Hierarchical design of structures and multiphase material cells

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ABSTRACT

This paper presents a hierarchical topology optimization method to simultaneously achieve the optimum structures and multiphase material cells for minimum system thermal compliance. Macro design variables and micro phase design variables are introduced independently, and coupled through elemental phase relative density. Based on uniform interpolation scheme with multiple materials, the sensitivities of thermal compliance with respect to the design variables on the two scales are derived. Correspondingly, the hierarchical optimization model of structures and multiphase material cells is built under prescribed volume fraction and mass constraints. The proposed method and computational model are validated by several 2D numerical examples. The superiority of multiphase materials in hierarchical optimization is presented through the comparison of single phase materials. The optimized results of periodic structure, hierarchical structure and traditional continuous structure are compared and analyzed. At last, the effects of volume fraction and mass constraints are discussed.

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1. Introduction

Structural optimization is drawing more attention than ever before with the shortage of global resources and higher levels of world-wide competition in industries. Comparing shape optimization and size optimization, topology optimization is considered more efficient in decreasing weight at the conceptual design stage. Currently, topology optimization technology has been widely applied in various fields since it was proposed by Bendsoe and Kikuchi [1] in 1988. Periodic porous material like cellular materials and truss-like materials are always of high interest in automobile, aerospace and other industrial applications for exhibiting properties such as a high ratio of stiffness to weight, excellent energy absorption and thermal isolation characteristics. In recent years, the topology optimization method has been a powerful tool to design the multifunctional material microstructures [2–7] since inverse homogenization method was proposed by Sigmund [8]. In the past, the topology optimization method was mainly used to solve single scale optimization problems either for the optimal design of macrostructures to improve structural performance or for the material design to develop new microstructures with prescribed or extreme properties. However, the material microstructures with certain equivalent properties are not always guaranteed to be efficient when constructing structures, since both structural shapes and boundary conditions always vary in practical use. That is, we need a kind of system-level optimization technology which can embody the structural performance and material properties together. Although macro structural optimization and material microstructure design are at two different scales, they have a common feature, which is that they both focus on material distribution. The common feature supplies a possibility to integrate macrostructure and material microstructure into one system, which can offer its own set of strengths. To some extent, the integrated optimization can be understood as a material microstructure design method which can satisfy macro structure performance. Rodrigues et al. [9] proposed a hierarchical optimization method of structure and porous material, and Coelho et al. [10] extended this hierarchical approach in 3D elastic structures. However, this work strongly aims to achieve optimal material microstructures and allows variation from point to point in macro-scale, which leads to low computational efficiency and some difficulties in manufacture. A universal approach for concurrent design at two scales was proposed by Liu et al. [11]. Material is assumed to be uniformly distributed on the macro level. Penalization approaches are adopted at two scales to achieve clear topologies, i.e. porous anisotropic material penalization at the macro-scale, and conventional SIMP at the micro-scale. Optimizations at two scales are integrated into one system through homogenization theory. This method is recommended for easier manufacturing, although optimal design may not be achieved. Similar optimization models were applied by Yan et al. [12] and
Niu et al. [13] to account for thermo-mechanical loads and frequency optimization. Recently, Huang et al. [14] developed the BESO to the concurrent design of macrostructures and material microstructures, which demonstrated certain advantages over the continuum density-based method [15–18]. Zhang and Sun [19] and Yan et al. [20] investigated the size-effects of material microstructure on the concurrent optimization based on superelement technique and extended multiscale finite element method, respectively. Through numerical examples, Liu et al. [21] pointed out that structure design is oriented to structure efficiency, material design is oriented to multifunctional properties and hierarchical design is oriented to both structure efficiency and multifunctional properties. Deng et al. [22] and Yan et al. [23] indicated that, sometimes the two-scale optimization of structure and porous material is more advantageous than the sole macro structural optimization in terms of improving multi-objective performances.

Multiphase materials have been researched widely because of their advantages in complicated external conditions. A literature study points out that structural topology optimization with multiphase materials stemmed from Thomsen [24]. Sigmund and his co-worker [3,25] expanded the SIMP to interpolate material properties of two solid material phases and void. Up to now, multiphase materials have been studied by various methods [26–28]. In particular, Gao and Zhang [29] testified that the mass constraint is more effective than the volume constraint in the topology optimization of structures consisting of multiphase materials. Similar to Jacobi and Gauss–Seidel iteration processes, Tavakoli and Mohseni [30] split multiphase material problems into a series of binary material iterative problems. Derived from multiphase material topology optimization, a method labeled as Discrete Material Optimization (DMO) was proposed by Stegmann and Lund [31] to treat the laminate design of composite materials.

