Axial Cascade Technology and Application to Flow Path Designs. Part II—Application of Data to Flow Path Designs

The cascade considerations presented in Part I are extended to obtain interrelations between cascade parameters and machine similarity parameters, and to assess the effect of compressibility on these interrelations. These data are used to compute the efficiency potential and cavitation sensitivity of single cascaded and multiple cascaded axial machines in the medium to high specific speed regime. The calculated data show fair agreement with available test information. The effect of only a limited number of design parameters has been investigated. Further studies will be required to cover the performance potential adequately.

**Introduction**

Generally valid relations for profile, endwall, and clearance losses in cascades have been presented in Part I as function of the cascade parameters. These data are used to calculate the efficiency potential of axial flow machines as function of suitable machine parameters. The intent of these studies is to recognize the effect of the cascade design on the efficiency, and to explore means for improving the performance of axial flow machines in operating regimes of general interest.

**Turbomachine Parameters**

Convenient parameters for identifying a turbomachine are the similarity parameters, specific speed \( N_s \) and specific diameter \( D_s \), defined as

\[
N_s = \frac{N \sqrt{V}}{H_{ad}^{1/3}} \quad (1)
\]

and

\[
D_s = \frac{D H_{ad}^{1/3}}{V} \quad (2)
\]

when \( N \) denotes the rotative speed, \( V \) the flow rate at stage inlet for pumps and compressors (but stage exit for turbines), \( H_{ad} \) the head produced by compressors and pumps (but expanded by turbines), and \( D \) the tip diameter of the rotor. The cascade parameters \( \delta_u \) (blade load parameter) and \( \beta_m \) (mean vector angle) are interrelated with \( N_s \) and \( D_s \), the specific form of this interrelation depending on the assumptions made regarding the vortex distribution before and after the cascade. Assuming a pump or compressor rotor with “zero inlet swirl” at rotor inlet and a “free vortex” at rotor exit (constant head per streamline, constant axial velocity), the \( \delta_u \) and \( \beta_m \)-values for every streamline follow the relations

\[
cot \beta_{m-r} = D_s (1 - \lambda^2) \left[ \frac{7.5g}{r_a} \left( \frac{r_e}{r_a} \right) \left( \frac{\eta}{\beta_m} \right) - \frac{\pi^2 N_s D_s^2}{240 \left( \frac{r_e}{r_a} \right) \left( \frac{\eta}{\beta_m} \right) \xi D_s^2} \right] \quad (3)
\]

\[
\delta_u = \frac{15 g D_s (1 - \lambda^2)}{240 \left( \frac{r_e}{r_a} \right) \left( \frac{\eta}{\beta_m} \right) \xi D_s^2} \quad (4)
\]

when \( \lambda = d_{hub}/D \) denotes the hub ratio, \( \eta \) the hydraulic efficiency, and \( r_e/r_a \) the radius ratio, referring to the tip radius. At the same time the cascade parameters of the stator are directly interrelated with the rotor cascade parameters, e.g., for \( c_s = \text{const} \),

\[
\delta_{u\text{-sector}} = \delta_{u\text{-rotor}} \quad (5)
\]

\[
cot \beta_{m\text{-sector}} = \frac{-\delta_u}{2} \quad (6)
\]

if the assumption is made that the flow direction at stator exit is axial (zero exit swirl). With equations (3) to (6), the cascade parameters are determined for every streamline of the stage. Thus, the losses for every streamline, in rotor as well as stator, can be determined by the relations presented in Part I. Hence, the efficiency of every streamline and, by integration, the hydraulic efficiency of the stage can be calculated. The efficiency

---

**Nomenclature**

\[ A = \text{flow area, sq ft} \]
\[ a = \text{exponent for vortex distribution} \]
\[ c_s = \text{tangential velocity component, fps} \]
\[ D = \text{rotor tip diameter, ft} \]
\[ D_s = \text{specific diameter} \]
\[ H_{ad} = \text{adiabatic head, ft} \]
\[ H_s = \text{cavitation suppression head, ft} \]
\[ K = \text{cavitation parameter} \]
\[ k_s = \text{meridional acceleration factor} \]
\[ M = \text{Mach number} \]
\[ N = \text{rotative speed, rpm} \]
\[ N_s = \text{specific speed} \]
\[ p = \text{pressure, lb/sq ft} \]
\[ q = \text{head coefficient} \]
\[ R = \text{gas constant, ft./deg} \]
\[ r = \text{radius, ft} \]
\[ S^* = \text{suction specific speed} \]
\[ T = \text{temperature, deg F} \]
\[ u = \text{tip speed, fps} \]
\[ V = \text{flow rate, cu ft/sec} \]
\[ Z = \text{number of stages} \]
\[ \beta = \text{blade number} \]
\[ \eta = \text{efficiency} \]
\[ \lambda = \text{hub ratio} \]
\[ \kappa = \text{ratio of specific heats} \]
\[ \phi = \text{flow factor} \]
\[ \rho = \text{degree of reaction} \]

**Subscripts**

\[ h = \text{hydraulic} \]
\[ m = \text{meridional} \]
\[ o = \text{outer diameter} \]
\[ R = \text{rotor} \]
\[ at = \text{static} \]
\[ t = \text{total} \]
of the stage follows the relation
\[ \eta_h = 1 - \frac{\phi_v \Sigma^*}{2 q_{th}} \]  

(7)

the flow factor \( \phi_v \) being again a function of \( N_s \) and \( D_s \)
\[ \phi_v = \frac{240}{\pi^2 (1 - \lambda) N_s D_s^2} \]  

