6.1 PREFACE

The previous chapters have dealt with the vehicle motion under the assumption that the wheel is rigidly attached to the vehicle body, and there is no relative displacement between the body and the wheels. This is a good assumption, proven from practical experience, for understanding the basic vehicle motion characteristics. However, in the case of normal passenger cars, the vehicle body and wheels are connected to each other by soft and elastic connections to improve the vehicle ride comfort. This mechanism is generally called the suspension system, and the vehicle body is called the sprung mass, while the wheels are called the unsprung mass. The suspension system between the vehicle body and the wheels allows a relative up–down displacement between the vehicle body and the wheels. When the vehicle moves laterally, a centrifugal force acts at the vehicle center of gravity, causing the vehicle to tilt to the direction of the centrifugal force. This ‘tilt’ is called the vehicle roll. If the suspension system is considered, the vehicle will have a roll degree of freedom that is produced together with vehicle lateral motion.

This chapter will look into the roll mechanism, derive the vehicle equations of motion including the roll motion, and study the effects of suspension system characteristics and vehicle roll on the vehicle motion.

6.2 ROLL GEOMETRY

Eberan’s hypothesis of the roll center as the vehicle’s geometrical instantaneous rotation center, and assumption that this roll center is always fixed [1], have long been taken as the standard approach. This hypothesis is generally used due to its simplicity. Based on this hypothesis, the roll mechanism of the
vehicle will be studied with a constant lateral acceleration, which is caused by a constant centrifugal force.

### 6.2.1 Roll center and roll axis

In general, there are various types of suspension systems, from the simple rigid axle type to the independent suspension that is common in passenger cars. The relative vertical displacement or angular displacement between the sprung and unsprung masses is dependant on the structure of the suspension system.

The front and rear wheel roll centers are also determined by the suspension system configuration. The line that connects the front and rear roll centers is called the roll axis. The roll center is the vehicle’s instantaneous rotation center in the plane perpendicular to the vehicle’s longitudinal direction, which contains the left and right wheels’ ground contact point. The wheels are considered rigid in both up–down, left–right directions and the ground contact point is fixed.

**Figure 6.1** shows the axle type suspension system. The vehicle body at points A₁ and B₁ can only have vertical displacement relative to the unsprung mass due to the springs. Even if the sprung mass rolls, the unsprung mass including the wheels is assumed rigid and thus, doesn’t move, the roll center is at point O. In other words, when a rolling moment acts on the vehicle, the vehicle body will produce a roll angle, $\phi$, relative to the wheels with respect to the point O.

**Figure 6.2** shows a typical independent type of suspension – often called the double wishbone suspension. As its name implies, each wheel can move independently, relative to the vehicle body. If the vehicle body is fixed, the instantaneous rotation centers of the left and right unsprung mass relative to the vehicle body are the points O₁ and O₂, respectively. The point O₁ is the intersecting point of the extended lines of A₁–A₂ and A₃–A₄, while the point O₂ is the intersecting point of the extended lines of B₁–B₂ and B₃–B₄. Here, when the vehicle body rolls during cornering, the wheel contact points with the ground, A and B, are fixed and the unsprung masses must roll around them. The points O₁ and O₂ move in the direction perpendicular to O₁A and O₂B. O₁ and O₂ are the virtual points on the vehicle body as well as on the unsprung masses. Consequently, the vehicle body instantaneous rotating center, or the roll center is the intersection of the extended lines of O₁A and O₂B, which is the point O.

![Figure 6.1 Roll center for rigid axle suspension.](image-url)
Based on this way of thinking, the roll center for other types of suspension system is shown in Fig. 6.3.

It is clear that the vehicle roll center position is dependant on the structure of the suspension system. Usually, the suspension system and the vehicle are symmetrical on the left and right, and the roll center is always on the symmetric axis. In this case, it is the height of the roll center that is dependant on the suspension system structure.

The roll center is the vehicle instantaneous rotation center, and its position can move during suspension movement. The point O shown here is the roll center when roll angle is zero; if the vehicle rolls, the roll center will also move. To understand
this, the roll center $O'$ during body roll is shown for two types of suspension system – the wishbone and the swing axle suspension systems, in Fig. 6.4.

If the roll angle is not large, the movement of the roll center is small, and it is possible to assume that the roll centers are fixed at point $O$. It is still possible to understand the vehicle roll mechanism, even with a moving roll center. But the fixed roll center concept is easier to understand and gives a good understanding of the basic vehicle dynamics. Based on Eberan’s roll center hypothesis, the front and rear roll centers are determined, and if the vehicle body is rigid, the vehicle’s fixed roll axis is determined as shown in Fig. 6.5. The roll center at the front and rear may not have the same height above the ground and the roll axis is not necessarily parallel to the vehicle longitudinal axis.

Furthermore, when vehicle motion is accompanied by large roll angles, the fixed roll center and roll axis concepts are not suitable anymore. In such cases, vehicle roll is usually dealt with as the indeterminate problem of the vehicle’s four wheels.

### 6.2.2 Roll stiffness and load transfer

Now, the vehicle is assumed to have a constant lateral acceleration and centrifugal force acting at the vehicle center of gravity. The center of gravity
doesn’t normally coincide with the vehicle roll axis, but is usually above the roll axis, as shown in Fig. 6.6. The centrifugal force acting at the center of gravity produces a rolling moment around the roll axis resulting in a constant roll angle. If the vehicle body rolls, the left and right vertical springs of the suspension system will be stretched at one side and be compressed on the other side. This produces an equilibrium moment to the rolling moment due to the centrifugal force. The magnitude of the moment produced by the stretch and the compression of the spring per unit roll angle is called the roll stiffness.

Here, the respective roll stiffness for the front and rear suspension systems is defined as $K_{f_1}, K_{f_2}$, the roll center height from the ground as $h_f, h_r$, the front and rear tread as $d_f, d_r$, the distance between the vehicle center of gravity and the roll axis as $h_s$, and the distance between the front and rear axles to the center of gravity as $l_f, l_r$. The weight of the unsprung mass is small compared to the weight of the sprung mass and could be neglected. In this case, the vehicle weight is taken to be equal to the vehicle body weight, and written as $W_s$. The vehicle lateral acceleration is taken as $\ddot{y}$ (same as in subsection 3.3.3) and the centrifugal force acting on the vehicle is $\ddot{y}W_s$. Assuming that the vehicle is rigid, and the roll angle is small, the rolling moment by the centrifugal force is $\ddot{y}W_s h_s$ and the roll moment by the vehicle weight due to tilting of the vehicle body is $W_s h_s \phi$, the vehicle roll angle becomes

$$ (K_{f_1} + K_{f_2})\phi = \ddot{y}W_s h_s + W_s h_s \phi \quad \text{or} \quad \phi = \frac{\ddot{y}W_s h_s}{K_{f_1} + K_{f_2} - W_s h_s} \quad (6.1) $$

The centrifugal force, $\ddot{y}W_s$, acting on the vehicle requires tire cornering forces to achieve equilibrium. Distributing the $\ddot{y}W_s$ force acting at the center of gravity to the front and rear wheels, the forces $\ddot{y}W_s l_f/l$ and $\ddot{y}W_s l_r/l$ could be considered to act on the front and rear wheels, respectively, where $l = l_f + l_r$. These forces are equal to the front and rear wheel lateral forces.

If the vehicle body rolls, the left and right wheels at both front and rear axles will increase in load at one side and decrease at the other side. This is called the load transfer due to roll. Defining the load transfer for the front and rear as $\Delta W_f$ and $\Delta W_r$, respectively, the roll moment around the roll center at the front and rear wheels in the plane perpendicular to the vehicle longitudinal direction has to be in equilibrium, as shown in Fig. 6.7. The following equations are derived:
At this time, it is assumed that there is no load transfer between the front and the rear. Substituting Eqn (6.1) into Eqns (6.2) and (6.3) to find $\Delta W_f$ and $\Delta W_r$, gives the following equations:

$$K_{\phi f} \phi = \Delta W_f d_t - \frac{\ddot{y} W_{s f}}{l} h_f$$  \hspace{1cm} (6.2)

$$K_{\phi r} \phi = \Delta W_r d_t - \frac{\ddot{y} W_{s r}}{l} h_r$$  \hspace{1cm} (6.3)

These equations give the load transfer between the left and right wheels due to a constant lateral acceleration. The equations show that a higher vehicle center of gravity distance from the roll axis, $h_s$, results in a larger load transfer at the front and rear wheels. Furthermore, a load transfer at the front and rear wheels is basically proportional to the front and rear roll stiffness ratios to the total roll stiffness, respectively.

