Homogenization-based topology optimization for high-resolution manufacturable microstructures

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Summary
This paper presents a projection method to obtain high-resolution, manufacturable structures from efficient and coarse-scale homogenization-based topology optimization results. The presented approach bridges coarse and fine scale, such that the complex periodic microstructures can be represented by a smooth and continuous lattice on the fine mesh. A heuristic methodology allows control of the projected topology, such that a minimum length scale on both solid and void features is ensured in the final result. Numerical examples show excellent behavior of the method, where performances of the projected designs are almost equal to the homogenization-based solutions. A significant reduction in computational cost is observed compared to conventional topology optimization approaches.

KEYWORDS
high-resolution, homogenization, manufacturing constraints, topology optimization

1 | INTRODUCTION

Nowadays, the wide-spread availability of computational resources has resulted in topology optimization as a mature design method, with numerous applications in industry. Nevertheless, large-scale topology optimization still comes at a high computational cost, dominated by the finite element analysis. Furthermore, to ensure near optimal solutions, continuation approaches have to be used, which increase the number of design iterations and computational time. Theoretically, it is known that the optimal topology consists of periodic composites with infinitesimally small features. In homogenization-based topology optimization, the space of admissible designs is relaxed to allow such composites. In this way, that provided the basis for the original works in topology optimization, optimal solutions could be obtained at a much lower computational cost compared to density-based topology optimization.

Even more than an optimal design, a manufacturable structure is required, which can of course not consist of infinitesimally small features. To get manufacturable designs out of the relaxed solutions, several multiscale techniques have been used. Rodrigues et al proposed a hierarchical optimization scheme that combines free material optimization on the macroscale, with an optimization of the microstructure using inverse homogenization. This approach was later extended to allow for parallel computations, as well as for nonlinear elasticity. To reduce the computational cost, one can also consider to restrict the amount of unique microstructures throughout the domain; however, this comes at a reduction of optimality. A limitation of these methods is that the connectivity between neighboring microstructures is not guaranteed. Greifenstein and Stingl approached this problem by constraining the variation of the material. Multiscale techniques such as the aforementioned approaches have received rapid growing interest, in part spurred by the increased availability of additive manufacturing techniques. By far, most of the published work (all of which shall not be listed here) is based on “separation of scales”, i.e., it is assumed that microscopic patterns are much smaller than the macroscopic details, in turn allowing for separate modelling of the microscale and...
In connection with additive manufacturing, one should also mention the treatment of in-fill, i.e., porous structures that, apart from weight saving, also may increase buckling stability of topology optimized structures, see Clausen et al.9,10

In a different approach, Lazarov and Alexandersen use the multiscale finite element method to design high-resolution manufacturable structures.11,12 These works do not assume separation of scales and hence provide better analysis and design of structures with finite periodicity, although the computational cost is high compared to homogenization-based topology optimization.

In a very appealing approach, Pantz and Trabelsi introduced a method to project the microstructures from the relaxed design space to obtain a solid-void design with finite length scale.13,14 The local structure is oriented along the directions of lamination such that a well-connected design is achieved. This approach paves the way for coarse-scale topology optimization where the projection can be performed on a high-resolution mesh in a postprocessing step, without a need for cumbersome and expensive multiscale formulations. In a related study, Rumpf and Pazos show that any type of (also spatially varying) unit cell, represented by a Fourier series, can be projected on a fine scale mesh.15,16

This paper shall be seen as a simplification and improvement of the approach introduced by Pantz and Trabelsi.13,14 We simplify the projection approach and introduce procedures for controlling the size and shape of the projected design, such that high-resolution (e.g., 1.3 million elements in 2D), near-optimal, and manufacturable lattice designs can be achieved within a few minutes in a single processor MATLAB code on a standard PC. This short time allows a designer to get high-resolution manufacturable designs at speeds, which potentially can make high-resolution topology optimization an integrated part of interactive design processes.17

The paper is organized as follows: the methodology of numerical homogenization, and how it can be used in the context of topology optimization is introduced in Section 2. The projection method and its implementation details are presented in Section 3. In Section 4, a method to control the shape of the projected designs is shown. Corresponding numerical examples are shown in Section 5. Finally, the most important conclusions of this study are presented in Section 6.

2  HOMOGENIZATION-BASED TOPOLOGY OPTIMIZATION

It is well-known that for many topology optimization problems, the optimal solutions can be found in the relaxed design space, ie, the space allowing for microstructural materials that have an infinitely fast variation in solid and void regions.18-20 At the microscopic scale, these microstructures are assumed to be uniform, hence they can be represented by periodic unit cells. While at the macroscopic scale, geometries and orientations are spatially varying such that optimal structural properties can be obtained.

In the original works on topology optimization, microstructures were either described by square cells with rectangular holes or as layered materials.21 Similar to these works, we apply topology optimization to compliance minimization of linear-elastic plane problems subject to a single load case, where the main focus of this study is on the physical realization of these optimized topologies.

Avellaneda has shown that the optimal solution for minimum compliance problems is in the space of layered materials, the so-called rank-n laminates.20 Here, rank-2 laminates are optimal for plane problems subject to a single load case, and rank-3 laminates are optimal for plane problems subject to multiple load cases. Unfortunately, rank-n laminates require different length scales, which poses a challenge on their physical realization. Therefore, we restrict ourselves to the suboptimal (but close in performance) square unit cells with rectangular holes, used by Bendsøe and Kikuchi.2 An illustration of one unit cell, with its local coordinate system (y1,y2) and link to the global coordinate system (x1,x2), is seen in Figure 1.

2.1  Numerical homogenization

The macroscopic constitutive properties of the unit cell in its local coordinate system (y1,y2) form the homogenized elasticity tensor \( C^H \). These properties, which depend on the parameters describing the height \( a_1 \) and width \( a_2 \) of the hole*, can be obtained using numerical homogenization. A discussion on the theory of homogenization is outside the scope of this work; instead, the interested readers are referred to previous works,22-25 in which detailed descriptions of the theory and implementation can be found.