(8)

and the theoretical head coefficient \( q_{th} \) following the relation (zero
inlet swirl assumed)
\[ q_{th} = \frac{- \delta h_t}{\cot \beta_{a-o} - \delta h_t} \]  

(9)

The term \( \Sigma^* \) in equation (7) represents the overall losses, i.e., the
sum of profile, endwall, and clearance loss coefficients of the rotor
and stator. With equation (7), the original assumption for the
hydraulic efficiency in equations (3) and (4) can be checked so
that eventually, by iteration, the cascade geometry for rotor and
stator at every flow line, as well as the efficiency of every flow
line, and thus, by integration, the overall hydraulic efficiency of
the stage is determined as function of specific speed and specific
diameter. Varying then the specific diameter for a given specific
speed, eventually the maximum obtainable efficiency for a given
specific speed, i.e., the optimum \( D_s \)-value is found. In these
calculations the hub ratio has to be varied for every specific speed
and specific diameter calculations until the optimum \( \lambda \)-value is
found. In addition, an optimum blade number for rotor and
stator can be established by trial and error calculations. This
parameter enters the calculation when the endwall and clearance
losses are calculated, and when the relation
\[ h = \frac{z(1 - \lambda)}{2 \pi r_i / r_o} \]  

(10)
is observed, \( z \) denoting the blade number. The hub ratio \( \lambda \) in
equation (10) is interrelated with the cascade parameters by \( \lambda = 1
- 2 h / D \). In addition to the efficiency potential, the cavitation
sensitivity of pumps can be determined. For convenience, the
cavitation sensitivity may be expressed by the suction specific speed
defined as
\[ S^* = \frac{N \sqrt{V_{in}}}{H_{in}} \]  

(11)

when \( H_i \) denotes the cavitation suppression head. Of interest
usually is the maximum obtainable suction specific speed, i.e.,
design and operating conditions where the required cavitation
suppression head becomes a minimum. The numerical value of
\( H_i \) can be calculated from the cascade characteristic if the
assumption is made that cavitation can be avoided as long as the
minimum suction surface pressure is just equal to the vapor pres-
sure. This point usually is referred to as "incipient cavitation." 
Thus, the cavitation suppression head becomes equal to the
square of the effective maximum suction surface velocity (re-
ferred to the rotating element), i.e.,
\[ H_i = \frac{v_{r-i}}{2 g} \left( v_{r-i} \right)^2 - \frac{v_{r-i}}{2 g} ^2 \]  

(12)

\( v_{r-i} \) denoting the relative approach velocity to the rotor.
Expressing now \( v_{r-i} \) in terms of tip speed, flow factor, and radius
ratio, accounting for the difference between the relative inlet
velocity at cascade inlet \( v_i \) and the maximum suction surface
velocity \( v_{i-max} / v_i \), the cavitation suppression head can be
quoted in the form
\[ H_i = \frac{v_{i-max}^2}{2 g} \left( \frac{K^{*2} - 1}{(r_i / r_o)^2} + \phi_k^2 K^{*2} \right) \]  

(13)

with
\[ K^{*} = 1 + 0.9 \frac{\delta_{max}}{C} + 0.3 \phi_k \]  

(14)

the parameter \( \delta_{max} \) being a function of the cascade parameters \( \delta_i \) and
\( \phi_k \) (equation (12) of Part I). Introducing equation (13) into
equation (11), the expression for the suction specific speed for
every streamline reads
\[ S^* = \frac{30(2g)^{3/2}}{\sqrt{\pi}} \sqrt{(K^{*2} - 1) \left( \frac{r_o}{r_i} \right)^3 + \phi_k K^{*2}} \]  

(15)

By replacing the flow factor in equation (15) by equation (8), the
suction specific speed can also be quoted as function of \( N_s \) and \( D_s \).

The value of \( K^* \) depends on the rotor-cascade parameters, i.e.,
differs from streamline to streamline. Hence, different suction
specific speed values are obtained for different streamlines.

It is important to realize that the maximum suction surface
velocity is a function of the aerodynamic blade load coefficient
\( \beta_{a} \). A study of this interrelation indicates that the maximum
suction surface velocity decreases with decreasing aerodynamic
blade load coefficients. Thus, the suction specific speed, in
general, can be increased by selecting small values for the aerody-
namic blade load coefficient. It must, however, be considered
that decreasing aerodynamic blade load coefficients increase the
frictional surface, and thus decrease the cavitation efficiency. This
then means that, in general, cascades optimized for maximum
suction specific speed will tend to have a different geometry and
lower efficiencies than cascades optimized for highest efficiency.

The interrelations between cascade and machine parameters
for turbines are similar to the compressor relations, e.g., \( r_a = \) const,

\[ \cot \beta_{a-o} = D_s (1 - \lambda^2) \left( \frac{r_o}{r_i} \right)^3 - \frac{7.5 \phi_k \eta_h}{N_s} \]  

(3a)

\[ \delta h_t = \frac{15 \phi_k D_s (1 - \lambda^2) \eta_h}{N_s} \]  

(4a)

\[ \delta h_{stator} = \delta h_{rotor} \]  

(5a)

\[ \cot \beta_{a-stator} = \frac{\delta h_t}{2} \]  

(6a)

when similar assumptions regarding the vortex distribution are
made, swirl free inlet and exit flow, and free vortex distribution
between stator and rotor. The turbine efficiency becomes
\[ \eta_h = \frac{1}{1 + \frac{\phi_k \Sigma^*}{2 q_{th}}} \]  

(7a)

with
\[ q_{th} = \frac{- \delta h_t}{\cot \beta_{a-o} + \delta h_t / 2} \]  

(9a)