Different with literature [15,18], the proposed hierarchical optimization approach is to find optimum thermal conductive configurations for structures and porous multiphase material cells under prescribed volume fraction and mass constraints. There is only one finite element model. Structure and material cell are coupled through elemental phase relative density instead of homogenization theory. A uniform interpolation scheme is used to deal with multiple material problems. The layout of the paper is as follows. A hierarchical optimization model for thermal conduction is established and described in Section 2. The two-scale sensitivity analysis based on finite element method is presented in Section 3. In Section 4, the numerical treatments are given. In Section 5, several 2D numerical examples are presented to demonstrate the effectiveness of the proposed optimization algorithm. In Section 6, a summary and all drawn conclusions are provided.

2. Hierarchical topology optimization model with multiphase materials

In this research, it is assumed that structure is assembled by uniform multiphase material cells. The optimization at two scales is integrated into one system and resolved simultaneously. As shown in Fig. 1, the 2D designable domain is divided into \( M \times N \) finite elements, where \( M \) stands for the total number of unit cells and \( N \) stands for the total number of finite elements within each cell. To ensure the structural periodicity, all unit cells should be meshed consistently. Two classes of design variables are independently defined, i.e. macro design variable \( P_i \ (i = 1, 2, \ldots, M) \) in structural design domain and micro phase design variable \( r_j^{(q)} \ (j = 1, 2, \ldots, N, q = 1, 2, \ldots, S, \) where \( S \) stands for the varieties of solid materials) in a unit cell, both ranging from 0 to 1. \( P_i = 1 \) if unit cell \( i \) is occupied by multiphase material cell. \( P_i = 0 \) if no multiphase material cell exists in unit cell \( i. \) \( r_j^{(q)} = 1 \) if element \( j \) is full of phase material \( q. \) \( r_j^{(q)} = 0 \) if phase material \( q \) doesn’t exist in element \( j \) within the material cell.

In every finite element, micro phase design variables should satisfy

\[
\sum_{q=1}^{S} r_j^{(q)} = 1 \tag{1}
\]

Eq. (1) reflects the mutex relationship between \( r_j^{(q)} \ (q = 1, 2, \ldots, S), \) i.e. if \( r_j^{(1)} = 1, \) \( r_j^{(q)} = 0 \ (q \neq 1). \) Eq. (1) cannot be used directly in the optimization process because huge combinatorial problem is involved. Here a uniform interpolation model is used to solve the problem of multiphase materials which will be introduced in Section 3.

For a discretized structure, each element is assigned \( S \) relative densities and can be expressed as the combination of two-scale design variables [21]

\[
x_j^{(q)} = P_j r_j^{(q)} \tag{2}
\]

In Eq. (2), \( i \) and \( j \) indicate the unit cell number and the elemental number within the unit cell, respectively. Elemental phase relative density \( x_j^{(q)} \) imposes the design variables \( P_i \) and \( r_j^{(q)} \) associated with each element in designable domain. From Eq. (2) we know that the status of any element is identical to its corresponding ones in other existent unit cells, which guarantees the uniformity of porous material cell at a macro-scale.

To seek the minimum thermal compliance for structure and porous cell with multiphase materials, the hierarchical topology optimization can be formulated as

---

Fig. 1. Schematic figure of hierarchical optimization with multiphase materials: (a) design domain; (b) unit cell.
\[
\begin{align*}
\text{Find : } & P_i, \quad r_j^{(q)} (i = 1, 2, \ldots; M; j = 1, 2, \ldots; N; q = 1, 2, \ldots, S) \\
\text{Minimize } & C = F^T T \\
\text{Constraint I : } & V(x) = \sum_{i=1}^{M} \sum_{j=1}^{N} s \frac{x_i^{(q)} V_j}{V_0} \leq f \\
\text{Constraint II : } & m(r) = M \sum_{j=1}^{N} \rho_j(r)V_j \leq m \\
\text{Constraint III : } & 0 < \delta \leq P_i, \quad 0 < \delta \leq r_j^{(q)} \leq 1
\end{align*}
\]

where \( C \) denotes the structural thermal compliance. \( F \) and \( T \) represent the external heat load vector and the nodal temperature vector of the structure, respectively.

