Discrete material optimization of general composite shell structures

J. Stegmann* † and E. Lund

Institute of Mechanical Engineering, Aalborg University, Pontoppidanstraede 101, DK-9220 Aalborg East, Denmark

SUMMARY

A novel method for doing material optimization of general composite laminate shell structures is presented and its capabilities are illustrated with three examples. The method is labelled Discrete Material Optimization (DMO) but uses gradient information combined with mathematical programming to solve a discrete optimization problem. The method can be used to solve the orientation problem of orthotropic materials and the material selection problem as well as problems involving both. The method relies on ideas from multiphase topology optimization to achieve a parametrization which is very general and reduces the risk of obtaining a local optimum solution for the tested configurations. The applicability of the DMO method is demonstrated for fibre angle optimization of a cantilever beam and combined fibre angle and material selection optimization of a four-point beam bending problem and a doubly curved laminated shell. Copyright © 2005 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Over the last two or three decades the strive for lighter and stronger structures has resulted in an increasing use of advanced materials. Consequently, composite materials in general and fibre reinforced polymers in particular have gained an ever increasing popularity due to their very high strength to weight ratio. In structural applications fibre reinforced polymers are usually stacked in a number of layers, each consisting of strong fibres bonded together by a resin, to form a laminate. The fibres may be uniformly oriented in typically one or two directions or they may be oriented in no particular ordered fashion. The best use of the material is achieved when ordering the fibres in specific directions to obtain high stiffness in the loading directions and lower stiffness in other directions. Exploiting this ‘directionality’ of the material is at the core of efficient design with composite laminates. Furthermore, to obtain an optimal design

*Correspondence to: J. Stegmann, Institute of Mechanical Engineering, Aalborg University, Pontoppidanstraede 101, DK-9220 Aalborg East, Denmark.
†E-mail: js@ime.aau.dk

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the engineer must choose which materials to use in individual layers. However, proper choice of materials, stacking sequence and fibre orientation is a far from simple task since laminates can often consist of as many as 500 or more different layers. This brings forward the need for efficient and reliable numerical design tools—in particular when dealing with large scale structures involving complicated geometries, multiple layers, multiple materials and multiple load cases.

The objective of the present work is to solve general material optimization problems efficiently for general composite laminate shell structures where the objective is to maximize stiffness, i.e., minimize the compliance. This objective is three-fold since it requires simultaneous solution for material, orientation and stacking. As will be shown, this can be encompassed in the same parametrization and solved within a finite element framework. The platform for implementation has been the finite element based analysis and optimization code MUST (MUltidisciplinary Synthesis Tool) [1], developed at the Institute of Mechanical Engineering, Aalborg University.

The present paper is focussed on introducing the proposed parametrization and demonstrates the efficiency of the method for orientation optimization problems as well as combined material selection and orientation optimization of beams and composite laminate shell structures.

1.1. Composite laminates

Composite laminate shell structures are characterized as being composed of multiple materials and multiple layers and will, in general, be curved or doubly curved. Materials will be assumed to be linear elastic and either isotropic or orthotropic. To establish the basic notation used in the following consider the two-material composite laminate in Figure 1. Each layer, \( k \), in the laminate is characterized by the constitutive matrix, \( C_k \), which is a function of the spatial orientation, \( \theta_k \), of the fibres in that layer, i.e., \( C_k = C_k(\theta_k) \). Note that the superscript \( l \) refers to ‘layer’. In the present study laminates will be treated as equivalent single layers (ESL) which means that the layers are assumed to be perfectly bonded together and thus, displacements and strains will be continuous across the thickness. Consequently, interlaminar

![Figure 1. Schematic representation of composite laminate shell with two different base materials, Mat A and Mat B, where Mat A is oriented at two different fibre angles, \( \theta_1, \theta_2 \), while Mat B is isotropic; \( k \) is the layer number and \( N^l \) is the total number of layers.](image)
effects such as delamination are disregarded but ESL still provides a good approximation of the structural stiffness and thus a sufficient and well-established basis for doing global analysis and optimization.

1.2. Finite element formulation

The finite element method is employed to obtain the response of the structure, subject to a given set of boundary conditions. Displacements will presently be assumed small and material behaviour linear but the proposed parametrization can easily be used for non-linear problems as well.