With equations (3) to (9), the streamline operating points can be
identified in the cascade diagram, Fig. 1. This plot shows that,
in general, the blade load coefficient \( \beta_{a} \) increases with decreasing
specific speeds, and that the nozzle and diffuser operating points
are on a line with a 2:1 slope (\( \beta_{a} = 90 \) deg and \( \beta_i = 90 \) deg, Fig.
19, Part I). The turbine rotors operate at higher \( \cot \beta_{a} \) values
than the corresponding nozzles, thus requiring a lower solidity
than the nozzle (Fig. 10, Part I), with the exception of the design
operating at \( N_s = 50 \), \( D_s = 1.5 \). For this configuration, the
rotor hub operates close to "impulse" (\( \cot \beta_{a} = 0 \)). The com-

330 / OCTOBER 1968

Transactions of the ASME
Assumption determines the radial flow distribution in the flow passage. Forced vortex as well as free vortex distributions are possible. For the convenience of the calculations, a free vortex distribution has been assumed. With this stipulation, the degree of reaction decreases from tip to root, and can reach high negative values at the hub streamline. This means that the static pressure at rotor exit may be lower than the static pressure at rotor inlet, thus impairing the suction specific speed potential of the pump. This can be avoided when the minimum allowable degree of reaction for the hub streamline is restricted to a value of about zero, i.e., by stipulating that the head which corresponds to the absolute leaving velocity at rotor hub exit must not exceed the head produced by the rotor ($p = 0$)

$$\frac{c_2^2}{2g} \leq \frac{H_{ad}}{\eta_h}$$

(16)

This means that the minimum allowable hub ratio is a function of $N_s$ and $D_s$. The numerical values are determined by introducing the similarity parameters into equation (16). This leads to a relation for the minimum allowable $N_s$ value

**Calculated Pump Performance Data**

Several assumptions can be made for designing the flow path of an axial pump. Of particular importance is the assumption of the vortex distribution between rotor and stator, since this
It is sometimes sufficient to calculate the flow path performance by considering only the mean streamline. This procedure can be expected to give realistic values if the profile and endwall loss coefficient for every streamline have the same numerical value. Cascades with a comparatively high value of the hub ratio are likely to fall into this category, but not necessarily cascades with low hub ratios. This is demonstrated in Fig. 3 which shows the operating condition of the tip, mean, and hub streamline of a pump operating at a specific speed of $N_s = 1000$, and specific diameter of $D_s = 0.5$. Fig. 3 shows that the profile loss coefficient for the mean streamline does not represent a true average. Hence, the tip, mean, and hub streamline have to be evaluated separately by determining the suction specific speed and hydraulic efficiency for every streamline stator as well as rotor. The overall efficiency is then obtained by integrating the streamline efficiencies. Fig. 3 also shows that the suction specific speed varies with radius, and reaches a minimum at rotor tip. Thus, the $S^*$-value for the tip streamline is the maximum obtainable pump suction specific speed as long as $\psi_2$ is assumed constant for all streamlines. The results of these calculations for single cascaded pump designs, neglecting tip clearance effects ($\delta/h = 0$) and assuming a blade number of $z = 16$, are presented in Fig. 4 by showing lines of constant (maximum) efficiency, constant hub ratio, and constant suction specific speed as function of specific speed and specific diameter. These data are obtained by using $\psi_2 = 0.9$, i.e., by selecting the optimum value for the aerodynamic blade load coefficient, and by assuming that the initial endwall boundary layer thickness is above the "critical" value ($\tau_e = 3 \times \tau_{e,\min}$). The calculations were restricted to designs with zero swirl at rotor inlet and stator exit, and a constant blade height throughout the flow path. This means that the axial velocity has the same value at rotor inlet and stator exit, so that the calculated efficiency can be interpreted to represent the

$$N_{r,\text{min}} = \sqrt{\frac{5800}{\lambda^2 \eta_h D_s^4 \left[ 1 - \frac{\eta_h D_s^4}{40 \eta_h (1 - \lambda^2)} \right]}} \quad (17)$$

$\eta_h$ denoting the rotor efficiency. Fig. 2 shows that the minimum allowable hub ratio decreases with decreasing specific speeds, and that the minimum allowable specific diameter is limited to comparatively high values.

Fig. 4 shows that, in general, the suction specific speed increases with increasing specific speeds, and that the maximum $S^*$ regime is restricted to comparatively low values of the specific diameter. Likewise, the efficiency increases with increasing specific speeds, but reaches a maximum at specific diameters which are different from the specific diameter desired for obtaining maximum suction specific speeds. This demonstrates that cascade geometries optimized for maximum efficiency usually compromise the suction specific speed, whereas geometries optimized for obtaining maximum suction specific speed frequently will show a lower efficiency than the maximum obtainable ef-

Fig. 3 Loss distribution and suction specific speed distribution along the radius

Fig. 4 Calculated performance data for single cascaded pump stages, $s/h = 0$

Fig. 5 Influence of aerodynamic load coefficient on suction specific speed
efficiency for that specific speed. The optimum hub ratio decreases with increasing specific speeds, showing values as low as \( \lambda = 0.3 \) at high specific speeds. It is also evident from Fig. 4 that the performance of single cascaded axial pumps decreases rapidly for \( N_s < 400 \).

Increased suction specific speeds are obtained when the original assumption of \( \psi_z = \text{const} \) is modified. The desired modification becomes evident from Fig. 5 which shows the suction specific speed of the tip, mean, and hub streamline as function of the aerodynamic load coefficient for fixed \( N_r \) and \( D_s \) values. By decreasing \( \psi_z \) with increasing radii, a design geometry evolves which has \( S^* = \text{const} \) for all streamlines. This geometry is less efficient than the original \( \psi_z = \text{const} \) geometry. The trade-off between \( S^* = \text{const} \) and \( \psi_z = \text{const} \) designs is presented in Fig. 6 for pump designs along the shaded operating regime of Fig. 4. A significant increase in suction specific speed potential up to ratios of 2:1 is calculated when the efficiency is compromised by factors of 0.83 to 0.9.