The last term in Eqns (6.4) and (6.5) depends on the height of the roll axis from the ground, $h_f$ and $h_r$, and causes the jack-up effect.

6.2.3 Camber change and roll steer

If the ground contact point of the wheels is fixed, as the vehicle body rolls, the unsprung mass, including the wheels, tilts relative to the ground. This gives the
camber change of the wheel, which is measured relative to the ground and is due to body roll. The vehicle roll also gives the wheels an up-and-down displacement relative to the vehicle body. At such time, depending on the structure of the suspension system, the wheels may produce some angular displacement in the horizontal plane along with the up-and-down movement relative to the vehicle body. This is called the roll steer.

The camber change and roll steer are dependant on the structure of the suspension system. The suspension system is designed with keen consideration of these characteristics, often using them to affect the vehicle dynamics or sometimes trying to avoid them completely. This chapter will skip the detailed explanation of camber change and roll steer mechanism for various suspension systems, and only look at the basic characteristics of camber change and roll steer. The collective term for camber change and roll steer is sometimes called the alignment change due to roll.

In axle type suspensions, the wheel doesn’t produce any camber change due to vehicle roll. The camber change due to roll only occurs for independent suspension systems, where depending on the suspension structure, there could be one of two cases: camber change in the same direction as roll, which is called positive camber, or in the opposite direction, negative camber. This is shown in Fig. 6.8.

Independent suspension systems are constructed by a linkage mechanism, and the vehicle roll angle and camber change can be determined from geometrical analysis of the linkage. Figure 6.9 shows the actual measured value and calculated value of the camber change for a wishbone type suspension system. This relationship varies substantially with the arrangement of the links, even for suspension systems of the same type. From the figure, if the roll angle is not large, the camber change can be considered as nearly proportional to the roll angle. As the roll angle becomes large, this linear relation is lost, and non-linearity appears. This is generally for other types of suspension systems. The non-linear characteristic of the camber change is one of the main factors that

![Figure 6.8 Camber change due to body roll.](image)

FIGURE 6.8 Camber change due to body roll.
influences the vehicle motion at large lateral accelerations. For lower lateral accelerations, the camber change could be considered to be proportional to the roll angle, in the directions shown in either Fig. 6.8(a) or (b).

**Example 6.1**

Investigate the geometrical condition of the suspension mechanism such as in Figs 6.2 or 6.3, which basically determines the positive or negative camber to the body roll.

**Solution**

Any part of the left half of the vehicle body is going downward and the right half upward due to the body roll. While the unsprung mass rolls around the tire contact point, for example, point A in Fig. 6.2, which is a camber change. So any point of the left side of the unsprung mass relative to the center line of the left wheel is going downward and the right side to upward due to the positive camber of the unsprung mass. Due to the negative camber, the left side is going upward and the right side is going to downward.

As the virtual point, $O_1$, is the common point fixed to the vehicle body and the left hand side unsprung mass, it must move in the same direction as the body roll and the camber change during roll motion. If point $O_1$ is to the left of the body center and the right of the wheel centerline, then the suspension has negative camber. If the point $O_1$ is to the left of the body and to the right of the wheel centerline, or to the right of the body center and the wheel centerline, then the suspension has to show the positive camber.

![FIGURE 6.9 Chamber changes to suspension stroke.](image-url)
The roll steer due to the roll of the vehicle body is also dependant on the suspension system structure. For an axle type suspension, the sprung mass and unsprung mass are often connected using leaf springs. The mounting point of the spring at the vehicle axle moves in the rear-and-forward direction and causes the axle to produce an angular displacement relative to the vehicle body in the horizontal plane. This is roll steer for an axle type suspension, which is sometimes called axle steer due to roll.

For independent suspension systems, the amount of roll steer can be determined from geometrical analysis of the linkage. Similar to camber change, the roll steer direction and magnitude, relative to the roll angle, can vary substantially with the arrangement of the links. The suspension systems are usually designed to control the amount of steer by careful arrangement of the links. For independent suspensions, if the roll angle is small, the roll steer can be considered to be proportional to the roll angle. The direction of the roll steer can sometimes be in either positive or negative direction, depending on the suspension system structure.

The roll steer for an independent type suspension is sometimes called the toe change due to the vertical stroke of the suspension. Figure 6.10 shows an example of toe change due to suspension stroke. Roll steer in the direction toward the inner side of the vehicle is called toe-in and the one in the opposite direction is called toe-out.

### 6.3 BODY ROLL AND VEHICLE DYNAMICS

By examining the vehicle in steady-state cornering, the vehicle steer characteristics that fundamentally influence the vehicle dynamic performance have
been understood. During steady-state cornering, a constant centrifugal force acts at the vehicle center of gravity and if the suspension system is considered, the vehicle will produce a constant roll angle. The previous sections explained body roll geometrically. This section will try to study the effect of vehicle roll on the steer characteristics by considering vehicle steady-state cornering with body roll. The effect of suspension lateral stiffness on vehicle steer characteristics will also be looked at.

6.3.1 Load transfer effect

As described in Chapter 2, the tire lateral force changes with tire load in the form of a saturating curve. When there is a load transfer between the left and the right wheels, the sum of their lateral forces will be lower than when load transfer is not considered. The larger the load transfer, the greater the reduction in total lateral force.

Figure 6.11 shows a typical relationship between the load transfer and the lateral force. For an axle with two wheels and weight $W$, a load transfer of $\Delta W$ occurs between the left and right sides. This yields lateral forces of $P_1 A_1$ and $P_2 A_2$, and their sum is $2BA$. In contrast, the lateral force when there is no load transfer is $2PA$. The reduction of the lateral force at this axle, due to roll, is exactly equal to $2PB$.

During vehicle body roll, the lateral load transfer occurs at the front and rear axles, as described by Eqns (6.4) and (6.5). Consequently, the front and rear wheel cornering stiffnesses will decrease according to the magnitude of $\Delta W_f$ and $\Delta W_r$. In order to undergo steady-state cornering, at the same radius as without load transfer, with the same magnitude of centrifugal force, the front and rear wheel side-slip angles must increase according to the magnitude of $\Delta W_f$ and $\Delta W_r$. This will produce the lateral forces for equilibrium.

The vehicle steer characteristic is determined by the relative magnitude of the front and rear wheel side-slip angles. If $\Delta W_f > \Delta W_r$, the vehicle steer...
characteristic will tend to US, while if $\Delta W_f < \Delta W_r$, the steer characteristic will change to OS.

The load transfer for the front and rear wheels, as shown by Eqns (6.4) and (6.5) is dependent on:

- the front and rear roll center heights, $h_f$ and $h_r$,
- the ratio of front and rear roll stiffnesses, $K_{\phi f}/K_{\phi r}$, and
- the front and rear tracks, $d_f$ and $d_r$.

If the following conditions are true: $h_f \rightarrow$ large, $h_r \rightarrow$ small, $K_{\phi f}/K_{\phi r} \rightarrow$ large, $d_f \rightarrow$ small, and $d_r \rightarrow$ large, then $\Delta W_f$ will become larger than $\Delta W_r$, and the vehicle steer characteristic changes to US. In contrast, the vehicle steer characteristic will change to OS if $h_f \rightarrow$ small, $h_r \rightarrow$ large, $K_{\phi f}/K_{\phi r} \rightarrow$ small, $d_f \rightarrow$ large, and $d_r \rightarrow$ small.

Below is a further examination of the reduction in the axle equivalent cornering characteristics due to the load transfer between the left and right wheels.