The elements used are four node shell elements with full integration and assumed natural strains to circumvent problems with shear locking (for details see e.g. Reference [2]). The element stiffness matrix, \( K^e \), is obtained as the sum of layer stiffnesses over the total number of layers, \( N^l \):

\[
K^e = \sum_{k=1}^{N^l} \int_V \left( B^l_k \right)^T C^l_k B^l_k \, dV
\]

In (1), the matrix \( B^l_k \) is the strain–displacement matrix of the \( k \)'th layer and the integral is in practice solved for each layer using a full 3D Gauss quadrature. As usual the global stiffness matrix is obtained as the sum of element stiffnesses over all elements, \( N^e \), i.e.

\[
K = \sum_{m=1}^{N^e} K^e_m
\]

(the superscript \( e \) refers to 'element'). This stiffness matrix is used to solve the linear static equilibrium:

\[
K u = p
\]

where \( u \) and \( p \) are the global vectors of displacements and external forces, respectively. Hence the compliance, \( C \), which is the objective function of the optimization can be stated as:

\[
C(u) = u^T p = u^T K u = 2U
\]

where \( U \) is the total strain energy. These equations form the basis for implementing the proposed parametrization and will be addressed in more detail in Section 2.

1.3. Orientation optimization with orthotropic materials

The classical approach to solve for optimal orientation of orthotropic materials and minimum compliance has been to use the local orientation (denoted by local fibre angles, \( \theta \)) as design variables. In a finite element framework this can be depicted as shown in Figure 2, where each arrow represents the 1st principal material direction and is uniquely defined in each element (or layer) by the angle, \( \theta_m \), relative to some fixed frame. The design variables are then the continuous parameters, \( \theta_m \), and thus the method is often referred to as continuous fibre angle optimization (CFAO). The optimization problem may be stated as

Objective : \( \min \theta \ C(u) = u^T p \)

Subject to : \( Ku = p \)

\[
\theta_{\min} \leq \theta \leq \theta_{\max}
\]
Figure 2. Illustration of the classical concept of orientation optimization in a finite element framework. The arrows represent the 1st principal material direction (denoted by $\theta_m$) of the orthotropic material.

Figure 3. (a) Test example geometry with boundary conditions; and (b) optimum solution.

where $\theta_{\text{min}}$ and $\theta_{\text{max}}$ contain the lower and upper bounds on the design variables, respectively (typically $-90^\circ$ and $+90^\circ$). The design sensitivities (compliance sensitivities) can be obtained as the derivative of (3) and when assuming design independent loads, the following expression may be derived (see e.g. Pedersen [3] or Masur [4]).

\[
\frac{\partial C}{\partial \theta_m} = -u^T \frac{\partial K}{\partial \theta_m} u = -(u_m^c)^T \frac{\partial K_m^c}{\partial \theta_m} u_m^c
\]  

This expression is valid if $\theta_m$ only pertains to a single element. Later we will introduce so-called patch variables which affect several elements and in such cases the sensitivity $\partial C/\partial \theta_m$ must be evaluated as a sum over all elements affected by the variable. The expression in (5) is common to both CFAO and topology optimization and is in practice quite efficient to evaluate either analytically or more commonly for CFAO by a finite difference approximation. The major difficulty faced when using this type of formulation is that the global design space becomes non-convex. To illustrate this the simple problem in Figure 3 has been constructed and solved using a continuous formulation and the method of moving asymptotes (MMA) by Svanberg [5]. The result is the design space shown in Figure 4(a), where four extrema can be found—one global and three local (see also Table I). To obtain the global optimum solution the initial guess must be within the non-shaded part of the design space (the safe domain).
Figure 4. Test example (CFAO) objective function (normalized). The black squares mark the local extrema and the shaded area marks values that will lead to local optimum solution.

Table I. Extremum values for the design space in Figure 4.

<table>
<thead>
<tr>
<th>Optimum:</th>
<th>Global</th>
<th>Local #1</th>
<th>Local #2</th>
<th>Local #3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Element 1</td>
<td>24.2°</td>
<td>29.3°</td>
<td>−90°</td>
<td>−90°</td>
</tr>
<tr>
<td>Element 2</td>
<td>−41.6°</td>
<td>90.0°</td>
<td>90.0°</td>
<td>−47.3°</td>
</tr>
</tbody>
</table>

as illustrated in Figure 4(b). In a simple case like this it is fairly easy to determine how to start the optimization but for complicated geometries with multiple layers it becomes almost impossible to state the problem a priori within the safe domain. It follows that for generic problems the results obtained in this manner with CFAO will often be sub-optimal albeit better than the initial design. This is, of course, not a new realization and several methods have already been proposed to circumvent the problem of local optimum solutions. These methods roughly fall within one of four categories:

1. Analytical methods, which rely on the closed-form formulation of an optimality criteria as described by Prager [6]. This constitutes a limitation in the applicability of the methods but for simpler geometries the methods have been applied very successfully and often serve as benchmark solutions for purely numerical methods. The major contributor to optimality criteria methods in orientation design is Pedersen [3] who has used the method to optimize 2D continua, beams and plates. Luo and Gea [7] used a similar approach to solve for optimal orientation of plate bead stiffeners.