The trends shown in Fig. 4 change significantly when the clearance effect is accounted for. The effect of the clearance flow on the flow deflection of the tip streamline is expressed in Part I by the blade load correction factor \( A^* \). This factor increases significantly with increasing values of \( \beta_m \) and increasing values of the tip clearance ratio \( s/h \), as shown in Fig. 7. Thus, the blade load parameter \( \delta_k \), required for obtaining the desired head, will have to be increased, particularly toward the tip of the blade. This is shown in Fig. 8 by presenting the cascade parameters for the tip and hub streamline for two different operating points \( (N_r = 4000, D_s = 0.4 \) and \( N_r = 300, D_s = 1) \) for three different clearance ratios. A significant increase of the blade load parameter \( \delta_k \) at the tip and mean streamline, particularly for the high specific speed design, is evident. Thus, the profile, endwall, and leakage loss coefficient increases with increasing tip clearance ratios, \( \delta_k \)-effect in Figs. 9, 10, and 11. On this basis, an efficiency correction factor is calculated and shown in Fig. 12 for the
shaded operating regime of Fig. 4. This diagram shows that the efficiency correction for a tip clearance ratio of 0.002 is 0.98 to 0.94, but significantly larger for a tip clearance ratio of 0.01.

The suction specific speed is also influenced by the leakage effect since the factor $K^*$ increases with increasing blade load coefficients, as shown in Fig. 13. Hence, the suction specific speed decreases with increasing tip clearance ratios as shown by the dashed lines in Fig. 12, particularly at high specific speeds in the shaded operating regime of Fig. 4.

Fig. 7 demonstrates that the correction factor for the blade loading coefficient increases with increasing values of $\cot \beta_m$. This means that $\cot \beta_m$ becomes a criterion for the decrease in efficiency and suction specific speed due to clearance effects. Fig. 14 shows that the value of $-\cot \beta_m$ increases with increasing specific diameters, reaching values as high as $-100$ for a specific speed of 3000 and a specific diameter of 1. Considering now that the efficiency correction factor for $\cot \beta_m = -10$ ($N_s = 4000, D_s = 0.4$) was 0.94 for a tip clearance ratio of 0.002, and 0.24 for a tip clearance ratio of 0.01, it becomes evident that a sizable reduction in efficiency as well as suction specific speed occurs for high specific diameter designs. Hence the efficiency and suction specific speed data shown in Fig. 4 for operating points above the shaded regime in Fig. 4 are optimistic when leakage effects have to be considered. Remembering now that the minimum allowable specific diameter values are limited by the necessity of avoiding negative degrees of reaction (in cases where high $S^*$-values are desired) it follows that only a comparatively narrow regime of specific diameters will show a high efficiency potential, particularly in the high specific speed regime.

It is important to realize that the data presented in Figs. 4 to 14 are calculated with the assumption that the head is constant for every streamline, i.e., by assuming $r \times c_a = \text{const}$. By deviating from the free vortex distribution and, for example, using a distribution of the form $c_a \times r^a = \text{const}$ with $a \neq 1$ and observing the radial equilibrium condition for the $c_m$-distribution, different results will be obtained. The trends will be that decreasing $a$ (i.e., $c_a$-values are smaller) results in smaller specific diameters and maintaining a constant distribution of the meridional flow component. Another assumption for the performance calculation for multiple cascaded pumps was that the blade load is split evenly between the subcascades. Thus the mean vector angle and blade load coefficient of the subcascades follow the relations shown in Table 1. These relations also provide, for most designs, that sufficient pressurization is obtained in the first rotor subcascade to avoid cavitation in the subsequent rotor subcascade. This condition is met when the apparent suction specific speed of the second (or third) subcascade is

$$S^* = \frac{30(2g)^{0.75}}{\sqrt{\pi}} \times \left\{ \left[ (K^* - 1) - \left( \frac{1}{\mu^2} - 1 \right) \right] \left( \frac{r_0}{r_1} \right)^{1 + \psi \sqrt{-(K^* - 1 - \frac{1}{\mu^2})}} \right\}^{0.25}$$

$\mu^2$ denotes the deceleration in the preceding subcascade.

Fig. 15 shows that double cascaded pump designs obtain the same efficiency and the same suction specific speed at specific speeds of $N_s = 250$ as single cascaded pump designs obtain at a specific speed of $N_s = 450$. Thus, the general trend is that the operating regime, where efficient pump designs are obtained, moves to lower specific speeds with increasing numbers of subcascades. Remembering now that a specific speed regime be-
between 150 and 400 is usually considered the optimum operating regime for "mixed flow" pumps, it follows that double and triple cascaded pump designs offer the same or higher efficiencies than "mixed flow" designs at specific speeds between 150 and 400. Since the overall diameter of mixed flow pump designs is larger than the rotor diameter of axial designs, it becomes also apparent that another advantage of multiple cascaded pump designs is the smaller envelope diameter.

Fig. 15 also shows the interrelation between the cascade parameters and machine similarity parameters for the tip streamline of the first rotor subcascade for a triple cascaded stage. The mean vector angle for highest suction specific speed designs is \( \cot \beta = -0.6 \). In contrast, the mean vector angle for the rotor of single cascaded pump stages (Figs. 4 and 14) for highest suction specific speeds is somewhat higher, \( \cot \beta = -0.8 \). Thus, multicasceded designs, in general, will tend to be less clearance sensitive than single cascaded designs.