In topology optimization, the parameters \( a_1 \) and \( a_2 \) are spatially varying and subject to change during each design iteration. Therefore, it is cumbersome to perform numerical homogenization for each variation in \( a_1 \) and \( a_2 \). Instead, we can calculate \( C^H \) for a large number of combinations of \( a_1 \) and \( a_2 \) and interpolate between them, as proposed by Bendsøe and Kikuchi.2 To do so,

*Hence, \( a_1 \) determines the stiffness of the unit cell in the \( y_1 \)-direction, and \( a_2 \) determines the stiffness of the unit cell in the \( y_2 \)-direction
FIGURE 1 Layout of the unit cell with a rectangular hole, in local $(y_1,y_2)$ and global $(x_1,x_2)$ coordinate system

FIGURE 2 The indices of the homogenized elasticity tensor $C^H$ for different values of $a_1$ and $a_2$, where $E_{\text{strong}} = 1$ and $\nu = 0.3$

the publicly available and easy-to-use MATLAB code by Andreassen and Andreassen is used. With this code, a mesh consisting of $100 \times 100$ bilinear finite elements has been created, on which $C^H$ has been determined for $51 \times 51$ different combinations of $a_1$ and $a_2$. The used material properties are $E = 1$ and $\nu = 0.3$, while a very compliant material plays the role of void, i.e., $E_{\text{void}} = 10^{-9}E$. To avoid discontinuities in elastic properties, we treat the hole as an infinitesimally thin crack when one of the parameters $a_i$ takes the value 0. The resulting curves for the 4 unique indices of the homogenized elasticity coefficients are shown in Figure 2, where linear interpolation is applied to obtain values between the data points.

With these curves known, the macroscopic elasticity tensor in the global reference frame $C$ can be calculated.

$$C(a_1,a_2,\theta) = R^T(\theta)C^H(a_1,a_2)R(\theta),$$

where $R$ is the well-known rotation matrix. Finally, the amount of material in unit cell $m$ can be calculated as

$$m = 1 - a_1 a_2.$$

2.2 Topology optimization formulation

Topology optimization is a material distribution problem, where the goal is to find an optimized material distribution that minimizes an objective function, subject to a number of constraints. In this study, we will restrict ourselves to minimizing compliance $J$, for single load-case problems, subject to an upper bound on the volume. The design domain is discretized in elements, in which the shape and orientation of the microstructure are assumed to be uniform. The local design variables $a_1$, $a_2$, and $\theta$ can thus be combined into design vectors $a_1$, $a_2$, and $\theta$.

The topology optimization problem is solved in nested form, by successive minimizations w.r.t. design variables $a_1$, $a_2$, and $\theta$; where for each design iteration, the equilibrium equations are satisfied by finite element analysis. As is shown by Pedersen, the optimal orientation of an orthotropic composite coincides with the principal stress directions, hence $\theta$ is aligned accordingly for each minimization step. Subsequently, design vectors $a_1$ and $a_2$ are updated at each minimization step based on their gradients. The discretized optimization problem can thus be written as
\[
\begin{align*}
\min_{a_1, a_2, \theta} : & \ J(a_1, a_2, \theta, U), \\
\text{s.t. :} & \ K(a_1, a_2, \theta)U = F, \\
& \ v^Tm(a_1, a_2) - V_{\text{max}} \leq 0, \\
& \ 0 \leq a_1, a_2 \leq 1,
\end{align*}
\]  

(3)

where \(m(a_1, a_2)\) describes the amount of material within an element (see Equation 2), \(v\) is the vector containing the element volumes, and \(V_{\text{max}}\) is the maximum allowed volume of the material in the design domain. Stiffness matrix \(K\) is a function of \(a_1, a_2, \text{ and } \theta\); \(F\) describes the loads acting on the domain; and \(U\) describes the solution of the equilibrium equation. For the design update of \(a_1\) and \(a_2\), the MATLAB implementation of the method of moving asymptotes is used.\(^{30}\)

As shown in Díaz and Sigmund,\(^{31}\) checkerboard patterns can occur since the strain energy density of these patterns is over-estimated using bilinear finite elements. To prevent the checkerboarding, the solution space is restricted using a density filter applied to \(a_1\) and \(a_2\) independently.\(^{32,33}\) A small filter radius \(r_{\text{min}}\) of 1.5 times the element width \(h_e\) is used, since the filter only has to prevent the occurrence of checkerboard patterns and should not impose a length scale on the design. The physical dimensions of the hole \(\bar{a}_1\) and \(\bar{a}_2\) are the filtered design variables. These values are used to determine the homogenized elasticity tensor \(C^H\) and to determine the amount of material within a unit cell \(m\).

### 2.3 Test problems

In this study, 2 test problems are considered. The first is a cantilever problem, subject to a distributed unit load over 20% of the right boundary, shown in Figure 3A. A volume constraint of 0.5 is set, and a coarse discretization of 80 \(\times\) 40 elements is used to solve the homogenization-based topology optimization problem. The second problem is a Michell-type problem with circular support, subject to a distributed unit load over 10% of the right boundary, shown in Figure 3B. For this problem, a volume constraint of 0.25 is set, while a mesh of 80 \(\times\) 60 elements is used to perform the topology optimization. For both problems, a solid material with unit stiffness, and a Poisson’s ratio of \(v = 0.3\), is used.

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To verify the homogenization-based topology optimization algorithm, the cantilever beam problem is solved, which results in a compliance of 58.35. This value is very close to the value of 56.73 obtained using the more optimal rank-2 laminate as microstructure, reported in Sigmund et al. The unit-cell densities of the optimized topology can be seen in Figure 4A. A close-up of this design can be seen in Figure 4B, where the red and blue lines correspond to $1 - a_1$ and $1 - a_2$, respectively. The challenge is now to convert this result to a practically realizable structure with the highest possible precision using the least computational effort.