2. Mathematical programming techniques are purely numerical and are concerned with tuning the optimizer itself rather than reformulating the problem or change the parametrization. Among others Bruyneel and Fleury [8] and Moita and co-workers [9] have applied such methodology with success. Still, these methods do not ensure convergence to the globally optimum solution.
3. Parametrization methods start from the realization that the problem of local optimum solutions is inherent in the CFAO method. The goal of the methods is to change the parametrization and obtain a convex design space. Such methods include the lamination parameter method introduced by Tsai and Pagano [10] and used by e.g. Miki and Sugiyama [11] for orientation optimization of plates and Foldager et al. [12] for plates and cylindrical shells. The method requires closed-form analytical formulation of the feasible domain of the lamination parameters which has so far only been achieved for relatively simple geometries.

4. Evolutionary techniques, attributed to Holland [13], are fundamentally search methods but employ ideas derived from genetics and Darwin’s ‘survival of the fittest’ principle to more effectively select the best solution from a population of solutions. Genetic Algorithms (GAs) are particularly useful for discrete problems and problems where the sensitivities are impossible or extremely difficult to compute. If the population is sufficiently large the method reduces the risk of obtaining a local optimum solution but still cannot guarantee convergence to the global optimum. The major problem faced with GAs is the amount of computational effort involved when solving problems with a large number of design variables, particularly if the analysis itself is computationally expensive. GAs have been successfully applied to e.g. stacking sequence optimization by Le Riche and Haftka [14], Adali et al. [15] and others.

The work-horse of engineering design in industry today is the mathematical programming techniques, which have been integrated in e.g. the commercial code BOSS Quattro [16]. Such software cannot guarantee that the obtained solution is the global optimum but is still a valuable design tool since it can provide engineers with a significant performance increase. The goal of the present work is to provide a novel method that obtains a global optimum solution, or at least a close approximation to it, while being applicable to problems of industrial interest. To this end a new parametrization is proposed.

2. THE DISCRETE MATERIAL OPTIMIZATION METHOD

The basic idea in the Discrete Material Optimization (DMO) parametrization is essentially an extension of the ideas used in structural topology optimization but instead of choosing between solid and void we want to choose between any distinct number of materials. This methodology can be stated as: for all elements in the structure find one distinct material from a set of pre-defined candidate materials such that the objective function is minimized. This was first introduced as multiphase topology optimization by Sigmund and co-workers who used it for designing materials with extreme thermal expansion and later also for designing materials with extreme bulk modulus [17, 18]. The multiphase method has also been applied to actuator design [19] which is closely related to compliant mechanism design [20]. Common to the parametrization used in [17–20] is that materials are assumed to be isotropic, the elements are single layered and the maximum number of phases involved is three—two distinct materials and void. Furthermore, the structures considered have been either 2D continua or plates. The DMO formulation extends the scope of application by considering multiple phases and 3D structures and furthermore by allowing the structures to be multi-layered and of orthotropic materials.
However, the method is still limited to operate on a fixed design domain, i.e. thicknesses and shape are not changed during optimization.

In the context of orientation optimization, different materials simply means the same material oriented at various angles in space (fibre angles) but in general, it might as well mean carbon or glass fibre reinforced plastic (CFRP/GFRP), polymer foam, steel, aluminium or any other material at any orientation. As such, the proposed formulation is very versatile and can be used to optimize the material constitution of structures in general and composite structures in particular.

2.1. The methodology

As in topology optimization the parametrization of the DMO formulation is invoked at the finite element level. The element constitutive matrix, $C^e$, is expressed as a weighted sum of candidate materials, each characterized by a constitutive matrix, $C_i$. In general, this may be expressed as a sum over the element number of candidate materials, $n^e$:

$$C^e = \sum_{i=1}^{n^e} w_i C_i = w_1 C_1 + w_2 C_2 + \cdots + w_{n^e} C_{n^e}, \quad 0 \leq w_i \leq 1 \quad (6)$$

It follows that the number of candidate materials is also the number of element design variables and if $N^e$ is the number of elements, the total number of design variables for single layered structures is $n^e \times N^e$. Note that the ‘classical’ topology optimization formulation having one design variable per element is obtained by setting $n^e = 1$ in (6).