Constraint I sets an upper bound on the total available solid materials by defining relative volume \( V(x) \) smaller than a prescribed value \( f \). \( V_j \) is the volume of element \( j \) in a material cell. \( V_0 \) is the initial volume of design domain.

Constraint II sets an upper bound of mass on material cells. \( m(r) \) denotes the mass density related to element \( j \) within a material cell, which is interpolated from the set of available solid material densities. \( m \) is the upper bound controlling the material cell mass.

Constraint III sets bounds for design variables at two scales to avoid numerical singularity of optimization, where \( \delta \) is a small pre-determined value such as 0.001.

When considering Eq. (2), the Constraint I can be extended as

\[
\begin{align*}
\sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{q=1}^{S} p_i^{(q)} r_j^{(q)} V_j/V_0 & \leq f \\
\end{align*}
\]

Eq. (4) involves both macro and micro design variables, which can be understood from two aspects. One is to impose a volume constraint on the structure. The other is to coordinate relationship between two scales.

The material density \( \rho_j(r) \) in Constraint II can be obtained according to the linear uniform interpolation model [29]

\[
\rho_j(r) = \sum_{q=1}^{S} \rho_0^{(q)} r_j^{(q)}
\]

where \( \rho_0^{(q)} \) is the density of phase material \( q \).

Taking Eq. (5) into Eq. (3), the Constraint II can be formulated as

\[
\sum_{j=1}^{N} \sum_{q=1}^{S} \rho_0^{(q)} r_j^{(q)} V_j \leq m
\]

Clearly, the hierarchical optimization can be degenerated into the optimization of traditional continuous structure with multi-phase materials. When \( M = 1 \), and macro design variable \( P \equiv 1 \), binary constraints are degenerated into one single constraint. In this paper, only the mass constraint is reserved. Correspondingly, the traditional thermal conduction problem can be stated as

\[
\begin{align*}
\text{find : } & r_j^{(q)} (j = 1, 2, \ldots; N; q = 1, 2, \ldots, S) \\
\text{min } & C = F^T T \\
\text{Constraint I : } & m(r) = M \sum_{j=1}^{N} \rho_j(r)V_j \leq m \\
\text{Constraint II : } & 0 < \delta \leq r_j^{(q)} \leq 1
\end{align*}
\]

3. Sensitivity analysis based on finite element method

The elemental thermal conductive coefficient of phase material \( q \) can be interpolated as [31]

\[
k_{ij}^{(q)} = x_i^{(q)} \prod_{\zeta=1}^{S} \left( 1 - x_{ij}^{(q)} \right) k_{0j}^{(q)}
\]

where \( p \) is the penalization factor whose classical value is 3. \( k_{ij}^{(q)} \) is the thermal conductive coefficient of solid material phase \( q \). \( k_{ij}^{(q)} = k_{0j}^{(q)} \) only when \( x_i^{(q)} = 1 \) and \( x_j^{(q)} = 0 \ (q) \); \( k_{ij}^{(q)} = 0 \ (q) \), if \( x_i^{(q)} = 0 \) or \( x_j^{(q)} = 1 \). Thus, if \( k_{ij}^{(q)} = k_{ij}^{(q)} \), \( k_{ij}^{(q)} = 0 \ (q) \), this means that when element \( j \) of unit cell \( i \) consists of only one single phase material \( q \), other phase materials will automatically become inactive, i.e. the mutex relationship expressed in Eq. (1) is achieved through Eq. (8). Cooperating with the mass constraint, phase materials can be distributed reasonably in material cell [29].

The elemental thermal conductive matrix can be expressed as

\[
K_{ij} = \left( \sum_{q=1}^{S} k_{ij}^{(q)} \right) K_0
\]

where \( K_0 \) is the basic elemental thermal conductive matrix when thermal conductive coefficient equals 1.

The expression of \( K_0 \) is

\[
K_0 = \int_{V_e} B^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} B dV
\]

where \( B = LN \), \( L \) is differential operator, \( N \) is shape function matrix; \( V_e \) denotes the volume of an element.

The global thermal conductive matrix can be assembled by the elemental thermal conductive matrix

\[
K = \sum_{i=1}^{N} \sum_{j=1}^{N} G K_{ij} G
\]

where \( G \) is transformation matrix.