Curve OP in Fig. 6.12 is the relationship between the lateral force and tire side-slip angle for a tire with the weight $W$. The lateral force for the tire with the extra weight of $\Delta W$ is shown by curve OP$_1$, and the lateral force for the tire with less weight of $\Delta W$ is shown by curve OP$_2$. The sum of these two curves is the lateral force for the axle when there is a load difference of $\Delta W$ between the left and right wheels. This is equal to two times of the curve OB.

If there is no load difference, the lateral force for that axle is shown by the curve OP, which is larger than the curve OB. These two curves are the axle lateral force divided by the axle weight.

At lateral accelerations of $\ddot{y}_1$, $\ddot{y}_2$, and $\ddot{y}_3$, there are load transfers of $\Delta W_1$, $\Delta W_2$, and $\Delta W_3$, respectively, at the vehicle axle. The curves of the axle force divided by the axle weight to side-slip angle for each load transfer are seen in Fig. 6.13. As described in subsection 3.3.3, the vertical axis in Fig. 6.13 is identical to the vehicle lateral acceleration. Projecting points $\ddot{y}_1$, $\ddot{y}_2$, and $\ddot{y}_3$ from the vertical axis of Fig. 6.13 onto the curve for load differences $\Delta W_1$, $\Delta W_2$, and $\Delta W_3$, respectively, and connecting all the points on the curves, give a new curve of equivalent cornering force of the axle when lateral load transfer occurs according to the lateral acceleration.

![Figure 6.12 Axle lateral force and vertical load.](image-url)
The above investigation infers that a change of tire characteristics due to load transfer is expected and the change of vehicle motion characteristics due to this will become more obvious at large lateral accelerations because of the non-linear tire characteristics.

### 6.3.2 Camber change effect

The wheel camber could occur in either the same or the opposite direction to the roll direction, as described in subsection 6.2.3. Here, it is assumed that the first case gives positive camber change, and the second gives a negative camber change. In either case, the camber change results in a force that acts in the lateral direction (camber thrust). This is proportional to the camber angle, as described in Chapter 2. In steady-state cornering, the camber thrust becomes one of the forces that balance the centrifugal force at the CG. Positive camber angles produce a camber thrust that acts in the same direction as the centrifugal force. In this case, larger wheel side-slip angles are needed to achieve steady-state cornering at the same radius and speed as when camber change is not considered. In contrast, negative camber angles produce a camber thrust that acts in the opposite direction to the centrifugal force. Here the cornering force and the wheel side-slip angles can become smaller.

The vehicle steer characteristics are determined by the relative magnitude of the front and rear wheel side-slip angles. Consequently, the positive camber change alters the vehicle steer characteristic to US at the front wheels and OS at the rear wheels. Negative camber changes have an opposite effect and change the vehicle steer characteristic to OS at the front wheels and US at the rear wheels.

### 6.3.3 Roll steer effect

The angular displacement of the wheel due to roll is defined as roll steer. A positive roll steer acts in the same direction as the actual steer angle while a negative roll steer acts in the opposite direction.
Figure 6.14 shows the geometry of a steady-state cornering vehicle with roll steer. Here, $\alpha_f$ and $\alpha_r$ are the front and rear roll steers. The geometrical relation of steady-state cornering excluding roll steer is given by Eqn (3.34). With roll steer as in Fig. 6.14, the equation becomes

$$\rho = \frac{l}{\delta - \beta_f + \beta_r + \alpha_f - \alpha_r}$$  \hspace{1cm} (6.6)$$

Equation 6.6 shows that the vehicle steer characteristic is determined by the front and rear roll steer angles, $\alpha_f$ and $\alpha_r$ as well as the front and rear wheel side-slip angles, $\beta_f$ and $\beta_r$. When a cornering radius at a constant steer angle increases with speed or lateral acceleration, the steer characteristics is termed as US. If the radius decreases, the steer characteristic is called OS. If the front roll steer, $\alpha_f$, is positive, it will change the vehicle steer characteristics to OS and if it is negative, the vehicle will tend to US. On the contrary, if the rear roll steer, $\alpha_r$, is positive, it will change the vehicle steer characteristics to US, and a negative roll steer will result in an OS vehicle. Figure 6.15 shows the effect of roll steer on the vehicle steer characteristic. Rewriting Eqn (6.6) gives:

$$\delta = \frac{l}{\rho} + \beta_f - \beta_r + \alpha_r - \alpha_f$$  \hspace{1cm} (6.6)'$$

As described in subsection 3.3.3, $\beta_f - \beta_r$ is determined in relation to the lateral acceleration, $\ddot{y}$, during cornering. Also, the front and rear roll steers, $\alpha_f$ and $\alpha_r$, are found from the roll angle (or the suspension stroke), which is proportional to the lateral acceleration, $\ddot{y}$.

Based upon the above, Eqn (6.6)' can be used to investigate the vehicle steer characteristics in the relationship between the steady-state steer angle, $\delta$, and the lateral acceleration, $\ddot{y}$, when roll steer is considered.
Example 6.2

Derive the equation to show the effect of the roll steer on the steady-state turning radius when the front and rear suspensions both have the roll steer proportional to the roll angle.

Solution

As is described at the end of subsection 3.3.1 (2), the side-slip angles of the front and rear tires during steady-state turning with the lateral acceleration, $mV^2/\rho$, are described as:

$$\beta_f = \frac{mV^2 l_f}{2IK_f} \frac{1}{\rho}$$

$$\beta_r = \frac{mV^2 l_r}{2IK_r} \frac{1}{\rho}$$
From Eqn (6.1), the roll angle in the steady-state turning is

\[ \phi = \frac{\dot{y}W_s h_s}{K_\phi} = \frac{m_s V^2 h_s}{K_\phi} \frac{1}{\rho} \]

where \( K_\phi = K_{\phi f} + K_{\phi r} - W_s h_s \).

The roll steer angles of the front and rear wheels proportional to the roll angle are described as:

\[ \alpha_f = \frac{\partial \alpha_f}{\partial \phi} \frac{\partial \phi}{\partial \alpha_f} m_s V^2 h_s \frac{1}{K_\phi} \frac{1}{\rho} \]  
\[ \alpha_r = \frac{\partial \alpha_r}{\partial \phi} \frac{\partial \phi}{\partial \alpha_r} m_s V^2 h_s \frac{1}{K_\phi} \frac{1}{\rho} \]  

(E6.1)  

(E6.2)

Substituting the above \( \beta_f, \beta_r, \alpha_f, \) and \( \alpha_r \) into Eqn (6.6):

\[ \rho = \left[ 1 + \left\{ -\frac{m(l_f K_f - l_r K_r)}{2l^2 K_f K_r} - \frac{m_s h_s \partial \alpha_f}{l K_\phi} \frac{\partial \phi}{\partial \alpha_f} + \frac{m_s h_s \partial \alpha_r}{l K_\phi} \frac{\partial \phi}{\partial \alpha_r} \right\} V^2 \right] / \delta \]  

(E6.3)

### 6.3.4 Suspension lateral stiffness and its effect

Section 6.1 described how the vehicle body and the wheels are connected elastically by the suspension. This suspension system gives the vehicle wheels a displacement relative to the vehicle body mainly in the vertical direction; however, the vehicle body and the wheels are not completely connected rigidly in the lateral direction.

This section will look at the effect of this suspension system lateral stiffness on the vehicle steer characteristics.

Figure 6.16 shows the connection of the vehicle body with the wheel in the horizontal plane. The lateral force doesn’t usually act through the lateral stiffness center of the suspension system. Therefore, it produces an angular displacement of the wheel in the horizontal plane. This is called the compliance steer and influences the vehicle steer characteristic.

The compliance steer is generated by the lateral tire force, which depends on the lateral acceleration. Consequently, the suspension system compliance steer is due to the lateral acceleration during cornering. If \( \alpha_f \) and \( \alpha_r \) in Fig. 6.15 are considered as the compliance steers, the effect of the compliance steer on the vehicle steer characteristic can be studied in exactly the same way as the roll steer effect.

