0.05x_i^{(1)} + 0.05x_i^{(2)}) (i = 1, 2, \ldots, 4)\). Let the communication topology be given as in Fig. 1. Choose \(a_1 = 4, a_2 = 6, a_3 = 4, \) and \(a_4 = 1\). Then, we have

\[
\hat{A} = \begin{bmatrix} -4 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

Solving the Eq. (8) by using function lyap of MATLAB gives the positive solution matrix

\[
\hat{P} = \begin{bmatrix} 0.3750 & 1.0000 & 1.1250 & 0.5000 \\ 1.0000 & 5.1250 & 5.5000 & 2.3750 \\ 1.1250 & 5.5000 & 8.3750 & 4.0000 \\ 0.5000 & 2.3750 & 4.0000 & 3.1250 \end{bmatrix}
\]

We choose \(\omega_c = 5.9692\) to satisfy the requirement of Lemma 3. Using function are of MATLAB, we can obtain the solution matrix \(P\) of the Eq. (12) and get the feedback matrices \(\Gamma\) and \(K\) as:

\[
P = \begin{bmatrix} 2.2572 & 2.0475 & 0.7071 \\ 2.0475 & 3.9145 & 1.5961 \\ 0.7071 & 1.5961 & 1.4478 \end{bmatrix}
\]

\[
\Gamma = \begin{bmatrix} 0.5000 & 1.1286 & 1.0273 \\ 1.1286 & 2.5475 & 2.3108 \\ 1.0237 & 2.3108 & 2.0961 \end{bmatrix}
\]

\[
K = \begin{bmatrix} -0.7071 & -1.5961 & -1.4478 \end{bmatrix}
\]

The results are shown in Figs. 2–4. The consensus errors \(x_i' - x_i'^{(1)} (i = 2, 3, 4, t = 1, 2, 3)\) are shown in Fig. 2, which implies that the states of all agents come into agreement, i.e., the consensus is reached. Fig. 3 shows that LESO Eq. (4)
novel consensus distributed disturbance rejection protocols have been developed for high-order multi-agent systems with directed graphs and matched disturbances in this paper. Different from the existing disturbance rejection consensus schemes, the proposed adaptive LESO-based protocols depend only on the agent dynamics and the relative observer state information of neighboring agents, which indicates that they can be designed in a fully distributed way. Based on the results presented in this paper, it is interesting to extend the proposed distributed adaptive protocols to multi-agent systems with mismatched disturbances and further consider the consensus disturbance rejection problem for leaderless multi-agent systems without requiring the strongly connected graph in the future.

6. Conclusions

Novel consensus distributed disturbance rejection protocols have been developed for high-order multi-agent systems with directed graphs and matched disturbances in this paper. Different from the existing disturbance rejection consensus schemes, the proposed adaptive LESO-based protocols depend only on the agent dynamics and the relative observer state information of neighboring agents, which indicates that they can be designed in a fully distributed way. Based on the results presented in this paper, it is interesting to extend the proposed distributed adaptive protocols to multi-agent systems with mismatched disturbances and further consider the consensus disturbance rejection problem for leaderless multi-agent systems without requiring the strongly connected graph in the future.

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