For the steady thermal conduction problem, the finite element equilibrium equation corresponds to

\[
KT = F
\]

From Eqs. (2), (8)–(12), we know that there is only one finite element model. Instead of homogenization theory, elemental phase relative density \( x_i^{(q)} \) is used to establish a link between macro scale and micro scale.

Considering Eq. (12), the objective function in Eq. (3) can be written as

\[
C = T^T K T = \sum_{i=1}^{M} \sum_{j=1}^{N} T_i^p K_{ij} T_j^p
\]

where \( T_i^p \) represent the elemental temperature vector.

With the help of Eqs. (2), (8)–(13), the derivation of the objective function with respect to the macro design variable \( P_i \) can be expressed using the adjoint variable method [32]

\[
\frac{\partial C}{\partial P_i} = -\sum_{j=1}^{N} \left( pP_i^{(p-1)} r_j^{(q)} \right) + \sum_{m=12}^{S-1} \left( (-1)^n(p + 1)pP_i^{(n+1)(p-1)} r_j^{(q)} \right) \times \sum_{\zeta=1}^{S} r_{j(\eta-m)}^{(n)} \prod_{\xi=1}^{n-1} r_{j(\zeta)}^{(n)} R_{\eta-m}^{(1)}
\]

\[
\prod_{\zeta=1}^{S} \left( 1 - x_{ij}^{(q)} \right) k_{0j}^{(q)}
\]

where \( c_q \) is the elemental thermal compliance and can be expressed as \( c_q = T_i^p K_{ij} T_j^p \). When the subscript \( (n - m) \leq 0 \), \( r_{j(\eta-m)}^{(n)} \) and the items after it will be 1. \( \xi, \eta, \ldots, \zeta \) cannot be alternated.
Similarly, the sensitivity of the thermal compliance against the micro phase design variable \( r_{ij}^{(q)} \) can be written as

\[
\frac{\partial C}{\partial r_{ij}^{(q)}} = -\sum_{l=1}^{M} \left[ p^{p_{l}^{(p-1)(q)}} \sum_{\zeta \neq q} \left( 1 - r_{ij}^{(p_{l}^{(p-1)(q)})} \right) k_{ij}^{(p_{l}^{(p-1)(q)})} c_{ij} / \sum_{q=1}^{S} k_{ij}^{(p_{l}^{(p-1)(q)})} \right] + \sum_{l=1}^{M} \left[ p^{p_{l}^{(p-1)(q)}} \sum_{\zeta \neq q} \left( 1 - r_{ij}^{(p_{l}^{(p-1)(q)})} \right) k_{ij}^{(p_{l}^{(p-1)(q)})} c_{ij} / \sum_{q=1}^{S} k_{ij}^{(p_{l}^{(p-1)(q)})} \right]
\]

The derivatives of \( V(x) \) with respect to design variables can be expressed as

\[
\frac{\partial V(x)}{\partial \mathbf{r}} = \sum_{j=1}^{N} \left( \sum_{q=1}^{S} r_{ij}^{(q)} \right) V_{j}/V_{0}; \quad \frac{\partial V(x)}{\partial \mathbf{r}} = \sum_{l=1}^{M} P_{l} V_{j}/V_{0}
\]

Similarly, the derivatives of \( m(r) \) with respect to design variables can be expressed as

\[
\frac{\partial m(r)}{\partial \mathbf{r}_{ij}^{(q)}} = 0; \quad \frac{\partial m(r)}{\partial \mathbf{r}_{ij}^{(q)}} = M p_{l}^{(q)} V_{j}
\]

The sensitivities of the constraints with respect to the design variables can be obtained easily by using Eqs. (16) and (17).

When macro design variable \( P_{l} \equiv 1 \), the hierarchical optimization can be understood as a periodic topology optimization. Similarly, volume and mass constraints are degenerated into a single mass constraint. The derivation of thermal compliance with respect to design variable \( r_{ij}^{(q)} \) can be simplified as

\[
\frac{\partial C}{\partial r_{ij}^{(q)}} = -\sum_{l=1}^{M} \left[ p^{p_{l}^{(p-1)(q)}} \sum_{\zeta \neq q} \left( 1 - r_{ij}^{(p_{l}^{(p-1)(q)})} \right) k_{ij}^{(p_{l}^{(p-1)(q)})} c_{ij} / \sum_{q=1}^{S} k_{ij}^{(p_{l}^{(p-1)(q)})} \right] + \sum_{l=1}^{M} \left[ p^{p_{l}^{(p-1)(q)}} \sum_{\zeta \neq q} \left( 1 - r_{ij}^{(p_{l}^{(p-1)(q)})} \right) k_{ij}^{(p_{l}^{(p-1)(q)})} c_{ij} / \sum_{q=1}^{S} k_{ij}^{(p_{l}^{(p-1)(q)})} \right]
\]