The data presented in Fig. 15 are calculated by assuming \( \psi_z = \text{const} = 0.9 \) for every streamline and subcascade. Modifying this assumption in order to obtain higher \( S^* \)-values (by decreasing \( \psi_z \) for the mean and tip streamlines), the trade-off between \( S^* \) and \( \eta \) is calculated and shown in Fig. 16 for triple cascaded designs, assuming again that \( s/h = 0 \). Suction specific speed gains result which are similar to the \( S^* \)-gains for single cascaded pumps.

It can now be argued that a triple cascaded pump requires the same number of blades as a three-staged single cascaded pump, so that, aside from potential mechanical design advantages, no obvious benefits are realized by multiple cascaded designs. The validity of this argument may be checked by calculating the performance potential of a three-stage single cascaded axial pump which produces the same head and volume flow, and has the same envelope diameter as a single-staged triple cascaded pump design. It may also be assumed that all stages

### Table 1

<table>
<thead>
<tr>
<th>Overall Stage</th>
<th>Rotor</th>
<th>Stator</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>First row</td>
<td>Second row</td>
</tr>
<tr>
<td>( \delta h )</td>
<td>( \delta_{H1} = \frac{\delta h}{2} )</td>
<td>( \delta_{H1} = \frac{\delta h}{2} )</td>
</tr>
<tr>
<td>( \cot \beta_m )</td>
<td>( \cot \beta_{m1} = \cot \beta_m - \frac{\delta h}{4} )</td>
<td>( \cot \beta_{m1} = \cot \beta_m + \frac{\delta h}{4} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Overall Stage</th>
<th>Rotor</th>
<th>Stator</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>First row</td>
<td>Second row</td>
</tr>
<tr>
<td>( \delta h )</td>
<td>( \delta_{H1} = \frac{\delta h}{3} )</td>
<td>( \delta_{H1} = \frac{\delta h}{3} )</td>
</tr>
<tr>
<td>( \cot \beta_m )</td>
<td>( \cot \beta_{m1} = \cot \beta_m )</td>
<td>( \cot \beta_{m1} = \cot \beta_m )</td>
</tr>
</tbody>
</table>
of the three-staged design are identical. In order to realize these requirements, the specific speed \( N_a \) and specific diameter \( D_{1/2} \) for every stage of the three-staged single cascaded pump must follow the relations \( N_a = N_{a1} = 163 \) and \( D_{1/2} = D_{1/21} = 163 \) where \( N_a \) refers to the triple cascaded design, and where \( Z \) denotes the number of stages. Assuming a specific speed of \( N_{a1} = 180 \) and specific diameter of \( D_{1/21} = 1 \) for the triple cascaded design, it is found that the corresponding values for the three-staged designs are \( N_{a1s} = 430 \), \( D_{1/2s} = 0.76 \), and that the stage efficiency, according to Fig. 4, becomes 0.88. The efficiency of a corresponding triple cascaded design is calculated to be 0.88, i.e., 1.5 points higher than that of the three-staged design. Likewise, the suction specific speed for the triple cascaded design is calculated to be about 5 percent higher than the suction specific speed of the single cascaded design. These considerations demonstrate a somewhat higher efficiency potential for the triple cascaded design. Thus, it results from this comparison that multiple cascaded designs offer performance advantages in the low specific speed regime which will be difficult to match with multistaged single cascaded designs having identical stage designs.

It is to be noted that the efficiency potential of multiple cascaded pump designs is computed by neglecting any interference effects between the subcascades. This would appear to be a conservative assumption when the data of references [22] and [23] of Part I are considered. These data demonstrate that a certain optimum relative stagger between the subcascades exists, and that the positionings of the subcascades at the optimum stagger give overall losses which are lower than the sum of the individual losses. Thus, it would appear that the calculated performance potential of multiple cascaded designs, presented in Figs. 15 and 16, is conservative.

It is important to realize that Figs. 4 and 15 were calculated by assuming a constant blade number \( z = 10 \) and a high initial boundary layer thickness. These assumptions cause the calculated efficiency values in the low specific speed regime to be conservative since \( z > 16 \) is optimum in this operating regime, and since the assumed low blade number yields a large chord length and consequently high endwall losses. This applies to single cascaded as well as multiple cascaded designs. Additional studies will be required to investigate the blade number effect in more detail.

Comparison of Calculated Data with Test Results

References [1-7] quote experimental data on the efficiency and cavitation aspects of medium to high specific speed axial machines. A comparison of these data with the calculated results is presented in Fig. 17, which shows that the calculated data reflect the general trends of the test data fairly well, namely increasing efficiencies and increasing suction specific speeds with increasing specific speeds. Also, the calculated and tested values of the efficiency level agree fairly closely. Discrepancies are, however, evident in the values of the specific diameter. The likely reason is that most of the efficiency data are obtained in test runs. Since cavitation considerations are of no concern for blowers, negative degrees of reaction ("subimpulse" condition) can be tolerated. Thus, no restriction is imposed on the minimum feasible value for the specific diameter, in contrast to pump designs. It is therefore not surprising that most of the tested test points occur at specific diameters which are lower than those required to avoid subimpulse conditions at the hub streamline. It is to be noted also that some of the tested blowers were not designed for a free vortex distribution, i.e., they represent a design philosophy different from that applied to the calculated data. The test data for tandem cascaded blowers are also obtained at lower specific diameters than those considered optimum on the basis of the present calculations. Also, for most of these designs, the flow distribution behind the vanes was not of the free vortex type, but followed the relation \( c_s = \sqrt{2} \) const.