3 PROJECTION OF MICROSTRUCTURES ON THE FINE SCALE

Almost any type of periodic microstructure can be represented by a complex exponential Fourier series with spatially varying parameters. This allows one to project a complex microstructure on a fine scale, yet maintaining a smooth and continuous lattice.

3.1 Projecting a uniform microstructure

The unit cell with a rectangular hole, used in the topology optimization problem, is simple enough to be represented by just 2 orthogonal cosine waves. The first cosine wave describes the part of the unit cell aligned with $y_1$, while the second cosine wave describes the part aligned with $y_2$. Each of these cosine waves can be obtained independently from each other, thus we here restrict ourselves to the derivation of the first cosine wave $\tilde{\rho}_1$. This cosine wave is oriented using unit vector $e_1$, such that it is constant in the direction of $y_1$, hence $e_1$ is orthogonal to $y_1$.

$$\tilde{\rho}_1(x) = \frac{1}{2} + \frac{1}{2} \cos(P(e_1 \cdot x)), \quad (4)$$

where $e_1$ can be written for the local orientation angle $\theta$ as

$$e_1 = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}. \quad (5)$$

$P$ is a term that scales the periodicity based on the size of the unit-cell $\epsilon$:

$$P = \frac{2\pi}{\epsilon}. \quad (6)$$

Homogenization-based topology optimization is based on an infinite periodicity, i.e., $\epsilon \to 0$; however, this can of course not be realized. Therefore, the microstructures have to be magnified towards a fine but realizable scale on the macroscopic level, using a finite $\epsilon$. To take into account the height of the void ($a_1$), and to get a clear solid-void design, a Heaviside step function is used. Here $\tilde{\rho}_1$ is projected to a a physical design $\rho_1$, where the threshold parameter $\eta_1$ is based on $a_1$.

$$\rho_1(a_1(x)) = H(\tilde{\rho}_1(x) - \eta_1(a_1(x))), \quad (7)$$

$$\eta_1(a_1(x)) = \frac{1}{2} + \frac{1}{2} \cos(\pi(1 - a_1(x))). \quad (8)$$

The shape of the projected unit cell can then be retrieved using

$$\rho(x) = \min\{\rho_1(x) + \rho_2(x), 1\}. \quad (9)$$

where the second cosine wave $\rho_2$ is oriented using unit vector $e_2$, which is orthogonal to $e_1$. The projection procedure of a sequence of uniform rectangular unit cells can be seen in Figure 5. Figure 5A and Figure 5B show cosine waves $\tilde{\rho}_1$ and $\tilde{\rho}_2$ obtained using Equation 4, while the projected shape of the lattice obtained using Equation 9 is seen in Figure 5C. The domain has unit width and height, with $\epsilon = 0.25$.

3.2 Projecting a spatially variant microstructure

Equation 4 does not hold when the orientation of the microstructures is spatially varying. A unit cell cannot be square if the spatial variation in angles has to be satisfied. Instead, we use a mapping function $\phi_1$, which maps the optimization domain $\Omega$ onto a periodic set in $\mathbb{R}^2$ that describes the composite, i.e., $\phi_1 : \Omega \to \mathbb{R}^2$. Using this mapping function, we can reformulate Equation 4 such that the cosine wave $\tilde{\rho}_1$ can be described by
The challenge is thus to find a suitable parameterization $\phi_1$, such that each point in $\Omega$ corresponds to the correct composite shape. We can solve for $\phi_1$ by minimizing the least-squares error between unit vector $\mathbf{e}_1$ and the gradient of $\phi_1$:

$$\min_{\phi_1} : I(\phi_1(x)) = \frac{1}{2} \int_{\Omega} \| \nabla \phi_1(x) - \mathbf{e}_1(x) \|^2 d\Omega.$$  

(11)

However, this is a best fit that tries to take both the enforcement of the angle and lattice spacing into account simultaneously.\(^{16}\) We will argue that the projected shape resembles the homogenization-based topology best if the angles are enforced exactly, at the cost of a relaxed lattice spacing. To further enforce that the unit cells are oriented corresponding to the output of the homogenization problem, $\phi_1$ has to be constant in the direction of $\mathbf{e}_2$. Equation 11 can thus be reformulated in the following constrained least-squares minimization problem.

$$\min_{\phi_1(x)} : I(\phi_1(x)) = \frac{1}{2} \int_{\Omega} \| \nabla \phi_1(x) - \mathbf{e}_1(x) \|^2 d\Omega,$$

s.t. $: \nabla \phi_1(x) \cdot \mathbf{e}_2(x) = 0.$  

(12)

Finally, it has to be mentioned that domain $\Omega$ can be split up into 3 parts: a void domain $\Omega_v$, a part of the domain that is completely solid $\Omega_s$, and a part that describes a lattice structure $\Omega_l$. These 3 different parts are defined as

$$x \in \begin{cases} 
\Omega_v & \text{if } m(x) = 0, \\
\Omega_s & \text{if } m(x) = 1, \\
\Omega_l & \text{if } 0 < m(x) < 1.
\end{cases}$$  

(13)

The mapping $\phi_1$ should be just described accurately in $\Omega_l$, hence we can relax the accuracy of $\phi_1$ in $\Omega_v$ and $\Omega_s$ to make sure the best projection is obtained. To do this, we reformulate the constrained least-squares minimization problem into a spatially weighted problem.