4. Numerical treatments

4.1. Filtering

Numerical instabilities such as checkerboard pattern and mesh-dependency problem are common phenomena in the topology optimization. The mesh independent filter is adopted in the current procedure. Considering the relationship between sensitivity and elemental thermal compliance \( c_{ij} \) in Eqs. (14) and (15), the elemental thermal compliance \( c_{ij} \) is filtered. Instead of using explicit filter form, the filter can be implicitly represented by the solution of a Helmholtz partial differential equation (PDE) with homogeneous Neumann boundary conditions [33]

\[
-\nabla^2 c + \alpha c = \epsilon
\]

where \( c \) and \( \alpha \) are the unfiltered and filtered thermal compliance field, respectively.

The parameter \( R \) in Eq. (19) plays a similar role as the given filter radius \( r_{max} \) in convolution filter. A relationship between two parameters can be expressed as

\[
R = r_{max}/2\sqrt{3}
\]

The filter solved by PDE has the advantages of field preserving and high filter efficiency [33]. The filtering process can be viewed as the redistribution of elemental thermal compliance. After solving Eq. (19), \( c_{ij} \) is replaced by the new elemental thermal compliance \( c_{ij} \). Then by virtue of Eqs. (14) and (15), the sensitivities can be obtained.

The optimization process stops if the following convergence condition is achieved

\[
||D^{(l+1)} - D^{(l)}||/||D^{(l)}|| \leq \epsilon
\]

where \( D = P \cup r \) (\( P = \{P_{l}, i = 1, \ldots, M\}; \quad r = \{r_{ij}^{(q)} \}, \quad j = 1, 2, \ldots, N; \quad q = 1, 2, \ldots, S\)), \( D^{(l)} \) and \( D^{(l)} \) are design variables vectors of the \( l + 1 \) and \( l \) iteration, respectively. \( \epsilon \) is the precision of convergence. Grey transition regions between solid and void parts will be produced by the filtering technique. To avoid this undesirable effect, the following iterative strategy is employed. The filtering method is adopted to eliminate checkerboard pattern and mesh dependence until \( \epsilon \leq 0.5\% \). After that, the optimization model is performed without filtering until convergence to achieve a clear material distribution. The topological configuration is supposed to approach the optimal solution after \( \epsilon \leq 0.1\% \).

4.2. Numerical procedure

The method of moving asymptotes (MMA) is utilized as the optimizer [34]. The major numerical procedures are briefly summarized as follows.

**Step1**: Define basic parameters such as phase material properties, the target volume fraction \( f \), upper bound of mass constraint \( m \). Calculate basic elemental thermal conductive matrix \( K_{ij} \) according to Eq. (10). Prepare matrix \( L_{p}, T_{p} \) which are used in PDE filter according to Matlab code in Ref. [35]. Initialize macro and micro design variables \( P_{l} \) and \( r_{ij}^{(q)} \).

**Step2**: Obtain elemental phase relative density \( x_{ij}^{(q)} \) according to Eq. (2).

**Step3**: Calculate global thermal conductive matrix \( K \) according to Eqs. (8)–(11). Solve the nodal temperature vector \( T \) using Eq. (12). Obtain elemental thermal compliance \( c_{ij} = T_{ij} \).

**Step4**: Filter elemental thermal compliance \( c_{ij} \) according to Eq. (19).

**Step5**: Compute the sensitivities of the objective function and constraints with respect to the macro and micro design variables using Eqs. (14)–(17).

**Step6**: Adopt MMA for optimum search and update the design variables \( P_{l} \) and \( r_{ij}^{(q)} \). If convergence, go to Step 7. Otherwise, go to Step 2.