An evaluation of the suction specific speed data is even more difficult since the tested suction specific speed refers in many cases to cavitation "breakdown," whereas the calculated suction specific speed refers to "incipient" cavitation. \( S^* \)-values for cavitation "breakdown," even when thermodynamic effects are neglected, are higher (by factors as high as 1.8) than \( S^* \)-values for "incipient" cavitation. Actually, "incipient" cavitation conditions are difficult to detect with conventional pump testing procedures. In view of these differences, only a moderate agreement between calculated and tested \( S^* \)-values can be expected.

Compressibility Effects

As long as the loss coefficients, calculated in Part I, are approximately valid for compressible media, the diagrams shown in Figs. 4 and 15 will serve as an indicator for the efficiencies obtainable from compressor stages. An assessment of the likely effect of the compressibility on these data is made by the following considerations:

It was one of the conclusions of Part I that the deceleration rate \( \mu = e_v e_1 \) is one of the most significant parameters for the cascade losses. Its value determines the boundary layer growth, and consequently the referred momentum thickness \( \theta \). The \( \mu \)-value of a fixed cascade geometry will decrease with increasing Mach Numbers for decelerating cascades, due to increase in density, but increase for accelerating cascades. Since \( \theta \) increases with decreasing \( \mu \)-values, the losses would be expected to increase with increasing Mach numbers in compressor cascades, but to decrease with increasing Mach numbers in turbine cascades. For a numerical valuation of the compressibility effect it may be assumed that the interrelation between \( D_i \) and \( \theta \), described in Part I, is still valid for compressible media, and that the "incompressible form" of the loss relation (equation (28) in Part I) can still be used. With these arguments, the losses for cascades operating with compressible media can be calculated when \( \mu \) is interrelated with the cascade parameters \( \delta_l \) and \( \beta_m \), while accounting for the compressibility effect on these interrelations. This changes the originally rather simple relations to more complex forms. They are derived on the following basis: For the convenience of the calculations, the interrelation between the blade load parameter \( \delta_l \) and mean vector angle \( \beta_m \) is restated by introducing a meridional acceleration factor, defined for compressors as

\[
k_c = \frac{c_{m2}}{c_{m1}}
\]

where \( c_{m1} \) and \( c_{m2} \) denote the meridional flow component at cascade inlet and cascade exit, Fig. 18. The blade load coefficient is defined as

\[
k = \frac{c_{m2}}{c_{m1}}
\]
Fig. 18 Definition of $\delta u$ and $\beta_m$ for compressible media

$$\delta u = \frac{\Delta v_0}{c_{m-m}}$$  \(20\)

c_{m-m}, denoting the average meridional velocity component, i.e.,

$$c_{m-m} = \frac{c_{m-1} + c_{m-2}}{2} = c_{m-1} + k c$$  \(21\)

With these definitions the following interrelations result:

$$\delta u = \frac{2}{1 + k c} (k \cot \beta_2 - \cot \beta_1)$$  \(22\)

$$\cot \beta_1 = \frac{1 + k c}{2} \left( \cot \beta_m - \frac{\delta u}{2} \right)$$  \(23\)

$$\cot \beta_2 = \frac{1 + k c}{2 k c} \left( \cot \beta_m + \frac{\delta u}{2} \right)$$  \(24\)

Equations (22), (23), and (24) are equivalent to the corresponding incompressible relations for $k = 1$. In order to assess the influence of the compressibility on the loss coefficients, the meridional acceleration factor $k c$ has to be interrelated with the cascade parameters $\delta u$ and $\beta_m$. The following considerations provide a basis: A general valid relation for the meridional acceleration reads

$$k c = \frac{\gamma_1 A_{m-1}}{\gamma_2 A_{m-2}}$$  \(25\)

when $\gamma$ denotes the density, $A_m$ the meridional flow area, subscript 1 refers to the cascade inlet, and 2 to the cascade exit. In equation (25), the area ratio represents a geometrical effect, and $\gamma_1/\gamma_2$ the compressibility effect, which can be written in the form of a generally valid relation.

$$\gamma_1 = \frac{p_{1-1}}{p_{1-2}} \frac{T_{1-2}}{T_{1-1}} \frac{T_{2-1}}{T_{2-2}} \frac{p_{2-1}}{p_{2-2}}$$ \(26\)

The last two terms in equation (26) represent the change in total temperature and total pressure (subscript t). The ratio $p_{1-1}/p_{2-2}$ is a function of the frictional losses in the cascade passage and approaches the value of unity in cases where the frictional losses are comparatively small. For convenience of the calculations, an isotropic process and ideal gas relations are assumed. With this simplification, the density ratio can be directly evaluated and solved for $k c$ after introducing equation (25). This gives an equation of the fourth degree which can be reduced to a simple equation of the first degree by the approximation $4k c^2(1 + k c)^2 = k c$. Thus it results for compressors

$$k c = 1 + \frac{M_m}{\kappa + 1} \left[ \left( \frac{\cot \beta_m + \frac{\delta u}{2}}{2} \right) - A_{m-2} \right]$$  \(27\)

$M_m$ denoting the average meridional Mach number defined as

$$M_m = \frac{c_{m-m}}{\sqrt{\frac{2}{\kappa + 1} g R T_i}}$$  \(28\)

$T_i$ denoting the total temperature.