$$\min_{\phi_1(x)} : I(\phi_1(x)) = \frac{1}{2} \int_{\Omega} a_1(x) \| \nabla \phi_1(x) - \mathbf{e}_1(x) \|^2 d\Omega,$$

s.t. $: a_2(x) \nabla \phi_1(x) \cdot \mathbf{e}_2(x) = 0.$  

(14)

where

$$a_1(x) = \begin{cases} 
0.01 & \text{if } x \in \Omega_v, \\
0.1 & \text{if } x \in \Omega_s, \\
1 & \text{if } x \in \Omega_l.
\end{cases} \quad a_2(x) = \begin{cases} 
0 & \text{if } x \in \Omega_v, \\
0 & \text{if } x \in \Omega_s, \\
1 & \text{if } x \in \Omega_l.
\end{cases}$$  

(15)

The term $a_1$ is introduced to relax the projection of $\phi_1$ in $\Omega_v$ and $\Omega_s$, where the low values still ensure some regularization to the lattice spacing. Furthermore, the term $a_2$ is used to turn off exact angular enforcement in these regions. If equally weighted, requirements to the enforcement of angles in $\Omega_s$, where orientations may be badly determined, would severely influence the outcome in important higher density regions. Similarly, for $\Omega_v$, the orientation of the unit cell is less important since it will neither affect the constitutive properties nor the projected shape. The Lagrangian corresponding to Equation 14 can thus be written as
\[ L(\phi_1(x), \lambda(x)) = \int_{\Omega} \frac{\alpha_1(x)}{2} \| \nabla \phi_1(x) - e_1(x) \|^2 - \lambda(x) \alpha_2(x) (\nabla \phi_1(x) \cdot e_2(x)) d\Omega, \]  

where \( \lambda \) is the Lagrange multiplier corresponding to the constraint.

### 3.3 Projecting a rotationally symmetric spatially variant microstructure

The optimization problem solved in Equation 14 works well for a smoothly varying angle field \( \theta \). Unfortunately, the principal stress directions used to calculate \( \theta \) are rotationally symmetric, hence there may be jumps of size \( \pi \) in the optimized angle field. This rotational symmetry has neither an influence on the shape of the microstructure nor on its constitutive properties. However, it means that Equation 12 cannot be used, unless vectors \( e_1 \) and \( e_2 \) are oriented in a consistent fashion. To circumvent this problem, Pantz and Trabelsi\(^\text{13}\) reformulated the problem and solved for vector fields \( v_1 \) and \( v_2 \) instead of the scalar fields \( \phi_1 \) and \( \phi_2 \).

We argue that the approach proposed by Pantz and Trabelsi overly complicates the procedure and propose that the vector fields \( e_1 \) and \( e_2 \) can be oriented consistently in a straightforward fashion, in turn avoiding the computational burden in solving for vector fields \( v_1 \) and \( v_2 \). To do so, we use a well-known image-processing technique called connected component labeling.\(^{35}\) The main idea is to divide the image into separate components, where the pixels in each of the components have uniform or near-uniform properties. Using this technique, it is possible to find the components in the design domain where the angle field \( \theta \) is rotated with \( \pi \). To identify if voxel \( i \) and adjacent voxel \( j \) are in the same component, the following 2 conditions should be satisfied.

1. The relative difference between \( \theta_i \) and \( \theta_j \) should be smaller than \( \pi/2 \).
2. Voxels \( i \) and \( j \) should both be in \( \Omega_\ell \) or should both be in \( \Omega_\ell \cup \Omega_\ell \).

The first condition assumes that the principal stress field is smooth and should not rotate more than \( \pi/2 \) between 2 adjacent elements. The second condition is used to find the boundaries of regions in which there is no material. This is important when considering the Michell-type problem with the circular support. Here, we want the support to be a separate component that does not influence the orientation of angle field \( \theta \) outside of the support. Connected component labeling is a well-known image-processing task, for which various efficient implementations exist.\(^{35}\) Furthermore, it has to be noted that the technique works equally well on irregular meshes, provided that the adjacent elements are known.

To demonstrate the procedure, consider the vector field \( e_1 \) shown in Figure 6A. We assume a nonvoid domain, i.e., condition 2 is always satisfied, and perform the connected component label algorithm described above. A component label is assigned to each of the elements, and it can be seen in Figure 6B that the edges of the components correspond to the places where angle field \( \theta \) is rotated by \( \pi \). The different components can then be oriented consistently by making sure that the difference in \( \theta \) at these boundaries is close to \( k\pi \), where \( k \) is an integer. This aligns the angle field in a consistent fashion, and hence the projection can be performed using Equation 14.

![FIGURE 6](image)  
(A) \( e_1 \) plotted on top of \( \theta \)  
(B) Component label assigned to each element

**FIGURE 6**  
Angle field \( \theta \), which exhibits rotational symmetry, and the corresponding connected component labeling.
3.4 | Numerical implementation of the projection method

The solution of the topology optimization problem, obtained on coarse mesh \( T^c \), is projected onto a fine scale mesh \( T^f \), where \( h_f < h_c/15 \) to obtain a high-resolution design at a low computational cost. The calculation of \( \phi_1 \) and \( \phi_2 \) is done on an intermediate mesh \( T^i \), where the mesh-size is a trade-off between the computational cost and the quality of the result; however, a general rule of thumb is that \( h_i < h_c/3 \). An overview of these 3 different meshes and how they are related can be found in Figure 7; where in the coarse mesh \( T^c \), the red lines correspond to \( 1 - a_1 \), while the blue lines correspond to \( 1 - a_2 \).