The compliance steer can be assumed to be proportional to the lateral force acting on the tire within a relatively small lateral acceleration range. If the cornering stiffness is \( K \), and the lateral force is proportional to the side-slip angle, then
Here, $c$ is a compliance coefficient of the suspension system. Eliminating $a$ in the above equation gives:

$$F = K + cK \frac{b}{C0} = eK \frac{b}{C0}$$

whereby $e = 1/(1 + cK)$. In other words, the compliance steer has the effect of changing the tire equivalent cornering stiffness from $K$ to $eK$. This effect is similar to the effect of the steering system stiffness as described in subsection 5.3.1.

The above concept can be extended beyond the region where the compliance steer is proportional to the lateral force, to the non-linear region where the lateral force is not proportional to the side-slip angle. Figure 6.17 shows how the tire cornering characteristics considering both the suspension system lateral stiffness and the steering system stiffness equivalently vary from the original tire cornering characteristics by the compliance steers.

A tire with real side-slip angle, $\beta$, suspension system compliance steer, $\alpha_1$, and steering system compliance steer, $\alpha_2$, has a nominal side-slip angle of $\beta + \alpha_1 + \alpha_2$. This lateral force at side-slip angle equal to $\beta$ is equivalently regarded as the lateral force when the side-slip angle is equal to $\beta + \alpha_1 + \alpha_2$. In this manner, the equivalent tire cornering characteristics with consideration of the compliance steers are obtained, and their effect on the vehicle steer characteristics can be investigated.

### 6.4 EQUATIONS OF MOTION INCLUSIVE OF ROLL

Until here, the vehicle roll mechanism has only been dealt geometrically, or from a static point of view, where the roll is produced by a constant lateral acceleration.

Next, this knowledge will be used to derive the vehicle equations of motion that include rolling motion. This will enable us to investigate the vehicle motion further. These equations of motion are based on those proposed by Segel, using the fixed roll axis concept [2].
The choice of the coordinate system for dealing with the motion is not necessarily standardized, and can vary. Here, the coordinate system that the author thinks is most easily understood is used.

6.4.1 Coordinate system and dynamic model

A coordinate system with the $X$–$Y$ plane parallel to the ground is fixed in absolute space as shown in Fig. 6.18.

Point P in Fig. 6.18 is where a vertical line through the CG crosses the roll axis. This is the origin of a coordinate system $x$–$y$–$z$ that is fixed to the vehicle body, i.e., the sprung mass. The vehicle longitudinal direction, parallel to the ground, is taken as the $x$-axis (the forward direction is positive), the lateral
direction perpendicular to this is taken as the $y$-axis (the left hand side when the vehicle is facing to the front as positive), and the vertical direction as the $z$-axis (up direction as positive).

A coordinate system of $x' - y' - z'$ is fixed to the unsprung mass with the same origin $P$. The vehicle longitudinal direction is the $x'$-axis, the lateral direction perpendicular to this is the $y'$-axis, and the vertical direction is the $z'$-axis.

The vehicle roll axis doesn’t always coincide with the $x$-axis, as mentioned in subsection 6.2.1. For simplicity, the vehicle body is assumed to roll around the $x$-axis. Furthermore, the roll angle is assumed to be small and the vehicle is assumed to yaw around the $z$-axis. The unsprung mass roll motion is neglected, and the unsprung mass is assumed to produce the same yawing motion as the vehicle body around the $z'$-axis (therefore, the $x$-axis and the $x'$-axis always coincide). All the angular displacements and angular accelerations are taken as positive in the direction shown in Fig. 6.18.

Figure 6.19 shows the equivalent mechanic model for the sprung mass and the unsprung mass. The mass of the vehicle body, or the sprung mass, is assumed to be distributed symmetrically to the $x-z$ plane and the center of gravity is taken as the point $S$ in the $x-z$ plane. The height of the unsprung mass in the $z'$-direction is neglected, and its mass is assumed to be distributed in the $y'-z'$ plane with the center of gravity at point $U$ on the $x'$-axis.

Here, $m_S$ is the mass of the sprung mass, $m_U$ is the mass of the unsprung mass and $m$ is the mass of the whole vehicle. $h_s$ is the distance between the sprung mass center of gravity to the $x$-axis, $c$ and $e$ are the distances of the sprung mass and the unsprung mass center of gravity to the $z$-axis, respectively, while $r$ and $p$ are the vehicle yaw angular velocity and roll angular velocity, respectively. $\phi$ is the vehicle roll angle.

FIGURE 6.19 Equivalent dynamic model.
6.4.2 Inertias

Rigid body motion can be divided into both the translational motion of the center of gravity and the rotational motion around the center of gravity. Here, these motions will be considered for the sprung mass and the unsprung mass, which are both rigid bodies.

(1) TRANSLATIONAL MOTION

The translational motion of the center of gravity of a rigid body is equal to the motion of the point where the entire body mass is concentrated at the center of gravity.

The coordinate system $X–Y–Z$ is fixed in the absolute space and the coordinate system $x–y–z$ fixed on the moving rigid body as shown in Fig. 6.20.

The position vector of the center of gravity $C$ relative to the $x–y–z$ coordinates is taken as $\mathbf{r}$. If points $P$ and $C$ are described by position vectors $\mathbf{r}, \mathbf{R}$, relative to the $X–Y–Z$ system, then

$$\mathbf{r} = \mathbf{R} + \mathbf{\rho}$$

The $x–y–z$-coordinate system has a translational velocity, $\dot{\mathbf{R}}$, relative to the $X–Y–Z$ coordinates, and also a rotational motion with an angular velocity of $\mathbf{\omega}$. Thus,

$$\dot{\mathbf{r}} = \dot{\mathbf{R}} + \dot{\mathbf{\rho}} + \mathbf{\omega} \times \mathbf{\rho}$$

Here, $\dot{\mathbf{\rho}}$ is the relative velocity of the point $C$ to the point $P$ and if the point $C$ is fixed on the $x–y–z$ coordinate, then $\dot{\mathbf{\rho}} = 0$, so

$$\dot{\mathbf{r}} = \dot{\mathbf{R}} + \mathbf{\omega} \times \mathbf{\rho}$$

The translational motions of the sprung and unsprung mass centers of gravity, $S$ and $U$ in the $X–Y–Z$-coordinate system are $\mathbf{r}_S$ and $\mathbf{r}_U$. The position vectors of the point $S$ and the point $U$ relative to the $x–y–z$ system are $\mathbf{\rho}_S$ and $\mathbf{\rho}_U$, and the angular velocities of $x–y–z$ and $x'–y'–z'$ coordinates are $\omega_S$ and $\omega_U$. The unit
vectors in the $x$–$y$–$z$ direction are $i$, $j$, and $k$ and the unit vectors in the $x'$–$y'$–$z'$ direction are $i'$, $j'$, and $k'$. These are shown in Fig. 6.21.

The points $S$ and $U$ are fixed on the $x$–$y$–$z$ and $x'$–$y'$–$z'$ coordinates, respectively, and the following equations are formed based on Eqn (6.7):

$$
\dot{\mathbf{r}}_S = \dot{\mathbf{R}} + \omega_S \times \rho_S \tag{6.8}
$$

$$
\dot{\mathbf{r}}_U = \dot{\mathbf{R}} + \omega_U \times \rho_U \tag{6.9}
$$

Assuming that point $P$ has a velocity component of $u$ in the $x$-direction and $v$ in the $y$- or $y'$-direction when $\phi$ is small, then

$$
\dot{\mathbf{R}} = ui + vj = ui' + vj' \tag{6.10}
$$

The $x$–$y$–$z$ coordinates that move with the sprung mass has a roll velocity of $p$ around the $x$-axis and a yaw velocity of $r$ around the $z$-axis, hence,

$$
\omega_S = pi + rk \tag{6.11}
$$

the $x'$–$y'$–$z'$ coordinates, which move together with the unsprung mass have a yaw velocity of $r$ around the $z'$-axis, so

$$
\omega_U = rk' \tag{6.12}
$$

and, $\rho_S$, $\rho_U$ can be written as follows:

$$
\rho_S = ci + hsk \tag{6.13}
$$

$$
\rho_U = -ei' \tag{6.14}
$$

FIGURE 6.21 Motion of points $S$ and $P$. 
Substituting Eqns (6.10), (6.11), and (6.13) into Eqn (6.8) and Eqns (6.10), (6.12), and (6.14) into Eqn (6.9) gives $\dot{r}_S$ and $\dot{r}_U$ as follows:

$$\dot{r}_S = ui + (v - h_sp + cr)j$$  \hspace{1cm} (6.15)

$$\dot{r}_U = ui' + (v - er)j'$$  \hspace{1cm} (6.16)