For some applications the cascade inlet Mach number $M_1$, or exit Mach number $M_2$, may be the governing criteria. These values are interrelated with the average meridional Mach number by

$$M_1 = M_m \sqrt{\left( \frac{\cot \beta_m - \frac{\delta u}{2}}{2} \right)^2}$$  \(29\)

and $M_2 = M_2 \mu$. With these relations the meridional acceleration factor can be evaluated as function of the cascade parameters $\delta u$ and $\beta_m$, the ratio of specific heats $\kappa$, and the meridional Mach number or inlet and exit Mach number. Thus, the cascade acceleration rate $\mu$ can be expressed as a function of the cascade parameters and Mach numbers

$$\mu = \sqrt{\left( \frac{2 k c}{1 + k c} \right)^2 + \left( \cot \beta_m + \frac{\delta u}{2} \right)^2} \left( 1 + k c \right) \frac{\sin \beta_m}{\sin \beta_{m-1}}$$  \(30\)

In order to evaluate the profile losses numerically, the interrelation for the pitch-chord ratio has to be restated for compressible media. Using the average density together with the average meridional velocity for the actual tangential force (equation (7) of Part I), and the exit density for the ideal tangential force (equation (8) of Part I), the relation

$$\frac{1}{\sigma} = \frac{\psi_z \left( \frac{2 k c}{1 + k c} \right)^2 + \left( \cot \beta_m + \frac{\delta u}{2} \right)^2}{\delta u 2 k c} \sqrt{1 + \cot^2 \beta_m \left( 1 + A_{m-1}/A_{m-2} \right) \frac{1 + k c}{\sin \beta_m}}$$  \(31\)

is obtained when $A_m = (A_{m-1} + A_{m-2})/2$ is assumed for compressible media.

Fig. 19 Calculated compressibility effect on optimum solidity
Comparing equation (31) with the corresponding incompressible relation (equation (9) of Part I) it is found that the solidity \( \sigma \) increases slightly with increasing Mach numbers (for equal \( \delta u \) and \( \cot \beta_e \)-values) for decelerating and accelerating cascades when \( A_{n-2} = A_{n-1} \) is assumed, Fig. 19. This trend becomes plausible for decelerating cascades when it is considered that \( \mu \) decreases, and consequently the "adverse pressure gradient" along the blade surface increases, with increasing Mach numbers. Thus, a longer chord (higher \( \sigma \)-values) is desired to avoid separation. An explanation for the calculated \( \sigma \)-trend in accelerating cascades may be found in the fact that the ratio of the maximum suction surface velocity to inlet velocity \( (\beta_{max}/\beta_i) \) increases with increasing \( \mu \)-values (since \( \beta_{max} \) increases with increasing Mach numbers, equation (12) of Part I), i.e., increasing Mach numbers. This implies that the "adverse pressure gradient" increases with increasing Mach numbers as long as \( \beta_{max}/\beta_i \) increases at a higher rate than \( \mu \).

Having thus interrelated \( \mu \) and \( \sigma \) with the cascade parameters and Mach numbers, the loss coefficient for the profile losses can be calculated, using \( c_{n-m} \) as reference velocity. Since, for the incompressible case, the reference velocity was \( c_{n-m} \) for all loss coefficients, the loss equations of Part I have to be multiplied by a correction factor

\[
k' = \left( \frac{1 + k_e}{2k_e} \right)^2 \quad (32)
\]

after the proper \( k_e \)-values have been used. Typical values for the influence of compressibility on the profile loss coefficients are shown in Fig. 20 up to the condition where the Mach number of the maximum suction surface velocity \( (M_{\beta_{max}}) \) is unity, i.e., where the modified "incompressible" loss relations would be expected to become invalid. An increase in profile loss coefficients (for \( A_{n-2}/A_{n-1} = 1 \)) with increasing Mach numbers for decelerating cascades (\( \cot \beta_e \) negative), but the opposite trend for accelerating cascades, is calculated as expected. This trend is particularly noticeable for \( \cot \beta_e = -1 \), i.e., in an operating regime where comparatively high profile loss coefficients are calculated for incompressible media.

In order to evaluate the endwall losses numerically, the deflection angle \( \Delta \beta \) has to be calculated. Using the definitions for the blade load coefficient and mean vector angle as stated in Fig. 18, a "compressible" relation for the deflection angle reads

\[
\cos \Delta \beta = \frac{B}{\sqrt{1 + B^2}} \quad (33)
\]

with

\[
B = 1 + \frac{(1 + k_e)^2}{4k_e} \left( \cot \beta_e - \delta u \right) \left( \cot \beta_e + \delta u \right)
\]

\[
\left( 1 + \frac{1}{k_e} \left( \cot \beta_e + \frac{\delta u}{2} \right) - \left( \cot \beta_e - \frac{\delta u}{2} \right) \right) \quad (34)
\]

Equations (33) and (34) indicate that the deflection angle decreases with increasing Mach numbers for accelerating as well as decelerating cascades when \( \delta u \) and \( \beta_e \) are kept constant (assuming \( A_{n-2} = A_{n-1} \)). This, in general, tends to decrease the referred endwall loss coefficient. Combining this trend with the effect of the deceleration rate \( \mu \) on the losses, it follows that the endwall loss coefficient increases with increasing Mach numbers in decelerating cascades, but at a lower rate than the profile loss coefficient, and decreases for accelerating cascades at a higher rate than the profile losses, Fig. 21.