**Step7**: Output the final topology result. Stop.

5. Numerical examples and discussions

In this section, several 2D numerical examples are given to validate the proposed optimization method. As shown in Fig. 2, the initial structure has a dimension of 3.6 m × 3.6 m × 0.01 m. The applied concentrated heat flow has a value of \( F = 1 \) W/m². The origin of the coordinate is located at the upper left corner, and the zero reference temperature is set at the four corners. The domain is divided into 12 × 12 unit cells and each unit cell has 30 × 30 elements (4-node Planar Element). Three positions are heated separately. Position I applies heat at position (171, 161), Position II applies heat at position (71, 211), and Position III applies heat at position (193, 191).

The solid materials involved in numerical examples are shown in Table 1. Each various color represents a distinct phase material: black stands for phase material M1; blue stands for phase material...
M2; purple stands for phase material M3; red stands for phase material M4; and green stands for voids.

5.1. Numerical example 1

The effects of material properties on the optimized solution are investigated in this example. The volume fraction \( f \) is set to be 0.5 and the upper bound of mass constraint is \( m = 70 \text{ kg} \). The heat load \( F \) is applied at Position I.

Table 2 shows the comparison of four various material combinations. Generally, the phase materials in the material cells are oriented to efficiently carry the applied heat load in the macro-scale. It can be found that some material cells are blurred. This blurring is caused by the intermediate value of macro design variables. This phenomenon occurs by strictly meeting the volume fraction and mass constraints. From the material cell of Case C, it can be found that phase material M1 with higher thermal conductive coefficient is distributed near the center and at the four corners to reinforce material cell itself and the connections between material cells. Compared with Case A and B, Case C has the lowest thermal compliance, which indicates that multiphase materials have superiority in hierarchical structure design. Table 2 also indicates that the optimized material cells of Case A and D are different, but the objective values are very close. This example well demonstrates that the proposed model can distribute the multiphase materials reasonably in macro and micro levels.

Fig. 3 plots the iteration histories of the objective function, relative volume \( V(x) \) and mass of material cells \( m(r) \) of Case C. It is seen that the objective function, relative volume and mass of material cells are convergent to stable solutions.

5.2. Numerical example 2

In order to discuss the characteristics of hierarchical optimization, the optimization of periodic structure and traditional continuous structure are employed in this example. Because the optimization problems of periodic structure and traditional continuous structure are standard minimization problems with one constraint, the optimality criteria (OC) method is used as the optimizer for simplicity and efficiency [36]. Phase materials M1 and M4 are used. The heat flow \( F \) is applied at Position II and III respectively. As to hierarchical optimization model, the volume fraction is \( f = 0.5 \), and the upper bound of mass constraint is \( m = 43.2 \text{ kg} \). In order to guarantee the constraints are equivalent, the volume fraction and mass constraints are transformed into the equivalent

### Table 1

<table>
<thead>
<tr>
<th>Phase material</th>
<th>Thermal conductive coefficient ( k_0 ) (W m(^{-1}) K(^{-1}))</th>
<th>Density ( \rho_0 ) (kg/m(^3))</th>
<th>Ratio of thermal conductive coefficient to density ( k_0/\rho_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>20</td>
<td>1000</td>
<td>0.0200</td>
</tr>
<tr>
<td>M2</td>
<td>17</td>
<td>800</td>
<td>0.0213</td>
</tr>
<tr>
<td>M3</td>
<td>4</td>
<td>400</td>
<td>0.0100</td>
</tr>
<tr>
<td>M4</td>
<td>10</td>
<td>400</td>
<td>0.0250</td>
</tr>
</tbody>
</table>

### Table 2

The optimized structures and porous material cells with various phase materials.

<table>
<thead>
<tr>
<th>Case</th>
<th>Phase material</th>
<th>Thermal compliance ( C[J] )</th>
<th>Structure</th>
<th>Material cell</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>M1</td>
<td>20.3485</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
</tr>
<tr>
<td>B</td>
<td>M2</td>
<td>19.7986</td>
<td><img src="image3.png" alt="Image" /></td>
<td><img src="image4.png" alt="Image" /></td>
</tr>
<tr>
<td>C</td>
<td>M1 and M2</td>
<td>19.2502</td>
<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
</tr>
<tr>
<td>D</td>
<td>M1 and M3</td>
<td>20.3351</td>
<td><img src="image7.png" alt="Image" /></td>
<td><img src="image8.png" alt="Image" /></td>
</tr>
</tbody>
</table>
Table 3 shows the optimized results of different optimization models. Where Case A1, A2 are the optimized results of periodic structures, Case B1, B2 are the optimized results of hierarchical structures and Case C1, C2 are the optimized results of traditional continuous structures.