Differentiating Eqns (6.15) and (6.16) gives the acceleration vector of the point S and the point U:

$$\ddot{r}_S = (\ddot{u} - vr + h_sp^2 + c\ddot{r})i + (\ddot{v} + ur - h_sp\dot{\beta} + c\ddot{r})j + (vp + h_sp^2 + cpr)k$$  \hspace{1cm} (6.17)

$$\ddot{r}_U = (\ddot{u} - vr + er^2)i' + (\ddot{v} + ur - e\ddot{r})j'$$  \hspace{1cm} (6.18)

From Eqns (6.17) and (6.18), the lateral acceleration, in other words, the acceleration in the $y$- and $y'$-direction, the lateral direction, for the sprung mass and the unsprung mass, $\alpha_S$ and $\alpha_U$, is

$$\alpha_S = \dot{v} + ur - h_sp\dot{\beta} + c\ddot{r}$$

$$\alpha_U = \dot{v} + ur - e\ddot{r}$$

If the side-slip angle of the point P, $\beta$, is $|\beta| \ll 1$ and the velocity magnitude of the point P, $V$, is always constant, then $u \approx V$, $v \approx V\beta$, and

$$\alpha_S = V\ddot{\beta} + Vr - h_sp\dot{\beta} + c\ddot{r}$$

$$\alpha_U = V\ddot{\beta} + Vr - e\ddot{r}$$

The masses for the sprung mass and the unsprung mass are $m_S$ and $m_U$, respectively. The inertia forces in lateral direction, $Y_S$ and $Y_U$, of the sprung mass and the unsprung mass are:

$$Y_S = m_S\alpha_S = m_SV(\ddot{\beta} + r) - m_Sh_sp\dot{\beta} + m_Sc\ddot{r}$$

$$Y_U = m_U\alpha_U = m_UV(\ddot{\beta} + r) - m_Ue\ddot{r}$$

Consequently, the total vehicle inertia force in lateral direction $\Sigma Y$ is

$$\Sigma Y = Y_S + Y_U = (m_S + m_U)V(\ddot{\beta} + r) - m_Sh_sp\dot{\beta} + (m_Sc - m_Ue)\ddot{r}$$
Here, \( m_S + m_U = m \), and CG is the entire vehicle center of gravity, therefore, \( m_{SC} - m_Ue = 0 \). Thus,
\[
\Sigma Y = mV(\dot{\beta} + r) - m_S h_s \dot{\beta}
\]  
(6.19)

**Example 6.3**

Differentiate Eqns (6.15) and (6.16) and derive Eqns (6.17) and (6.18).

**Solution**

From Eqns (6.15) and (6.16):
\[
\ddot{r}_S = \dot{u}i + u\dot{i} + (\dot{v} - h_s \dot{p} + cr)j + (v - h_s p + cr)\dot{j} \quad \text{(E6.4)}
\]
\[
\ddot{r}_U = \dot{u}i' + u\dot{i}' + (\dot{v} - e\dot{r})j' + (v - er)\dot{j}' \quad \text{(E6.5)}
\]

It is known that:
\[
i = \omega_S \times i, \quad j = \omega_S \times j
\]
\[
i' = \omega_U \times i', \quad j' = \omega_U \times j'
\]

Substituting Eqns (6.11) and (6.12) into the above gives
\[
\dot{i} = (pi + rk) \times i = rij
\]
\[
\dot{j} = (pi + rk) \times j = pk - ri
\]
\[
\dot{i}' = rk' \times i' = rj'
\]
\[
\dot{j}' = rk' \times j' = -ri'
\]

Using the above, Eqns (E6.4) and (E6.5) can be rewritten as follows:
\[
\ddot{r}_S = (\dot{u} - vr + h_s pr - cr^2)i + (\dot{v} + ur - h_s \dot{p} + cr)j + (vp + h_s p^2 + cpr)k
\]  
(6.17)
\[
\ddot{r}_U = (\dot{u} - vr + er^2)i' + (\dot{v} + ur - e\dot{r})j' \quad \text{(6.18)}
\]
(2) ROTATIONAL MOTION

Generally, the moment of momentum, \( H_C \), around center of gravity, \( C \), of a rigid body could be written as follows:

\[
H_C = I \times \omega \\
= (I_{xx} \omega_x - I_{xy} \omega_y - I_{xz} \omega_z)i + (-I_{yx} \omega_x + I_{yy} \omega_y - I_{yz} \omega_z)j \\
+ (-I_{zx} \omega_x - I_{zy} \omega_y + I_{zz} \omega_z)k
\]

whereby \( I \) is the inertia tensor of the rigid body around point \( C \). The elements \( I_{xx}, I_{xy}, \ldots \) are the inertia moments, or inertia products, around the axis passing through point \( C \) parallel to the \( x-, y-, z- \) axis. \( \omega_x, \omega_y, \omega_z \) are the components of the angular velocity \( \omega \) in the \( x-, y-, z- \) direction.

The moment of momentum of the sprung mass centre of gravity, \( H_S \), with angular velocity, \( \omega_S \), given by Eqn (6.11) is

\[
H_S = I_S \times \omega_S = (I_{xxS}p - I_{xzS}r)i + (-I_{zzS}p + I_{zzS}r)k
\]

(6.20)

Here, \( I_S \) is the inertia tensor of the sprung mass around the point \( S \). Considering that the sprung mass is symmetrical in the \( xz \) plane, \( I_{yxS} = I_{xzS} = 0 \).

Similarly, with the angular velocity, \( \omega_U \), given by Eqn (6.12), the moment of momentum, \( H_U \), around the unsprung mass center of gravity, \( U \), is

\[
H_U = I_{zzU}r'k'
\]

(6.21)

Here, \( I_{zzU} \) is the moment of inertia around the axis passing through the point \( U \) and parallel to the \( z' \)-axis. If the unsprung mass is symmetrical to the \( x' \)-axis, then the height is neglected and the mass is assumed to be distributed in the \( x'-y' \) plane; all the inertia products are zero.

The time variation of moment of momentum of the moving body is caused by the external moment. Differentiating Eqns (6.20) and (6.21) with respect to time gives

\[
\dot{H}_S = (I_{xxS}\dot{p} - I_{xzS}\dot{r})i + [I_{zzS} \dot{p}^2 + (I_{xxS} - I_{zzS})p \dot{r} - I_{zzS} \dot{r}^2]j \\
+ (-I_{zzS}\dot{p} + I_{zzS}\dot{r})k
\]

(6.22)

\[
\dot{H}_U = I_{zzU}\dot{r}'k'
\]

(6.23)

From these equations, the yaw and rolling moments, \( N_S \) and \( L_S \), of the sprung mass around the axis parallel to the \( x- \) and \( z- \) axis and passing through the point \( S \) are

\[
N_S = -I_{zzS}\dot{p} + I_{zzS}\dot{r}
\]

\[
L_S = I_{xxS}\dot{p} - I_{xzS}\dot{r}
\]
The yawing moment, $N_U$, around the axis passing through the point U parallel to the $z'$-axis for the unsprung mass is:

$$N_U = I_{zzU} \dot{r}$$

From the above, the total vehicle yawing moment, $\Sigma N$, around the $z$- or $z'$-axis and the total rolling moment, $\Sigma L$, around the $x$- or $x'$-axis are given in Eqns (6.24) and (6.25), where $Y_S$ and $Y_U$ act at the sprung and unsprung mass centers of gravity, S and U, respectively:

$$\Sigma N = N_S + N_U + cY_S - eY_U$$

$$= (I_{zzS} + I_{zzU} + mSc^2 + mUe^2)\dot{r} - (I_{xzS} + mSh_c)c \dot{p}$$

$$+ (msc - mUe)V(\dot{\phi} + r)$$

$$= I_z \dot{r} - I_{zx} \dot{p}$$

(6.24)

$$\Sigma L = L_S - h_sY_S$$

$$= (I_{xxS} + mSh_s^2)\dot{p} - (I_{xzS} + mSh_s)c \dot{r}$$

$$+ mSh_s V(\dot{\phi} + r)$$

$$= I_x \dot{p} - I_{zx} \dot{r} - mSh_s V(\dot{\phi} + r)$$

(6.25)

whereby $|\phi| \ll 1$

$$I_z = I_{zzS} + I_{zzU} + mSc^2 + mUe^2$$

$$I_{zx} = I_{xz} = I_{xzS} + mSh_s c$$

$$I_x = I_{xxS} + mSh_s^2$$

$I_z$ is the total yaw moment of inertia around the vertical axis passing through the vehicle center of gravity, and $I_x$ is the rolling moment of inertia of the sprung mass around the $x$-axis.