The weights, $w_i$, in (6) must have values between 0 and 1 as no matrix can contribute more than the physical material properties and a negative contribution is physically meaningless. In this way, as in classical topology optimization, the weights on the constitutive matrices become ‘switches’ that turn on and off stiffness contributions such that the objective is minimized and a distinct choice of candidate material is made. This underlines that the DMO method relies heavily on the ability of the optimizer to push all weights to the limit values. Any element having intermediate values of the weights must be regarded as undefined since the constitutive properties are non-physical. For the same reason any element having more than a single weight of value 1 must be considered undefined as well. Consequently, the single most important requirement for the DMO method is that every element must have one single weight of value 1 and all other weights of value 0. Failing to comply with this essentially renders the results meaningless. It follows that the choice of weighting functions, $w_i$, is very important for the performance of the DMO method and several formulations have been developed and evaluated (as described shortly). The initial values of the design variables, $x_i$, may in principle be any set of numbers between 0 and 1 but in general the values should be chosen such that the initial weighting is uniform, i.e. $w_i = w_j$ for all $i, j = 1 \ldots n^e$. This provides the most ‘fair’ starting guess since no materials are favoured a priori. This is an important point since strong favouring may result from even a small change in design variables, depending on the expression of the weights. Ultimately, this may force the optimization into a local optimum and uniform initial weighting should therefore be used exclusively.

To illustrate the methodology for fibre angle optimization we solve the example in Figure 3 using DMO with the same orthotropic material oriented at 12 different angles ($0, \pm15, \pm30, \pm45, \pm60, \pm75, 90^\circ$) as the candidate materials in Figure 5(a). The possible material constitutions for the structure are then all combinations of the 12 candidate materials for two
elements, e.g. \(12 \times 12 = 144\) combinations in all \((\pm 90^\circ)\) are identical). These are marked by white triangles in Figure 5(b) where the obtained optimum solution \(30/\pm 45^\circ\) is marked by the large black dot. This solution is the ‘best fit’ to the global optimum solution \(24.2/\pm 41.6^\circ\) obtained with CFAO (see Table I). The normalized compliance of the DMO solution is 1.0040, which is very close to the global optimum solution.

2.2. Element level parametrization—single layered structures

The simplest choice of weight functions would be to extend the classical topology optimization parametrization to multiple design variables, \(x_i\), as

\[
C^e = \sum_{i=1}^{n_e} \left( x_i^e \right)^p C_i = \left( x_1^e \right)^p C_1 + \left( x_2^e \right)^p C_2 + \cdots + \left( x_{n_e}^e \right)^p C_{n_e}, \quad 0 \leq x_i \leq 1
\]

(7)

In this formulation each design variable scales only one constitutive matrix and has no influence on any of the other matrices. To push the design variables towards 0 and 1 the SIMP method has been adopted by introducing the power, \(p\), as a penalization of intermediate values of \(x_i\) (see e.g. Bendsøe and Sigmund [21] for details). The method in (7) is not very efficient as it fails to push the design to its limit values. This was also realized by Sigmund and co-workers who used the following formulation for two distinct materials (three phases) [17, 18]:

\[
C^e = \left( x_0^e \right)^p \left( \left[ 1 - \left( x_1^e \right)^p \right] C_1 + \left( x_1^e \right)^p C_2 \right), \quad 0 \leq x_i \leq 1
\]

(8)

This formulation is fundamentally different in that \(x_0\) scales the entire contribution to \(C^e\) while \(x_1\) ‘slides’ between \(C_1\) and \(C_2\). As such, the formulation encompasses simultaneous topology optimization (through \(x_0\)) and multiple material optimization (through \(x_1\)). The major difference
The difference from (7) is the term \((1 - x_i^p)\), which links a single design variable to more than one constitutive matrix. The benefit of that is that adding weight in one place automatically reduces weight in others, thus helping to push the weights towards 0 and 1. As can be seen from (8) the SIMP methodology is still encompassed in this formulation. However, several implementations of the power, \(p\), have been suggested but here we follow the method of Gibiansky and Sigmund [18].