From Case A1–B1, A2–B2, it can be found that the optimized hierarchical structure has smaller thermal compliance than that from the periodic structure. This is reasonable because only the material cell is optimized and the structure doesn’t change in the periodic optimization. That is, the design degrees of freedom of the hierarchical optimization are much more than that of the periodic optimization. From Case B1–C1, B2–C2, we can find that the thermal compliance obtained by a traditional continuous structure is lower than that obtained by a hierarchical structure. This indicates that the advantage of a hierarchical porous structure is its multifunctional properties. In addition, when the heat load is applied at the same position, the phase materials of hierarchical structure and periodic structure are distributed in a similar manner at the micro level; the topology configurations of hierarchical optimization and traditional continuous structure are similar at the macro level.

From Case B1, B2, it is seen that when the heat load is applied at different positions, the optimized material cells and the distribution of phase materials have significant differences, which indicate the necessity of hierarchical structure design.

### Table 3
The optimized structures and porous material cells of different optimization models.

<table>
<thead>
<tr>
<th>Case</th>
<th>Load position</th>
<th>Thermal compliance C/J</th>
<th>Structure</th>
<th>Material cell</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>II</td>
<td>37.3525</td>
<td><img src="image1" alt="image1" /></td>
<td><img src="image2" alt="image2" /></td>
</tr>
<tr>
<td>B1</td>
<td></td>
<td>25.1461</td>
<td><img src="image3" alt="image3" /></td>
<td><img src="image4" alt="image4" /></td>
</tr>
<tr>
<td>C1</td>
<td></td>
<td>15.8884</td>
<td><img src="image5" alt="image5" /></td>
<td><img src="image6" alt="image6" /></td>
</tr>
<tr>
<td>A2</td>
<td>III</td>
<td>39.2706</td>
<td><img src="image7" alt="image7" /></td>
<td><img src="image8" alt="image8" /></td>
</tr>
<tr>
<td>B2</td>
<td></td>
<td>25.8262</td>
<td><img src="image9" alt="image9" /></td>
<td><img src="image10" alt="image10" /></td>
</tr>
<tr>
<td>C2</td>
<td></td>
<td>16.0323</td>
<td><img src="image11" alt="image11" /></td>
<td><img src="image12" alt="image12" /></td>
</tr>
</tbody>
</table>

5.3. Numerical example 3

The effects of binary constraints on the final solution are analyzed respectively. The heat flow \( F \) is applied at Position I, and phase materials M1 and M4 are adopted in this example.

Table 4 lists the optimized topologies of structures and their material cells with various volume fractions when \( m = 45 \text{ kg} \). It can be seen from Table 4 that with various volume fractions, there is no significant difference for the optimized structures, but the...
numbers of material cells along the optimal thermal conductive path are different. Table 4 also indicates that optimized topologies of material cells are very different. In Fig. 4, we can see that with the increasing of \( f \), the thermal compliance decreases monotonically. Meanwhile, the amounts of M1, M4 decrease and increase monotonically respectively. It is because that more phase materials are demanding to be filled in the design domain with \( f \) increasing when \( m \) is fixed. In this case, M4 has a greater advantage for a higher ratio of \( \frac{k_{04}}{\rho_{04}} \).

Table 5 shows the optimized structures and material cells with various upper bounds of mass constraint when \( f = 0.5 \). With various \( m \), the optimal topologies of the structures and material cells show the similar features compared with the resulting topologies in Table 4. As shown in Fig. 5, with increment of \( m \), the thermal compliance decreases monotonically. At the same time, the amounts of M1, M4 increase and decrease monotonically respectively. When \( m \) is small, phase material M4 has an obvious advantage for higher ratio of \( \frac{k_{04}}{\rho_{04}} \). Phase material M1 shows a greater superiority for higher thermal conductive coefficient when requirement of \( m \) gradually relaxes.

Due to the volume fraction constraint and mass constraint playing different roles, the amounts of phase materials show different trends when \( f, m \) changes respectively. The volume fraction constraint plays a role in generating holes, and the mass constraint affects the amounts of phase materials.

### 5.4. Numerical example 4

The optimized topologies of three phase materials are presented in this example. Phase materials M1, M2, and M4 are used together. The heat flow \( F \) is applied at Position I. The volume fraction \( f \) is set to be 0.6, and upper bound of mass constraint \( m \) is equal to 45 kg.