### 6.4.3 External force

The external forces that act on the vehicle are the lateral tire forces. As described in Section 3.2, lateral forces, which are proportional to the tire side-slip angles, act at the front and rear tires of a moving vehicle. Where roll is considered, camber thrusts also act at the tires. When the vehicle equations of motion include roll motion, these forces must be included in the external forces acting on the entire vehicle.

The front and rear roll steers are $\alpha_f$ and $\alpha_r$. If they are assumed to be proportional to the roll angle, then

$$\alpha_f = \frac{\partial \alpha_f}{\partial \phi} \phi,$$
\[ \alpha_r = \frac{\partial \alpha_r}{\partial \phi} \]

where \( \partial \alpha_f / \partial \phi \) and \( \partial \alpha_r / \partial \phi \) are the front and rear wheel roll steer angles per unit roll angle. These are positive when the roll angle is positive, in other words, positive when the steer angle is in the anti-clockwise direction.

The front wheel steered angle is now changed by \( \alpha_f \) from the steer angle \( \delta \). Similarly, a change of \( \alpha_r \) is generated at the rear wheel. Using the same assumptions as in Section 3.2 and Eqns (3.6) and (3.7), the front and rear wheel tire side-slip angles are

\[
\beta_f = \beta + \frac{l_f}{V} r - \delta - \alpha_f = \beta + \frac{l_f}{V} r - \delta - \frac{\partial \alpha_f}{\partial \phi} \phi \\
\beta_r = \beta - \frac{l_r}{V} r - \alpha_r = \beta - \frac{l_r}{V} r - \frac{\partial \alpha_r}{\partial \phi} \phi
\]

Therefore, the lateral forces, \( 2Y_f \) and \( 2Y_r \), acting at the front and rear wheels are

\[
2Y_f = -2K_f \beta_f = 2K_f \left( \delta + \frac{\partial \alpha_f}{\partial \phi} \phi - \beta - \frac{l_f}{V} r \right) \\
2Y_r = -2K_r \beta_r = 2K_r \left( \frac{\partial \alpha_r}{\partial \phi} \phi - \beta + \frac{l_r}{V} r \right)
\]

where the lateral force changes caused by load transfer are neglected.

Assuming that the camber angle produced by the vehicle body roll is proportional to the roll angle, the camber thrust, \( 2Y_{cf} \) and \( 2Y_{cr} \), acting at the front and rear wheels, respectively, is:

\[
2Y_{cf} = -2K_{cf} \frac{\partial \phi_f}{\partial \phi} \phi \\
2Y_{cr} = -2K_{cr} \frac{\partial \phi_r}{\partial \phi} \phi
\]

where \( K_{cf} \) and \( K_{cr} \) are the front and rear wheel tire camber thrust coefficients. \( \partial \phi_f / \partial \phi \) and \( \partial \phi_r / \partial \phi \) are the front and rear wheel camber angles per unit roll angle. These are positive if the camber angle is in the same direction as the roll. It is also assumed that a camber thrust of the same magnitude and in the same direction is produced at the left and right wheels.

The total external forces acting on the entire vehicle in the lateral direction, in other words, the \( y \)-direction, are shown in Fig. 6.22.
\[ \Sigma F_y = 2Y_f + 2Y_r + 2Y_{cf} + 2Y_{cr} \\
= 2K_f \left( \delta + \frac{\partial \alpha_f}{\partial \phi} \phi - \beta - \frac{l_f}{Vr} \right) + 2K_r \left( \frac{\partial \alpha_r}{\partial \phi} \phi - \beta + \frac{l_r}{Vr} \right) - 2K_{cf} \frac{\partial \phi_f}{\partial \phi} \phi - 2K_{cr} \frac{\partial \phi_r}{\partial \phi} \phi \]  
(6.26)

The total yaw moment around the \( z \)-axis produced by the external forces acting on the entire vehicle, \( \Sigma M_z \), is:

\[ \Sigma M_z = 2l_fY_f - 2l_tY_t + 2l_fY_{cf} - 2l_rY_{cr} \\
= 2l_fK_f \left( \delta + \frac{\partial \alpha_f}{\partial \phi} \phi - \beta - \frac{l_f}{Vr} \right) - 2l_rK_r \left( \frac{\partial \alpha_r}{\partial \phi} \phi - \beta + \frac{l_r}{Vr} \right) - 2l_fK_{cf} \frac{\partial \phi_f}{\partial \phi} \phi + 2l_rK_{cr} \frac{\partial \phi_r}{\partial \phi} \phi \]  
(6.27)

When the vehicle body rolls, it will be subjected to the reaction force by the suspension system spring and shock absorber. These reaction forces produce the rolling moment to the vehicle body around the roll axis, in other words, the \( x \)-axis. From subsection 6.2.2, the rolling moment, which comes from the spring is \(-K_f\phi\). Assuming that the reaction force produced by the shock absorber is proportional to the roll angular velocity, the rolling moment produced by this reaction force is written as \(-C_{\phi p}\). Here, \( C_{\phi} \) is the equivalent damping coefficient of the rolling motion, which is the sum of the rolling moment per unit roll angular velocity at the front and rear shock absorbers.

A rolling moment around the \( x \)-axis by the weight \( m_S g \) also acts on the vehicle as shown in Fig. 6.23. If roll angle \( \phi \) is small, this can be approximated by \( m_S g h_s \phi \).

The total rolling moment around the \( x \)-axis by the external forces acting on the entire vehicle, \( \Sigma M_x \), is expressed as:

\[ \Sigma M_x = (-K_\phi + m_S g h_s) \phi - C_\phi p \]  
(6.28)
6.4.4 Equations of motion

The external forces and inertia forces of the vehicle motion inclusive of vehicle body roll can now be used in the equilibrium equations to derive the equations of motion:

\[ \Sigma Y - \Sigma F_y = 0 \] (equilibrium of lateral forces),
\[ \Sigma N - \Sigma M_z = 0 \] (equilibrium of yawing moment), and
\[ \Sigma L - \Sigma M_x = 0 \] (equilibrium of rolling moment).