Different numerical methods exist to find \( \phi_1 \) and \( \lambda \). Here, we solve the minimization problem of Equation 14 in finite difference form, where the solution vectors \( \phi_1 \) and \( \lambda \) contain the corresponding unknowns on the intermediate mesh. The resulting discretized Lagrangian can be written as

\[
\mathcal{L}(\phi_1, \lambda) = \frac{1}{2} \| A_1 (D \phi_1 - E) \|^2 - \lambda^T A_2 B D \phi_1.
\]  

where \( D \) is a finite difference matrix, vector, \( E \) contains the unit vectors \( e_1 \) at all discrete points, and matrix \( B \) holds the values of \( e_2 \). It has to be noted that \( A_1 \) and \( A_2 \) are diagonal matrices such that the spatial weighting terms are taken into account, hence \( A_1 \) contains values of \( \sqrt{\alpha_1} \) at the corresponding discrete points. Using the discretized Lagrangian and constraint, we can solve for \( \phi_1 \) and \( \lambda \) using the following KKT-system:

\[
\begin{bmatrix}
D^T A_1^T A_1 D & -D^T B^T A_2^T \\
A_2 B D & 0
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\lambda
\end{bmatrix}
=
\begin{bmatrix}
D^T A_1^T A_1 E \\
0
\end{bmatrix}.
\]

The solution of \( \phi_1 \) is prone to local high-frequency variations, as can be seen in Figure 8A. These variations can be prevented by solving for \( \phi_1 \) on a finer mesh; however, this increases the cost of the projection. Furthermore, it was observed that smoothing \( \phi_1 \) resolved the issue equally well. To smooth scalar field \( \phi_1 \), a density filter with a radius of \( 2h_i \) is used in a postprocessing step. The effect of this smoothing operation can be seen in Figure 8B.

3.5 | Scaling the periodicity

The projection procedure ensures that the local microstructure is oriented with \( \theta_i \), at the cost of the lattice spacing. The local lattice spacing is described by \( ||\nabla \phi_1|| \) and \( ||\nabla \phi_2|| \). If the angle is uniform throughout the domain, both values are 1, since \( e_1 \) and \( e_2 \) are unit vectors. Hence, \( ||\nabla \phi_i|| > 1 \) means that cosine wave \( i \) is locally compressed, while \( ||\nabla \phi_i|| < 1 \) means that the cosine wave is locally stretched. To make sure that the average size of the projected unit cell corresponds to \( \varepsilon \), we have to find the average lattice spacing \( \bar{p}_i \) in the part of the domain that consists of material \( (\Omega \setminus \Omega_e) \).

\[
\bar{p}_i = \frac{\int_{\Omega \setminus \Omega_e} ||\nabla \phi_i(x)|| d(\Omega \setminus \Omega_e)}{\int_{\Omega \setminus \Omega_e} d(\Omega \setminus \Omega_e)}
\]

Hence, \( \bar{p}_i > 1 \) means that cosine wave \( i \) is compressed in the domain of interest. Therefore, the periodicity for the \( i \)th cosine wave \( P_i \) has to be scaled using
The cantilever beam, for which the homogenization-based topology is shown in Figure 4, is treated as above. Here, \( T^c \) consists of 80 \( \times \) 40 elements, \( T^i \) consists of 320 \( \times \) 160 elements, and the final shape shown in Figure 9A is obtained on a fine mesh consisting of 1600 \( \times \) 800 elements, using \( \varepsilon = 15h_f \). The projection resembles the optimized shape well, and the corresponding compliance calculated on the fine mesh \( J_f = 63.6 \). Similarly, the test problem has been projected for \( \varepsilon = 30h_f \) as shown in Figure 9B, with the corresponding compliance of 61.2. It is interesting to note that in this case, a larger magnification leads to a better performing design. This is because of a larger number of small features being disconnected when \( \varepsilon = 15h_f \). How to resolve these small and sometimes disconnected features is the subject of the next section.

4 | CONTROL OF THE SHAPE OF THE PROJECTED MICROSTRUCTURE

The size of the microstructures can be varied using \( \varepsilon \). However, this parameter does not provide any control of the feature size of the individual structural members. Furthermore, if we take a closer look at Figure 9A, we can identify structural members that do not seem to carry any load, see close-ups in Figure 10A, as well as structural members that are so thin that they cease to exist, see Figure 10B.

To avoid these undesired features, some control of the projected design is required. Therefore, we present a heuristic method that enforces a minimum feature size \( f_{\text{min}} \) on both solid and void. First, we propose a continuation scheme that restricts the shape of the microstructures to prevent the occurrence of unit cells with very small structural members. Nevertheless, this approach still results in locations where the feature size of the projected design \( f \) is smaller than \( f_{\text{min}} \); therefore, material is added at these locations until manufacturability is ensured. Furthermore, we present a method that removes the material in nonload-carrying regions. Finally, we show an overview of all steps that have to be taken to project the homogenization-based structure.
FIGURE 10  Close-up of the cantilever beam, projected on a fine scale of 1600 × 800 elements, with $\varepsilon = 15h_f$.

FIGURE 11  Interpolation scheme plotted for the intervals where the behavior is non-linear, for different values of $\eta$ and $\beta$, that follow the order of the continuation approach.

4.1 Restriction of solution space

Microstructures with value of $a_1$ and $a_2$ close but not equal to 0 or 1 have to be prevented. These values describe unit cells with very thin strips of solid or void, which cannot be manufactured when the solution is projected on a fine scale. Instead, we want these values to either be 0, 1, or in a region $[\eta, 1 - \eta]$, which is bounded by length-scale $\eta$ on both the solid and void. To get a solution that satisfies these conditions, the following interpolation scheme is used, which links the filtered design variables $\tilde{a}_1$ and $\tilde{a}_2$ to the physical dimensions of the void $\bar{a}_1$ and $\bar{a}_2$.

$$\tilde{a}_1 = \bar{a}_1 (1 - \tilde{H}(\beta, (1 - \eta), \bar{a}_1)) \tilde{H}(\beta, \eta, \bar{a}_1) + \left(\frac{\beta - 1}{\beta} + \frac{\bar{a}_1}{\beta}\right) \tilde{H}(\beta, (1 - \eta), \bar{a}_1), \quad (21)$$

where $\tilde{H}$ is the smoothed Heaviside function,\(^{36}\)

$$\tilde{H}(\beta, \eta, \bar{a}_1) = \frac{\tanh(\beta\eta) + \tanh(\beta(\bar{a}_1 - \eta))}{\tanh(\beta\eta) + \tanh(\beta(1 - \eta))}. \quad (22)$$

$\beta$ controls the sharpness of the projection, and $\eta$ controls the threshold value. For a high value of $\beta$ and $\eta = 0$, we have that $\tilde{a}_1 = \bar{a}_1$. The interpolation function for different values of $\beta$ and $\eta$ can be found in Figure 11. The order of lines in the legend shows the continuation approach that is taken, using 50 iterations per step. First, the material interpolation scheme is close to a linear function, but gradually, $\eta$ is increased to enforce the length scale on solid and void. Finally, $\beta$ is increased to ensure that the far majority of physical dimensions of the hole are either 0, 1, or in the region $[\eta, 1 - \eta]$, such that the small feature sizes are avoided.