The formulation in (8) can be extended to include any number of materials by simply adding terms, e.g. for three distinct materials (four phases):

\[
C^e = (x_0^e)^p \left[ 1 - (x_1^e)^p \right] C_1 + (x_1^e)^p \left[ 1 - (x_2^e)^p \right] C_2 + (x_2^e)^p C_3 \]

\[
= (x_0^e)^p \left[ 1 - (x_1^e)^p \right] C_1 + (x_1^e)^p \left[ 1 - (x_2^e)^p \right] C_2 + (x_1^e)^p (x_2^e)^p C_3 \tag{9}
\]

where the limits \(0 \leq x_i \leq 1\) have been excluded for brevity. The expression in (9) becomes tedious to write out for larger number of variables but can be generalized for any number of candidate materials:

\[
C^e = (x_0^e)^p \sum_{i=1}^{n^e} \left[ \prod_{j=1}^{i-1} \left[ 1 - (x_j^e)^p \right] \right] \left[ (x_i^e)^p \right] C_i \tag{10}
\]

This expression is very easy to implement in a general way in existing finite element codes and has proven very efficient for up to three phases. However, when the number of phases is greater the formulation in (10) tends to get stuck in local optima. To counter this a modified SIMP-scheme has been tried in which a second power, \(q\), is introduced on the weights as \((1 - x_i^q)^q\) and \((x_i^e)^pq\). This improves the capabilities of the method but requires some tweaking to converge to the global optimum solution. Consequently, the interpolation in (10) is not generally employed when the number of candidate materials is large. In stead we use the following interpolation, which is a simple extension of (7):

\[
C^e = \sum_{i=1}^{n^e} \left[ \prod_{j=1}^{i-1} \left[ 1 - (x_j^e)^p \right] \right] \left[ (x_i^e)^p \right] C_i \tag{11}
\]

The difference from (7) is the term \((1 - x_i^e,_{\neq i})\), which is included so that an increase in \(x_i\), automatically involves a decrease in all other weights. This helps drive the design towards 0/1 and the method has proven quite effective for the problems tested. The difference between (10) and (11) becomes clearer when writing out the expression for e.g. three materials (phases) and comparing to (9):

\[
C^e = (x_1^e)^p \left[ 1 - (x_2^e)^p \right] [1 - (x_3^e)^p] C_1 + (x_2^e)^p [1 - (x_1^e)^p] \left[ 1 - (x_3^e)^p \right] C_2
\]

\[
+ (x_3^e)^p [1 - (x_1^e)^p] \left[ 1 - (x_2^e)^p \right] C_3 \tag{12}
\]
The disadvantage of (11) is that the weighting functions in general do not add up to unity—only for the converged case. For the general case of uniform initial weighting the element stiffness will be unrealistically low initially but slowly increase as the design variables are pushed towards their limits. The convergence in compliance is monotonic so this has no affect on the final result but increases the number of iterations to reach optimum and thus increases the computational cost. To circumvent this a scaled version of (11) could be used:

\[
C^e = \frac{1}{\sum_{k=1}^{n_c} w_k} \left( \prod_{j=1}^{n_e} \left[ 1 - \left( x_j^e \right)^p \right] \right)^{1/p} \sum_{i=1}^{n_c} \left( x_i^e \right)^p \prod_{j=1}^{n_c} \left[ 1 - \left( x_j^e \right)^p \right] C_i
\]  

This formulation gives faster convergence to a ‘near optimum’ compliance value but it cannot converge fully since it is less effective in driving the weights to 0/1. The reason is that the scaling to unity alters the effect of the penalization. However, in situations when dealing with constraints on physical properties (e.g. mass constraints) the weight functions must add up to unity to provide realistic results and the use of (13) is essential. It is possible to combine the good convergence of (11) and the scaling properties of (13) by using the latter to only interpolate the mass constraint. This scheme has been tried and works very well for the examples examined.

2.3. Element level parametrization—multi layered structures

For multi layered structures the interpolation method described in the previous section can be applied directly. The only difference is that the interpolation must be invoked for all layers in each element. Consequently, the interpolation scheme (11) is written by layer (denoted by \(l\)):

\[
C^l = \frac{1}{\sum_{k=1}^{n_l} w_k} \left( \prod_{j=1}^{n_l} \left[ 1 - \left( x_j^l \right)^p \right] \right)^{1/p} \sum_{i=1}^{n_l} \left( x_i^l \right)^p \prod_{j=1}^{n_l} \left[ 1 - \left( x_j^l \right)^p \right] C_i
\]  

where \(l\) denotes ‘layer’ and thus \(n_l\) is the number of candidate materials for the layer.

The number of element design variables, \(n^e\), for multi layered elements is then the sum of the number of design variables per layer, \(n_l\), over all layers, \(N^l\), i.e. \(n^e = \sum_{k=1}^{N^l} n_l^k\). As before, the total number of design variables in the problem is \(n^e \times N^e\), which for multi-layered structures implies a significant increase in the total number of design variables. To counter this, patches of design variables are introduced.