Substituting Eqns (6.19) and (6.24)–(6.28) into the above equations, derives the vehicle equations of motion inclusive of roll motion as below. Here, \( \frac{d\phi}{dt} = p \), and \( I \) and \( I_\phi \) are used in replacement of \( I_z \) and \( I_{xx} \). \( I \) is the entire vehicle yaw moment of inertia, while \( I_\phi \) is the roll moment of inertia around the roll axis when the roll axis is assumed to coincide with the \( x \)-axis.

\[
mV \left( \frac{d\beta}{dt} + r \right) - mSgh_s \frac{d^2\phi}{dr^2} = 2K_f \left( \delta + \frac{\partial \alpha_f}{\partial \phi} \phi - \beta - \frac{l_r}{V} \right) + 2K_r \left( \frac{\partial \alpha_r}{\partial \phi} \phi - \beta + \frac{l_r}{V} \right)
+ 2 \left( K_{cf} \frac{\partial \phi_f}{\partial \phi} + K_{cr} \frac{\partial \phi_r}{\partial \phi} \right)
\]

\[ (6.29) \]

\[
I \frac{dr}{dt} - I_{xz} \frac{d^2\phi}{dr^2} = 2K_f \left( \delta + \frac{\partial \alpha_f}{\partial \phi} \phi - \beta - \frac{l_r}{V} \right) l_r - 2K_r \left( \frac{\partial \alpha_r}{\partial \phi} \phi - \beta + \frac{l_r}{V} \right) l_r
- 2 \left( l_t K_{cf} \frac{\partial \phi_f}{\partial \phi} - l_r K_{cr} \frac{\partial \phi_r}{\partial \phi} \right)
\]

\[ (6.30) \]

\[
I_\phi \frac{d^2\phi}{dt^2} - I_{xz} \frac{dr}{dt} - mSgh_s \left( \frac{d\beta}{dr} + r \right) = (-K_\phi + mSgh_s)\phi - C_\phi \frac{d\phi}{dt}
\]

\[ (6.31) \]
rearranging the above equations gives

\[
mV \frac{d\beta}{dt} + 2 \left( K_f + K_r \right) \beta + \left[ mV \frac{2(l_f K_f - l_r K_r)}{V} \right] r - mS h_s \frac{d^2 \phi}{dt^2} = 2Y_\phi \phi \\
= 2K_f \delta
\]

(6.29)'

\[
2(l_f K_f - l_r K_r) \beta + \frac{l_r}{dr} \left[ 2(l_f K_f^2 + l_r K_r^2) - I_{xz} \right] \frac{d^2 \phi}{dt^2} - 2N_\phi \phi = 2l_f K_f \delta
\]

(6.30)'

\[-mS h_s V \frac{d\beta}{dt} - I_{xz} \frac{dr}{dt} - mS h_s Vr + I_\phi \frac{d^2 \phi}{dt^2} + C_\phi \frac{d\phi}{dt} + (K_\phi - mS g h_s) \phi = 0
\]

(6.31)'

where

\[
Y_\phi = \left( \frac{\partial \alpha_f}{\partial \phi} K_f + \frac{\partial \alpha_r}{\partial \phi} K_r \right) - \left( \frac{\partial \phi_f}{\partial \phi} K_{cf} + \frac{\partial \phi_r}{\partial \phi} K_{cr} \right)
\]

(6.32)

\[
N_\phi = \left( \frac{\partial \alpha_f}{\partial \phi} l_f K_f - \frac{\partial \alpha_r}{\partial \phi} l_r K_r \right) - \left( \frac{\partial \phi_f}{\partial \phi} l_f K_{cf} - \frac{\partial \phi_r}{\partial \phi} l_r K_{cr} \right)
\]

(6.33)

These are the equations of motion of the vehicle with body roll. Here, the change of rolling resistance due to the load transfer between the left and right wheels is neglected as its effect on the vehicle yaw motion is assumed to be small.

Applying Laplace transformation to Eqns (6.29)’–(6.31)’, the vehicle characteristic equation is obtained as follows:

\[
\begin{vmatrix}
  mV S + 2(K_f + K_r) & mV + \frac{2(l_f K_f - l_r K_r)}{V} & -mS h_s S^2 - 2Y_\phi \\
  2(l_f K_f - l_r K_r) & Is + \frac{2(l_f K_f^2 + l_r K_r^2)}{V} & -I_{xz} S^2 - 2N_\phi \\
  -mS h_s V S & -I_{xz} S - mS h_s V & I_\phi S^2 + C_\phi S + (K_\phi - mS g h_s)
\end{vmatrix} = 0
\]

(6.34)

The above characteristic equation is a little complicated in this form, thus, assuming

\[
K_f \approx K_r \approx K
\]

\[
l_f \approx l_r \approx \frac{l}{2}
\]
\[ I \approx ml_1l_z \approx m\left(\frac{l}{2}\right)^2 \]

and neglecting the inertia products:

\[ K_\phi - m_sgh_s \approx K_\phi \]

\[ I_{xz} \approx 0 \]

the characteristic equation becomes

\[
\begin{vmatrix}
  mVs + 4K & mV & -m_s h_s s^2 - 2Y_\phi \\
  0 & m\left(\frac{l}{2}\right)^2 s + \frac{K^2}{V} & -2N_\phi \\
  -m_s h_s V & -m_s h_s V & I_{\phi} s^2 + C_\phi s + K_\phi
\end{vmatrix} = 0 \quad (6.34)'
\]

Expanding this and rearranging gives

\[ A_4s^4 + A_3s^3 + A_2s^2 + A_1s + A_0 = 0 \]

where

\[
\begin{align*}
A_0 &= \frac{16K^2K_\phi}{mV} - \frac{32m_s h_s KN_\phi}{ml^2} V \\
A_1 &= 8K\left( K_\phi - \frac{m_s h_s Y_\phi}{m} \right) + \frac{16K^2C_\phi}{mV} \\
A_2 &= (mK_\phi - 2m_s h_s Y_\phi)V + 8KC_\phi + \frac{16K^2I_\phi}{mV} \\
A_3 &= mC_\phi V + 4K\left( 2I_\phi - \frac{m_s^2 h_s^2}{m} \right) \\
A_4 &= m\left( I_\phi - \frac{m_s^2 h_s^2}{m} \right)V
\end{align*}
\]

6.5 EFFECT OF BODY ROLL ON VEHICLE DYNAMICS

In the last section, the vehicle equations of motion with vehicle body roll were derived. The equation of motion is still too complicated for analysis of the vehicle motion characteristics. In order to analyze the basic effects of vehicle body roll on vehicle motion, a two degrees of freedom equation of motion (vehicle lateral side slip and yaw motion) that includes an equivalent roll effect will be derived referring to Ellis [3]. This is done by considering vehicle body roll due to constant lateral acceleration.
The steady-state vehicle body roll angle, from Eqn (6.31), is derived by assuming
\[
\frac{d\beta}{dt} = \frac{dr}{dt} = \frac{d^2\phi}{dt^2} = \frac{d\phi}{dt} = 0
\]
and the result is
\[
\phi = \frac{m_S h_s V}{K_\phi - m_S g h_s} r
\]  
(6.36)

Rewriting the right hand side of Eqns (6.29) and (6.30) with a steady-state roll angle gives
\[
mV \left( \frac{d\beta}{dt} + r \right) = 2K_f \left\{ \beta - \frac{l_f}{V} + \left( \frac{\alpha_f}{\phi} - \frac{K_{cf}}{K_f} \frac{\phi_f}{\phi} \right) \phi \right\} + 2K_r \left\{ - \beta + \frac{l_r}{V} + \left( \frac{\alpha_r}{\phi} - \frac{K_{cr}}{K_r} \frac{\phi_r}{\phi} \right) \phi \right\}
\]
\[
l \frac{dr}{dt} = 2K_f \left\{ \beta - \frac{l_f}{V} + \left( \frac{\alpha_f}{\phi} - \frac{K_{cf}}{K_f} \frac{\phi_f}{\phi} \right) \phi \right\} l_f - 2K_r \left\{ - \beta + \frac{l_r}{V} + \left( \frac{\alpha_r}{\phi} - \frac{K_{cr}}{K_r} \frac{\phi_r}{\phi} \right) \phi \right\} l_r
\]

\(\phi\) in these equations is proportional to \(r\), as given by Eqn (6.36), and the above equations are equivalent to the following equations:
\[
mV \left( \frac{d\beta}{dt} + r \right) = 2K_f \left( \beta - \frac{l'_f}{V} + \frac{l'_r}{V} r \right) + 2K_r \left( - \beta + \frac{l'_r}{V} r \right)
\]  
(6.37)
\[
l \frac{dr}{dt} = 2K_f \left( \beta - \frac{l'_f}{V} + \frac{l'_r}{V} r \right) l_f + 2K_r \left( - \beta + \frac{l'_r}{V} r \right) l_r
\]  
(6.38)

where
\[
l'_f = l_f (1 + B_f V^2)
\]  
(6.39)
\[
l'_r = l_r (1 + B_r V^2)
\]  
(6.40)

\[
B_f = \frac{-m_S h_s \left( \frac{\alpha_f}{\phi} - \frac{K_{cf}}{K_f} \frac{\phi_f}{\phi} \right)}{l_f (K_\phi - m_S g h_s)}
\]
When Eqns (6.37) and (6.38) are compared with Eqns (3.10) and (3.11), it is easily understood that the equivalent vehicle lateral side-slip motion and yawing motion with body roll are found by replacing the front and rear wheel distances from the vehicle center of gravity $l_{f}$ and $l_{r}$ with equivalent $l'_{f}$ and $l'_{r}$.