The cantilever beam problem has been solved using the continuation approach resulting in a compliance of 58.31 compared to the compliance of 58.35 using only density filtering. The difference between these values is negligible; similar differences have been observed for different mesh-sizes and different optimization problems. This shows that even though the solution space is restricted, equally well performing topologies can be found at different local minima. Hence, based on our experience, the presented approach does not negatively affect the performance of the design for the chosen parameters.
4.2 Satisfying the local feature-sizes

The presented restriction method greatly reduces the violation of length-scale $f_{\text{min}}$ in the projected design. Nevertheless, it does not enforce an explicit length scale on the projected shape. It is possible to identify the projected feature-sizes $f_s$ and $f_v$, corresponding to the solid and void, respectively, using the local periodicity and $a_1$ and $a_2$.

\[
 f_{s,i}(x) = \frac{\epsilon(1 - a_i(x))}{||\nabla \phi_i(x)||}, \quad (23)
\]

\[
 f_{v,i}(x) = \frac{\epsilon a_i(x)}{||\nabla \phi_i(x)||}. \quad (24)
\]

If $f_{s,i}(x)$ is smaller than $f_{\text{min}}$, the feature size of the solid is violated. To compensate for this, a new value $a_i^*$ is found such that the length scale is satisfied.

\[
 a_i^*(x) = 1 - \frac{f_{\text{min}}}{\epsilon} ||\nabla \phi_i(x)|| \quad (25)
\]

Similarly, if $f_{v,i} < f_{\text{min}}$, the hole is too small. However, we observed that removing material to satisfy the length scale has a significant negative effect on the optimality of the projected shape. Therefore, to make sure that the feature size is satisfied, the hole is closed instead, i.e., $a_i^* = 0$. This approach adds material to the structure and thus violates the volume constraint $V_{\text{max}}$ used in the optimization problem. Nevertheless, we argue that this violation is small, e.g., 5% of $V_{\text{max}}$, if reasonable values for $\epsilon$ and $f_{\text{min}}$ are used. The restriction of the solution space discussed previously makes sure that most of the unit cells in the domain do not have small feature sizes. Furthermore, the nonload carrying structural members shown in Figure 10A still have to be removed, which will reduce the violation of the volume constraint.

4.3 Removal of material at nonload carrying solids

Nonload carrying structural members, see, e.g., Figure 10A, can partially be explained by using the restriction method presented above, but mostly by the interpolation of $a_1$ and $a_2$ from $T^c$ onto $T^f$. Since a linear interpolation method is used, isolated structural members can arise at high-contrast regions. To remove these structural members, we use a simple iterative update scheme. In this scheme, a finite element analysis is performed at the fine scale; afterwards, the solid voxels that have a strain energy density $c_e$ lower than 0.1% of the mean strain energy density $\overline{c}_e$ are set to void. To make sure that the length scale $f_{\text{min}}$ on both the solid and void is still satisfied after each iteration, we perform an open-close filter operation as is proposed by Sigmund.37 This operation efficiently eliminates features smaller than the filter size but leaves other details virtually untouched.

The combination of removal of solid voxels with a low strain energy density, followed by the open-close filter operation, is a heuristic method to get smooth and manufacturable microstructures. Nevertheless, we have observed excellent behavior of this method and generally convergence is obtained within 5 to 10 iterations. Alternatively, one could use the projected microstructure as a starting guess for fine-scale topology optimization. However, even when an efficient topology optimization framework is used, e.g. Andreassen,48 the corresponding optimization will require significantly more time-consuming fine-scale iterations until convergence and probably result in no significant improvement in performance.

4.4 Overview of the projection procedure

The methods described in the previous sections are combined to obtain high-resolution manufacturable microstructures at low computational cost. The different steps that have to be taken are shown below:

1. Homogenization-based topology optimization for rectangular unit cells using the restriction approach presented in section 4.1, on coarse mesh $T^c$.
2. Connected component labeling and consistent alignment of $e_1$ and $e_2$, using the methodology presented in section 3.3 on $T^c$.
3. Calculating mapping functions $\phi_1$ and $\phi_2$ as shown in section 3.4, on intermediate mesh $T^i$.
4. Satisfying feature-size $f_i$ using the method presented in section 4.2, on fine mesh $T^f$.
5. Projection of the cosine waves using Equations 7 to 10 on fine mesh $T^f$.
6. Removal of the nonload carrying materials using the iterative method described in section 4.3 on mesh $T^f$. 

5 | NUMERICAL EXAMPLES

The solution of the cantilever beam problem has been projected on a fine mesh consisting of 1600 × 800 elements. The homogenization-based topology optimization took 199.4 seconds, and the subsequent connected component labeling took 0.1 seconds. Hence, the total time spent on the coarse mesh $T_c = 199.5$ seconds. The compliance of the optimized homogenization-based topology $J_c = 58.31$. The projection is performed for various sizes of the unit cell $\epsilon$, and several values of $f_{\text{min}}$. The corresponding compliance on the fine scale $J_f$ and volume $V_f$ of the projected structures can be found in Table 1. Furthermore, $T_i$ which is the time it took to calculate $\phi_1$ and $\phi_2$ on the intermediate mesh is shown, as well as $T_f$ the time it took to satisfy the feature size, project the cosine waves, and remove the nonload carrying members on the fine mesh. Finally, the time for the entire procedure $T_{\text{tot}}$ is given in Table 1.