2.4. Patch design variables

Collecting design variables in patches reduces the number of total design variables by merging several design variables from different layers and elements into a single variable—a patch design variable. The idea springs from the manufacturing process of composite laminates where fibre mats covering larger areas are often used. A single variable could then govern the orientation of the fibre mat even though it covers several elements. Of course, the layout of the patches is left up to the engineer \(a\ priori\) and consequently, the final result will also be dependent on the initial patch layout.
Patches of design variables may also be used to enforce laminate symmetry by assigning the same design variable to opposite laminae. This is a very convenient feature for many practical applications where laminate coupling effects may be unwanted.

An important note is that the patches must be comparable in size since the numerical value of the sensitivity increases with the number of elements in the patch, \( N^{ep} \):

\[
\frac{\partial C}{\partial \theta_m} = -\sum_{i=1}^{N^{ep}} (u^{e}_i)^T \frac{\partial K^e}{\partial \theta_m} u^{e}_i
\]

(15)

The total number of design variables when using patch design variables is \( n^p \times N^p \) where \( n^p \) is the number of candidate materials for the patch variable and \( N^p \) is the number of patches. This can provide a significant reduction in the number of design variables but at the same time requires some extra effort and insight on the part of the engineer.

3. THE OPTIMIZATION PROBLEM

The discrete material optimization problem can be stated in the same way as a topology optimization problem, subject to a optional constraint on total mass:

Objective : \( \min_x C(u) = u^T p \)

Subject to : \( Ku = p \)
\[
(m \leq m_c)
\]
\[
0 \leq x_{\text{min}} \leq x \leq 1
\]

(16)

where \( m \) is the mass of the structure and \( m_c \) is the allowable mass. This constraint is not active when doing pure fibre angle optimization since a change in fibre angle involves no change in mass and consequently, the constraint may be left out entirely. When doing multi material optimization the mass constraint is important since it effectively determines the amount of light material in the final structure.

3.1. DMO convergence

To determine whether the optimization has converged to a satisfactory result, i.e. a single candidate material has been chosen in all elements and all other materials have been discarded, the DMO convergence measure is defined. For each element the following inequality is evaluated for all weight factors, \( w_i \):

\[
w_i \geq \varepsilon \sqrt{w_1^2 + w_2^2 + \cdots + w_n^2}
\]

(17)

where \( \varepsilon \) is a tolerance level, typically 95–99.5%. If inequality (17) is satisfied for any \( w_i \) in the element it is flagged as converged. The DMO convergence, \( h_{\varepsilon} \), is then the ratio of converged elements to total number of elements:

\[
h_{\varepsilon} = \frac{N^e_{\varepsilon}}{N^e}
\]

(18)

The DMO convergence is denoted \( h_{99.5} \) if the tolerance level is 99.5% (and so forth) and full convergence, i.e. \( h_{99.5} = 1 \), simply means that all elements have a single weight contributing
more than 99.5% to the Euclidian norm of the weight factors. This provides a good measure of the convergence although it is not a rigorous mathematical definition. For multilayered structures the DMO convergence is simply computed layer-by-layer in stead of element-by-element, i.e. \( h_s = N_s^i / N_s^i \).

3.2. Implementation

The method has been implemented in the finite element based system MUST [1], which is aimed at analysis and design of structural problems as well as multidisciplinary problems such as fluid-structure interaction problems. The system is capable of reading input from an external preprocessor such as Ansys. To run the optimization it is only necessary to add the following information to the input file:

- Information about the analysis to perform (linear/non-linear, stress, buckling, etc.)
- Definition of the optimization problem to solve (objective and constraint functions)
- Specifications for the optimization algorithm about number of iterations, convergence criteria, penalization powers, etc.
- Definition of design variables. A specified design variable may be linked to a material set or a real constant set in the finite element model, and one design variable specification automatically generates a number of design variables for the problem. The design variables may be generated automatically either for each finite element or for a patch of elements (for example for a group of elements linked to the same material or the same constant set).
- Definition of the candidate materials to use. Each design variable is linked to a list containing information about the number of layers to use, the number of design variables for each layer, the candidate materials to use and their orientation for each layer. Finally, initial values of the design variables can be defined or computed automatically to ensure uniform initial weighting

Thus, it is a relatively simple task to run an optimization on an existing analysis model provided that the analysis model is well defined in terms of materials and fibre orientation.

Solution of the problem in (16) is done using analytical design sensitivities and the method of moving asymptotes by Svanberg [5]. Consequently, the DMO method requires full access to the finite element source code.

4. NUMERICAL EXAMPLES

To demonstrate the capabilities of the method several numerical examples will be presented. To give an impression of the computational effort required to use DMO, the approximate runtime on a desktop PC is stated for each example.