Rewriting Eqns (6.37) and (6.38), the equivalent vehicle two degrees of freedom equations of motion with roll can be expressed as follows:

$$mV \frac{d\beta}{dt} + 2(K_{f} + K_{r})\beta + \left[mV + \frac{2(l'_{f}K_{f} - l'_{r}K_{r})}{V}\right] r = 2K_{r}\delta \quad (6.37)'$$

$$2(l_{f}K_{f} - l_{r}K_{r})\beta + l\frac{dr}{dt} + \frac{2(l'_{f}l_{f}K_{f} + l'_{r}l_{r}K_{r})}{V} r = 2l_{f}K_{f}\delta \quad (6.38)'$$

In steady-state cornering, $\frac{d\beta}{dt} = \frac{dr}{dt} = 0$ can be substituted into Eqns (6.37)' and (6.38)'. The yaw velocity, $r$, in response to a constant steer angle is:

$$r = \frac{1}{1 - \frac{ml_{f}K_{f} - l_{r}K_{r}}{2l(l'_{f} + l'_{r})K_{f}K_{r}}V^{2} (l'_{f} + l'_{r})} \frac{V}{\delta} \quad (6.42)$$

from Eqns (6.39) and (6.40),

$$l'_{f} + l'_{r} = l(1 + BV^{2}) \quad (6.43)$$

where

$$B = \frac{l_{f}B_{f} + l_{r}B_{r}}{l} = \frac{m_{S}h_{s}}{l(K_{f} - m_{S}h_{s})} \left[ \frac{\partial\alpha_{r}}{\partial\phi} - \frac{\partial\alpha_{f}}{\partial\phi} + \frac{K_{cr}}{K_{f}} \frac{\partial\phi_{f}}{\partial\phi} - \frac{K_{cr}}{K_{r}} \frac{\partial\phi_{r}}{\partial\phi} \right]$$

(6.44)

Substituting Eqns (3.43), (6.43), and (6.44) into Eqn (6.42) gives

$$r = \frac{1}{1 + \frac{AV^{2}}{1 + BV^{2}}} \frac{1}{1 + BV^{2}} \frac{V}{l} \delta = \frac{1}{1 + A'V^{2}} \frac{V}{l} \delta \quad (6.45)$$

Here, $A'$ is the equivalent stability factor when roll is being considered.
\[ A' = A + B \]
\[ = -\frac{m(l_f K_f - l_r K_r)}{2l^2 K_f K_r} + \frac{m_S h_s}{l(K_f - m_S h_s)} \left[ \frac{\partial \alpha_f}{\partial \phi} - \frac{\partial \alpha_f}{\partial \phi} + \frac{K_{cr}}{K_f} \frac{\partial \phi_f}{\partial \phi} \right] \]
\[ \frac{\partial \phi_f}{K_f} \]

(6.46)

From the above, if \( B > 0 \) namely:

\[ \left[ \frac{\partial \alpha_f}{\partial \phi} - \frac{\partial \alpha_f}{\partial \phi} + \frac{K_{cr}}{K_f} \frac{\partial \phi_f}{\partial \phi} \right] > 0 \]

can be positive, having the effect of changing the vehicle steer characteristics toward US. More precisely, positive roll steer at the rear wheel and positive camber change at the front wheel, or negative roll steer at the front wheel and negative camber change at the rear wheel, have the effect of changing the vehicle steer characteristics toward US.

Figure 6.24 is an example of how the relation between yaw velocity, \( r \), and the traveling speed, \( V \), changes with the distance of the front axle from the vehicle center of gravity. The results are for steady-state cornering, for a normal passenger car, with rear wheel roll steer. The rear wheel roll steer is set so that it always changes the vehicle steer characteristic to either NS or US, regardless of the distance of the front axle from the vehicle center of gravity.

In the same figure, the relationship between the yaw velocity and the traveling speed for a two degrees of freedom model without vehicle roll is also shown. The figure shows that with the introduction of rear wheel roll steer, not only the vehicle steer characteristic is changed to US, but also the change in the vehicle steer characteristic due to the distance between front axle and the center of gravity also becomes less prominent.

Applying Laplace transforms to Eqns (6.37)’ and (6.38)’, the vehicle characteristic equation is obtained, which has a general form of:

\[ s^2 + 2D's + P^2 = 0 \]  

(6.47)
where
\[ 2D' = \frac{2m(l_l'K_f + l_r'K_r) + 2I(K_f + K_r)}{mIV} = 2D + \frac{2(l_l^2K_f + l_r^2K_r)V}{I} \] (6.48)

\[ p^2 = \frac{4K_fK_r(l_l' + l_r') - 2(l_lK_f - l_rK_r)}{mIV^2} = p^2 + \frac{4K_fK_rl^2}{ml}B \] (6.49)

2D and \( P^2 \) are the coefficients of the characteristic equation given by Eqns (3.56) and (3.57) when the roll is not being considered. Consequently, taking \( I = mk^2 \) and by Eqn (3.67), the vehicle natural frequency \( \omega'_n \) when the roll is being considered is

\[ \omega'_n = P' = \sqrt{P + \frac{4K_fK_rl^2}{ml}B} = \sqrt{\frac{4K_fK_rl^2}{m^2k^2}(\frac{1 + AV^2}{V^2}) + \frac{4K_fK_rl^2}{m^2k^2}B} \] (6.50)

\[ = \frac{2\sqrt{K_fK_rl}}{mk} \sqrt{1 + A'V^2} \]

Assuming \( l_l \approx l_r, \ K_l \approx K_r \), the damping ratio \( \zeta' \) when the roll is being considered is

\[ \zeta' = \frac{D'}{P'} = \left[ \frac{1 + k^2/l_ll_r}{2\sqrt{k^2/l_l'ls}} + \frac{1}{2\sqrt{k^2/l_r'l_r}}BV^2 \right] \cdot \frac{1}{\sqrt{1 + A'V^2}} \] (6.51)

**PROBLEMS**

6.1 Sometimes the roll angle of the vehicle subjected to the 0.5-g steady lateral acceleration is defined as the roll rate. Calculate the roll rate of the vehicle with the mass of the vehicle body \( m_S = 1400 \text{ kg} \), the body CG height from the ground \( h_S = 0.52 \text{ m} \), the front roll stiffness \( K_{\phi l} = 65.0 \text{ kN/m/rad} \), and the rear roll stiffness \( K_{\phi r} = 35.0 \text{ kN/m/rad} \).

6.2 Calculate the lateral load transfer at the front and rear suspensions, respectively, for the vehicle in Problem 6.1. Use the following vehicle parameters in addition to those in 6.1: the position of the front wheels from the CG \( l_f = 1.1 \text{ m} \), the position of the rear wheels \( l_r = 1.6 \text{ m} \), the front tread \( d_f = 1.5 \text{ m} \), and the rear tread \( d_r = 1.5 \text{ m} \).

6.3 Calculate how much percent of the cornering stiffness is equivalently reduced by the suspension compliance steer if the cornering stiffness of the original tire is 60 kN/rad and the compliance steer to a unit lateral force is 0.00185 rad/kN.

6.4 Investigate the relative value of the camber change rate and the roll steer rate, which gives us almost the same effects on the steer characteristics of the vehicle.

6.5 Find the roll steer rate at the rear suspension, which is needed to make the OS vehicle in Fig. 6.24 NS.
REFERENCES