A small unit-cell size combined with a large minimum length-scale leads to a large violation of the volume constraint. This is visualized for $\epsilon = 20h_f$ and $f_{\text{min}} = 4h_f$ in Figure 12A, where most structural members have a width corresponding to $f_{\text{min}}$. However, for reasonable combinations of $\epsilon$ and $f_{\text{min}}$, the volume constraint is hardly violated. A good rule of thumb is that $\frac{f_{\text{min}}}{\epsilon} > 10$ such that the violation of the volume constraint is within 2% of $V_{\text{max}}$. Furthermore, it is very interesting to see that for these projected structures excellent results are achieved, where the compliance is close to the performance of the homogenization-based topology. For $\epsilon = 40$, and a small feature-size $f_{\text{min}} = 2$, we obtain a structure that does not violate the volume constraint, see Figure 12B, while the performance is within 2% of the homogenization-based topology. Using a larger feature size $f_{\text{min}} = 4$, slightly increases the volume; however, this results in a topology for which the performance is almost identical to the homogenization-based topology. Furthermore, this larger feature-size ensures manufacturability as can be seen in Figure 13.

The presented procedure for calculating mapping functions $\phi_1$ and $\phi_2$ is highly efficient resulting in an average time of 10.6 seconds. The time it took to project the structure on the fine scale and satisfy the feature size is negligible compared to the cost of removing the nonload-carrying materials. The cost for this iterative procedure depends on the required number of iterations, where an average iteration costs approximately 28 seconds. This means that the total time to optimize and project the design $T_{\text{tot}}$ never exceeded 500 seconds, where it has to be noted that all experiments have been performed on a standard PC, using a single-core MATLAB code.

### TABLE 1  Performance and computational cost of the projection method, when the cantilever beam problem is projected on a fine mesh consisting of 1600 × 800 elements

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$f_{\text{min}}$</th>
<th>$V_f$</th>
<th>$J_c$</th>
<th>$J_f$</th>
<th>$T_c$</th>
<th>$T_i$</th>
<th>$T_f$</th>
<th>$T_{\text{tot}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20h_f</td>
<td>2h_f</td>
<td>0.503</td>
<td>58.31</td>
<td>60.27</td>
<td>199.5 s</td>
<td>10.7 s</td>
<td>145.2 s</td>
<td>355.4 s</td>
</tr>
<tr>
<td>20h_f</td>
<td>3h_f</td>
<td>0.519</td>
<td>58.31</td>
<td>58.21</td>
<td>199.5 s</td>
<td>10.2 s</td>
<td>260.6 s</td>
<td>470.3 s</td>
</tr>
<tr>
<td>20h_f</td>
<td>4h_f</td>
<td>0.540</td>
<td>58.31</td>
<td>56.23</td>
<td>199.5 s</td>
<td>10.8 s</td>
<td>266.6 s</td>
<td>476.9 s</td>
</tr>
<tr>
<td>30h_f</td>
<td>2h_f</td>
<td>0.500</td>
<td>58.31</td>
<td>60.93</td>
<td>199.5 s</td>
<td>10.7 s</td>
<td>150.3 s</td>
<td>360.5 s</td>
</tr>
<tr>
<td>30h_f</td>
<td>3h_f</td>
<td>0.509</td>
<td>58.31</td>
<td>59.72</td>
<td>199.5 s</td>
<td>11.2 s</td>
<td>233.6 s</td>
<td>444.3 s</td>
</tr>
<tr>
<td>30h_f</td>
<td>4h_f</td>
<td>0.518</td>
<td>58.31</td>
<td>58.71</td>
<td>199.5 s</td>
<td>10.5 s</td>
<td>285.3 s</td>
<td>495.3 s</td>
</tr>
<tr>
<td>40h_f</td>
<td>2h_f</td>
<td>0.500</td>
<td>58.31</td>
<td>59.55</td>
<td>199.5 s</td>
<td>10.7 s</td>
<td>115.5 s</td>
<td>325.7 s</td>
</tr>
<tr>
<td>40h_f</td>
<td>3h_f</td>
<td>0.505</td>
<td>58.31</td>
<td>59.03</td>
<td>199.5 s</td>
<td>10.2 s</td>
<td>145.6 s</td>
<td>355.3 s</td>
</tr>
<tr>
<td>40h_f</td>
<td>4h_f</td>
<td>0.510</td>
<td>58.31</td>
<td>58.57</td>
<td>199.5 s</td>
<td>10.0 s</td>
<td>148.7 s</td>
<td>358.2 s</td>
</tr>
</tbody>
</table>

**FIGURE 12**  Cantilever problem projected on a fine scale of 1600 × 800
FIGURE 13 Projection of the cantilever problem, on a fine scale of 1600 × 800 using $\epsilon = 40h_f$ and $f_{\text{min}} = 4h_f$. The compliance $J_f = 58.57$, while $V_f = 0.510$.