4.1. Cantilever beam with distributed top load—fibre angle optimization

The cantilever beam with distributed top load has become a standard test for minimum compliance fibre angle optimization. The beam has a length to height ratio of 3 and unit thickness and has been meshed using 768 four-node MITC stabilized shell elements. The DMO setup allows for 12 candidate materials in each element which results in a model having 9216 design variables in total. The candidate materials used are glass fibre reinforced epoxy with
the orthotropic properties $E_x = 54 \text{ GPa}$, $E_y = 18 \text{ GPa}$, $G_{xy} = 9 \text{ GPa}$ and $\nu_{xy} = 0.25$ oriented at $[90, \pm 75, \pm 60, \pm 45, \pm 30, \pm 15, 0^\circ]$. The optimization converges monotonically to full DMO convergence ($h_{99.5} = 1.0$) in 157 iterations taking just under 7 min on a desktop PC. The optimal fibre angle distribution determined is shown in Figure 6 and agrees very well with the results obtained by e.g. Pedersen [3]. To illustrate the patch variable methodology the problem has been solved using 48 patches of $4 \times 4$ elements, which reduces the number of design variables to 576 and reduces the runtime by approximately 16%. The resulting optimal fibre angle distribution is shown in Figure 7.

4.2. Four-point beam bending—fibre angle and material optimization

This example demonstrates the ability of the DMO method to simultaneously choose material type and material orientation. The domain is as defined for the previous example and a mesh of

![Figure 6](image)

Figure 6. Optimal fibre angle distribution in cantilever beam with uniformly distributed top load. Solved using 768 elements and a single candidate material at $[90, \pm 75, \pm 60, \pm 45, \pm 30, \pm 15, 0^\circ]$.

![Figure 7](image)

Figure 7. Optimal fibre angle distribution in cantilever beam with uniformly distributed top load. Solved using 768 elements in 48 patches of $4 \times 4$ elements with the same candidate materials as Figure 6.
Figure 8. Optimal material and fibre angle distribution in beam subjected to four-point bending (symmetric). Black indicates glass/epoxy, light grey indicates foam and intermediate values represent unconverged elements.

Figure 9. Geometry of hinged spherical cap example with actual thickness and distribution of layers.

768 nine node shell elements is used. As candidate materials the same orthotropic glass-epoxy as above is taken at [90, ±45, 0°] and furthermore, an isotropic polymeric foam material having $E = 125$ MPa and $v_{xy} = 0.30$ is used. This results in five design variables per element and thus 3840 in total. The mass constraint is set to $m_c = 4$ kg which effectively means that the foam must account for roughly 74% of the material usage when the densities of glass-epoxy and foam are 1900 and 100 kg/m$^3$, respectively. The optimization converges monotonically to $h_{05} = 0.96$ in 125 iterations taking just over 30 min.

The result of the optimization is shown in Figure 8. The edge of the optimum geometry (marked by black) is very similar to results obtained with classical topology optimization.
techniques. The difference is that instead of obtaining a frame-like structure, the DMO method uses the polymeric foam material (light grey in Figure 8) to form a sandwich structure. If the mass constraint is loosened the DMO method will tend toward distributing stiff material in a frame structure as well. The intermediate densities found in the area below the point load is a local effect and roughly outline would-be bars. Tightening the mass constraint will reduce this effect but the intermediate densities have been allowed here in order to illustrate the relationship of the DMO methods with classical topology optimization.

4.3. Hinged 8-layer spherical cap—fibre angle and material optimization

This doubly curved shell has been chosen to show the capabilities of the proposed method for general composite laminate shell structures. The base of the shell structure is spanned by a 1000×1000 (mm) square and the centre point rises 100 mm above the base. The four sides are prescribed by parabolas and the surface is generated by dragging one parabola along an identical parabola. In mathematical terms the surface may be described as \( z(x, y) = h - 2h/l^2(x^2 + y^2) \) where \( h \) is the height of the centre point and \( l \) is the side-length, here 100 and 1000 (mm), respectively. The thickness of the shell is 8 mm divided evenly over eight layers. The structure is loaded by a single load in the centre point and the model is hinged \((u_i = 0)\) on the four edge curves. The entire shell geometry is modelled using a 40×40 mesh of 4-node MITC-stabilized shell elements. The geometry and mesh are shown in Figure 9. As candidate materials we again use a glass/epoxy composite with \( E_x = 54 \text{ GPa}, \ E_y = 18 \text{ GPa}, \ G_{xy} = 9 \text{ GPa} \) and \( \nu_{xy} = 0.25 \) and the permissible fibre angles \([90, \pm 45, 0^\circ]\) as well as a polymeric foam with \( E = 125 \text{ MPa} \) and \( \nu_{xy} = 0.30 \). The orientation of the glass/epoxy is expressed relative to the global x-axis. The two skin layers are not allowed to choose the polymeric foam but the inner 6 layers can be either foam or composite. This results in 38 design variables per element, distributed as \([4, 5, 5, 5, 5, 5, 4]\) bringing the total number of design variables for the model to 64000. The mass constraint is \( \mu_c = 5.0 \text{ kg} \) which means that the foam must constitute just under 75\% of the total volume. The SIMP power is increased from 3.0 to 10.0 in steps of 1.0 every 10 iterations and convergence to \( h_{99.5} = 0.992 \) is reached in 50 iterations using roughly 80 min