Table 6 lists the objective function, optimized structure and material cell. The resulting topology of structure reflects the

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**Table 4**
The optimized structures and porous material cells with various volume fractions.

<table>
<thead>
<tr>
<th>Volume fraction</th>
<th>Structure</th>
<th>Material cell</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td><img src="image1" alt="Structure" /></td>
<td><img src="image2" alt="Material cell" /></td>
</tr>
<tr>
<td>0.6</td>
<td><img src="image3" alt="Structure" /></td>
<td><img src="image4" alt="Material cell" /></td>
</tr>
<tr>
<td>0.7</td>
<td><img src="image5" alt="Structure" /></td>
<td><img src="image6" alt="Material cell" /></td>
</tr>
</tbody>
</table>

**Table 5**
The optimized structures and porous material cells with various upper bounds of mass constraint.

<table>
<thead>
<tr>
<th>Upper bound of mass constraint/kg</th>
<th>Structure</th>
<th>Material cell</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td><img src="image7" alt="Structure" /></td>
<td><img src="image8" alt="Material cell" /></td>
</tr>
<tr>
<td>43.2</td>
<td><img src="image9" alt="Structure" /></td>
<td><img src="image10" alt="Material cell" /></td>
</tr>
<tr>
<td>50.4</td>
<td><img src="image11" alt="Structure" /></td>
<td><img src="image12" alt="Material cell" /></td>
</tr>
<tr>
<td>57.6</td>
<td><img src="image13" alt="Structure" /></td>
<td><img src="image14" alt="Material cell" /></td>
</tr>
</tbody>
</table>

**Table 6**
The optimized structure and porous material cell with three phase materials.

<table>
<thead>
<tr>
<th>Thermal compliance C/J</th>
<th>Structure</th>
<th>Material cell</th>
</tr>
</thead>
<tbody>
<tr>
<td>23.8388</td>
<td><img src="image15" alt="Structure" /></td>
<td><img src="image16" alt="Material cell" /></td>
</tr>
</tbody>
</table>

**Fig. 4.** Effects of volume fraction constraint on the thermal compliance and the amounts of M1, M4.

**Fig. 5.** Effects of mass constraint on the thermal compliance and the amounts of M1, M4.
optimal thermal conductive path. Phase materials M1 and M2 play supporting roles, which are distributed at the near central position and four corners within the material cell. It is reasonable that M2 distributes around M1 because M1 has a higher thermal conductive coefficient than M2.

6. Conclusions

This paper develops a hierarchical topology optimization model which can simultaneously achieve the optimum structure and the porous cell with multi-phase materials for minimum thermal compliance. Material cell is assumed to be uniform in macro-scale to reduce manufacturing cost. Macro and micro design variables are defined independently and connected by elemental phase relative density. The uniform interpolation scheme is adopted to deal with multiphase material problems. The two-scale optimization problem involves only one finite element model. The prescribed volume fraction on the solid materials and upper bound of mass on the material cells are considered as optimization constraints. Some conclusions can be concluded from 2D numerical examples.

(1) In all the numerical examples, the objective values decline and converge rapidly. The relative volume and mass of material cells meet the prescribed values in the end. Compared with single phase materials, multiphase materials have superiority in hierarchical optimization. Meanwhile, the distribution of multiphase materials at the two scales can be adjusted rationally to obtain an efficient structure and material cell.

(2) The hierarchical structure can obtain a lower thermal compliance than that from the periodic structure since the design degrees of freedom are increased significantly. The thermal compliance obtained by a traditional continuous structure is lower than that obtained by a hierarchical structure, which indicates that the advantage of a hierarchical porous structure is its multifunctional properties. Otherwise, structure, material cell and distribution of phase materials vary with external conditions, which indicate the necessity of hierarchical structure design.

(3) With various volume fractions and upper bounds of mass constraint, the optimized structures have no significant difference, but the optimized material cells and the amounts of phase materials change significantly. When volume fraction and upper bound of mass constraint change respectively, the amounts of phase materials show different trends. This is because volume fraction constraint and mass constraint play different roles. The volume fraction constraint plays a role in generating holes, and the mass constraint affects the amounts of phase materials. With the increasing of the volume fraction, the phase material which has higher ratio of conductive coefficient to density takes advantage gradually. When upper bound of mass constraint increases, the phase material with a higher conductive coefficient shows superiority gradually.

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References