TABLE 2 Performance and computational cost of the projection method, when the Michell-type problem with circular support is projected on a fine mesh consisting of 1200 × 900 elements

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$f_{\text{min}}$</th>
<th>$V_f$</th>
<th>$J_f$</th>
<th>$T_c$</th>
<th>$T_i$</th>
<th>$T_f$</th>
<th>$T_{\text{tot}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20$h_f$</td>
<td>2$h_f$</td>
<td>0.248</td>
<td>84.31</td>
<td>75.57</td>
<td>16.8 s</td>
<td>109.6 s</td>
<td>352.0 s</td>
</tr>
<tr>
<td>20$h_f$</td>
<td>3$h_f$</td>
<td>0.266</td>
<td>84.31</td>
<td>73.44</td>
<td>17.5 s</td>
<td>112.7 s</td>
<td>355.8 s</td>
</tr>
<tr>
<td>20$h_f$</td>
<td>4$h_f$</td>
<td>0.297</td>
<td>84.31</td>
<td>62.76</td>
<td>16.1 s</td>
<td>150.4 s</td>
<td>392.1 s</td>
</tr>
<tr>
<td>30$h_f$</td>
<td>2$h_f$</td>
<td>0.236</td>
<td>84.31</td>
<td>76.86</td>
<td>17.7 s</td>
<td>174.2 s</td>
<td>417.5 s</td>
</tr>
<tr>
<td>30$h_f$</td>
<td>3$h_f$</td>
<td>0.241</td>
<td>84.31</td>
<td>74.72</td>
<td>17.5 s</td>
<td>132.5 s</td>
<td>375.6 s</td>
</tr>
<tr>
<td>30$h_f$</td>
<td>4$h_f$</td>
<td>0.252</td>
<td>84.31</td>
<td>71.89</td>
<td>15.7 s</td>
<td>175.3 s</td>
<td>416.6 s</td>
</tr>
<tr>
<td>40$h_f$</td>
<td>2$h_f$</td>
<td>0.250</td>
<td>84.31</td>
<td>68.37</td>
<td>16.5 s</td>
<td>111.6 s</td>
<td>353.7 s</td>
</tr>
<tr>
<td>40$h_f$</td>
<td>3$h_f$</td>
<td>0.252</td>
<td>84.31</td>
<td>67.83</td>
<td>17.3 s</td>
<td>112.6 s</td>
<td>355.5 s</td>
</tr>
<tr>
<td>40$h_f$</td>
<td>4$h_f$</td>
<td>0.256</td>
<td>84.31</td>
<td>66.51</td>
<td>17.7 s</td>
<td>92.9 s</td>
<td>336.2 s</td>
</tr>
</tbody>
</table>

To require the same resolution using a standard density-based topology optimization approach, a minimum of 100 iterations is required, each costing approximately 25 seconds. Furthermore, for a standard topology optimization approach, a continuation strategy is required to get as close to the optimal solution as obtained by the suggested projection approach. Such a continuation approach requires approximately 1000 iterations, which would mean a total optimization time of approximately 25 000 seconds. Because of its optimization at a coarse scale, the presented method is thus able to be reduce the computational cost by more than 50 times.

The proposed scheme is also applied to the Michell-type problem with circular support. The coarse-scale optimization performed on a mesh of 80 × 60 elements resulted in a compliance of 84.31 obtained within 225.4 seconds. The shape is projected on a fine mesh of 1200 × 900 elements, for various sizes of the unit cell $\epsilon$, and several values of $f_{\text{min}}$. The corresponding volume, compliance, and times it took for each part of the procedure are shown in Table 2.

It is interesting to see that for $\epsilon = 30h_f$, the volume of the projected and postprocessed shape is smaller than the volume constraint in the topology optimization problem. The reason for this is that 2 peaks of the cosine waves are located exactly at the boundary of the void domain and are not well connected to the rest of the structure, as can be seen in Figure 14A. Therefore, these waves are nonload carrying and removed by the fine-scale optimization procedure as can be seen in Figure 14B. This unfortunate result is purely a problem of the periodicity $P$ having an unlucky value such that the projected structure at some positions is not well connected. If we use $\epsilon = 40h_f$, the peaks of the cosine waves are positioned in a better way and excellent results are obtained. The projection using 40$h_f$ and 3$h_f$ is shown in Figure 15, the violation of the volume constraint is negligible, and the corresponding compliance is 67.83. It is interesting that this value is much lower than the compliance of the homogenization-based topology. This difference can be explained by the enforcement of the circular boundary condition, which is known to have a very strong influence on the resulting compliance for this classical example. If the homogenization-based topology optimization is performed on a finer scale, the boundary is approximated more accurately, and it was observed that the compliance converges to the value obtained using the projection.

From the numerical examples, we can conclude that the presented method is able to provide high-resolution (more than a million elements), near-optimal, manufacturable topologies, at low computational cost. However, the exact locations of the peaks of the cosine waves are difficult to predict; therefore, the projected shape can have some poorly connected structural
members, as is demonstrated in Figure 14A and Figure 14B. This problem can be circumvented by shifting the cosine waves slightly, or by solving the projection procedure as a true multiscale approach. In such an approach, the topology optimized shape is also the shape that can be projected best.

Finally, a close inspection of Figures 13 and 15 shows that the projected shapes are not perfectly symmetric. The reason for this are angular fluctuations in low density regions. The influence of these regions on the projection is small; nevertheless, it can still result in a slight asymmetry as is best visible around the circular support in Figure 15. This asymmetry can be circumvented by reducing the computational domain at which the projection is performed to the nonvoid regions or by enforcing symmetry in all variable sets. We will leave further improvements in this direction to future works.

6 | CONCLUSION

An efficient approach to obtain near-optimal, high-resolution, manufacturable microstructures from coarse scale optimization studies has been presented. The methodology projects the unit cells of homogenization-based topology optimization on a fine mesh with minimum computational effort, where emphasis has been put on how to treat the rotational symmetry in the optimized angle field. Furthermore, an efficient method to control the shape of the projected design has been presented, such that the final topology satisfies a length-scale on both solid and void regions. Based on numerical experiments, this heuristic method has shown to produce near-optimal designs, at a speed which for the considered 2D examples is 50 times lower than conventional density-based topology optimization methods.

This overall promising performance paves the way for extending the proposed methodology to multiload problems, where more complex unit cells need to be considered. Furthermore, the excellent performance for single load-case minimum compliance problems allows for a natural extension of the methodology to 3D applications, where cubic unit cells with rectangular holes will have to be considered. We are confident that the presented methodology is robust enough to allow these extensions, such that its potential for incorporation in an efficient, interactive design process can be further revealed.
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