![Figure 10. Iteration history of objective function (left axis) and DMO convergence ratio(right axis) for spherical cap example.](image-url)
of computational time. The convergence is shown in Figure 10 and the resulting material distribution and fibre orientation is shown for layer 1–4 in Figure 11 and for layer 5–8 in Figure 12. The layers are numbered from the outside of the shell, i.e. layer 8 is on the centre of curvature side.

As shown in Figures 11 and 12 the fibre angle optimization problem has been solved for the skin layers (layers 1 and 8) and the combined material distribution and orientation problem has been solved for the internal layers (layers 2–7). The result for the skin layers resemble
Figure 12. Optimal material and fibre angle distribution in layers 5–8 of spherical cap.

those found in the literature for plates under similar boundary conditions (e.g. Pedersen [3]). A solution for the material distribution problem has not been reported in the literature but the solution corresponds well to known reinforcement techniques for sandwich panels [22]. In the final solution a cone has been formed through the thickness of the shell to support the local, concentrated load. At the bottom of the cone (the centre of curvature side) a wider reinforcement is obtained to distribute the transverse load over a larger area, thus reducing the local deformation in the lower skin.
5. CONCLUSIONS

In the present paper discrete material optimization (DMO) is introduced as a new gradient based technique for maximizing structural stiffness by optimizing material choice and material orientation. The method operates on a fixed domain, i.e. shape and thicknesses are defined a priori and remain fixed, and as such we deal entirely with solving a laminate lay-up problem. The DMO method is derived from multiphase material optimization in the sense that the element stiffness is computed from a weighted sum of candidate materials. The aim of the optimization is for each element (layer) to choose the material from the set of candidate materials that minimizes the objective the most. The candidate materials may be either isotropic or orthotropic with a given fibre angle.

In the three numerical examples presented the DMO method shows very good performance. The classical problem of finding the optimal orientation of an orthotropic material in a cantilever beam is solved and agrees very well with known results from the literature. The four-point bending problem is solved for material choice and orientation and resulted in a sandwich design with a hard skin and soft core. The contour of the skin resembles that obtained with classical topology optimization and as such seems to be in agreement with known results. The final example is a doubly curved composite laminate shell which also resulted in a sandwich design. For the skin layers the fibre angle problem is solved and for the internal layers the material distribution is obtained. The resulting design correlates well with known solutions for sandwich panels and therefore further validates the DMO method. Such design examples have not been solved before in the literature using design optimization approaches and the proposed method thus represents a novel extension of design methods available for laminate structures.

For the examples investigated no problems with local optima could be observed. This is very promising since the existence of local optima is the single most encumbering problem when optimizing with orthotropic materials. The checkerboard problem usually associated with topology stiffness design has not been encountered for the tested configurations but could arise in material selection problems involving large differences in material stiffnesses. Solution of the checkerboard problem can, if necessary, be handled using e.g. standard filtering techniques as described in Bendsøe and Sigmund [21]. The characteristics of the DMO method with respect to both local optima and the checkerboard problem will be investigated further in the near future.

So far the major drawback of the DMO methodology is the large number of design variables, which increases the computational cost. However, the authors have successfully solved problems involving just over 750,000 design variables and still, the optimization algorithm MMA solves the optimization problem in just under 3 s. The vast majority of time is spent on computing the design sensitivities for all design variables. However, patch design variables may be used to reduce the number of design variables, thus reducing computational cost, as demonstrated for the cantilever beam.

The discrete material optimization method is currently being extended to local criteria and geometrically non-linear structural behaviour. Only single load cases have been studied here, but facilities for multiple load cases have also been implemented using a weighted sum formulation for the compliance, such that a real life load spectrum can be taken into account.

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