The Mach number influence on the clearance effects is shown in Figs. 22 and 23. The increase in \( \sigma \) with increasing Mach numbers would tend to decrease the deflection angle \( \Delta \beta \), with increasing Mach numbers. This trend is reinforced for decelerating cascades since \( \cot \beta_e \) decreases with decreasing \( k_e \)-values

\[
\cot \beta_e = \frac{1}{1 + k_e} \left( k_e \cot \beta_i + \cot \beta_i \right)
\]

\[
= \frac{2k_e}{1 + k_e} \cot \beta_i - \delta u \quad (35)
\]

\( k_e \) decreasing with increasing Mach numbers for \( A_{n-2}/A_{n-1} = 1 \). It is also evident from equation (35) that the mean vector angle \( \beta_e \) increases with increasing \( k_e \)-values. This tends to increase the deflection angle (equivalent to correction factor \( A^* \), shown in Fig. 20 of Part I). This effect overcompensates the \( \sigma \)-effect in accelerating cascades (where \( k_e \) increases with increasing Mach numbers), thus causing \( \Delta \beta \) to increase with increasing Mach numbers, in contrast to decelerating cascades, Fig. 22. The clearance loss coefficient, Fig. 23, shows trends which are similar to the profile and endwall loss coefficients, increasing with increases.
ing Mach numbers for decelerating cascades, but decreasing for accelerating cascades. This trend becomes plausible when it is considered that increasing Mach numbers in decelerating cascades cause \( \mu \) to decrease, and the pressure difference between suction and pressure side of the blade to increase. Thus, the tip leakage flow increases. The opposite effects occur in accelerating cascades.

The calculated Mach number effects show that the optimum cascade geometry, i.e., soliditiy, for compressible media differs somewhat from the optimum geometry for incompressible media, and that, in general, the losses increase with increasing Mach numbers for decelerating cascades, but decrease for accelerating cascades. The most significant effect occurs for compressor designs with a mean vector angle of \( \cot \beta_m = -1 \), i.e., low specific speed single cascaded compressors designed for minimum specific diameters (Fig. 14). Hence, in these operating regimes, the derating factor for the efficiencies shown in Fig. 4 is comparatively large. Smaller derating factors will apply for high specific speed designs operating at high specific diameters. Since, in general, the blade loading of the individual cascades in multiple cascaded designs is smaller than that of the corresponding single cascaded designs, lower derating factors would be expected for multiple cascaded compressor designs than for single cascaded compressor designs. Thus, the general trends of the derating factors are that they increase with decreasing specific speeds and decreasing specific diameters. This trend would only be expected to occur when the meridional area ratio \( A_{m-i}/A_{m-i} \) is assumed unity. Actually, the term \( A_{m-i}/A_{m-i} \) becomes an optimizing parameter for compressible media since the derating factors can be minimized by selecting \( A_{m-i}/A_{m-i} < 1 \) for decelerating cascades.

A direct comparison of the calculated Mach number effects with test data is difficult since test data usually are obtained by varying the Mach number for a fixed cascade geometry, whereas the calculated data presume that the blade load coefficient \( \delta u \) and the mean vector angle \( \beta_m \) are kept constant. This implies that, for the present assumption of a constant aerodynamic load coefficient, the cascade geometry (solidity \( \sigma \)) changes with Mach number. The shaded regimes in Fig. 20 show the Mach number effect on compressor cascades according to the test information reported in [5–13]. Comparing these data with the calculated data, it is found that the trends are identical, namely, increasing Mach number effects with increasing solidity values \( \sigma \) when it is considered that the blade load coefficient increases with increasing \( \sigma \)-values (Fig. 10 of Part I). The test data show a significant influence of the chord thickness ratio \( d_{\text{max}}/C \) on the Mach number effect, and an influence of the thickness distribution and camberline curvature. The maximum chord thickness ratio for the calculated data is comparatively small (\( d_{\text{max}}/C = \psi_2 \delta u/9 \)).

Thus, the calculated data show the minimum Mach number effect to be expected for comparatively thin blades. Further studies, and a critical review of the test evidence, are required to determine the validity of the derived Mach number relations in more detail. It may well be found that the optimum value for the aerodynamic load coefficient is a function of the Mach number.

**Concluding Remarks**

It is shown that the machine similarity parameters, specific speed and specific diameter, are interrelated with the cascade parameters \( \delta u \) and \( \beta_m \) and that the cascade losses can be calculated as function of the cascade parameters on the basis of boundary layer considerations. These interrelations, at the same time, define the optimum geometry. Thus, a direct link between cascade boundary layer parameters and machine similarity parameters is established. A distinction has to be made between designs optimized for maximum efficiency and designs optimized for maximum suction specific speeds. Using the derived relations, design selection diagrams can be computed which present the maximum obtainable efficiency and the associated suction specific speed, or the maximum suction specific speed and the associated efficiency as function of the machine similarity parameters. These diagrams can also represent the optimum geometry data for the individual designs.

The maximum obtainable efficiencies and suction specific speeds for high specific speed designs are influenced by tip clearance effects and the assumed vortex distribution, so that these parameters affect the optimum data to a significant extent.

Multiple cascaded pump and compressor designs show a high efficiency potential in a specific speed regime of \( N_s = 150 \text{ to } 400 \), i.e., in an operating regime where the efficiency potential of single cascaded axial machines and centrifugal machines is low.

Evaluating the compressibility effect on the basis of simplified loss relations, it was found that the efficiency potential of highly loaded cascades of the decelerating type (low specific speed designs) will have higher derating factors than the efficiency potential of lightly loaded cascades (high specific speed designs). The compressibility effect increases with increasing Mach numbers for both turbine and compressor cascades, so that the optimum geometry for cascades operating with compressible media is different from that of incompressible designs.

The main purpose of the presented consideration is to demonstrate the interrelations between boundary layer parameters and machine similarity parameters, and to discuss the influence of significant design variables on these interrelations. Further studies will be required to obtain design diagrams which cover the different design aspects of axial flow machines in sufficient detail.
References

7 Ihlenfeld, H., "Highly Loaded Axial Blower With Slotted Vanes" (in German), Maschinenbautechnik, Vol. 15, No. 1, 1966.