Chapter 11
Limit Cycles

Aims and Objectives

• To give a brief historical background
• To define features of phase plane portraits
• To introduce the theory of planar limit cycles
• To introduce perturbation methods

On completion of this chapter, the reader should be able to

• prove existence and uniqueness of a limit cycle;
• prove that certain systems have no limit cycles;
• interpret limit cycle behavior in physical terms;
• find approximate solutions for perturbed systems.

Limit cycles, or isolated periodic solutions, are the most common form of solution observed when modeling physical systems in the plane. Early investigations were concerned with mechanical and electronic systems, but periodic behavior is evident in all branches of science. Two limit cycles were plotted in Chap. 10 when considering the modeling of interacting species.

The chapter begins with a historical introduction, and then the theory of planar limit cycles is introduced. Textbooks providing a more general introduction to the theory of limit cycles include [16, 17].

11.1 Historical Background

Definition 1. A limit cycle is an isolated periodic solution.

Limit cycles in planar differential systems commonly occur when modeling both the technological and natural sciences. Most of the early history in the theory of limit cycles in the plane was stimulated by practical problems. For example, the
differential equation derived by Rayleigh in 1877 [12], related to the oscillation of a violin string, is given by

$$\ddot{x} + \epsilon \left( \frac{1}{3}(\dot{x})^2 - 1 \right) \dot{x} + x = 0,$$

where $\ddot{x} = \frac{d^2x}{dt^2}$ and $\dot{x} = \frac{dx}{dt}$. Let $\dot{x} = y$. Then this differential equation can be written as a system of first-order autonomous differential equations in the plane

$$\dot{x} = y, \quad \dot{y} = -x - \epsilon \left( \frac{y^2}{3} - 1 \right) y.$$  \hspace{1cm} (11.1)

A phase portrait is shown in Fig. 11.1.

Following the invention of the triode vacuum tube, which was able to produce stable self-excited oscillations of constant amplitude, van der Pol [15] obtained the following differential equation to describe this phenomenon:

$$\ddot{x} + \epsilon (x^2 - 1) \dot{x} + x = 0,$$

which can be written as a planar system of the form

$$\dot{x} = y, \quad \dot{y} = -x - \epsilon (x^2 - 1) y.$$  \hspace{1cm} (11.2)

A phase portrait is shown in Fig. 11.2, and the shape of the van der Pol limit cycle is investigated in [10].
The basic model of a cell membrane is that of a resistor and capacitor in parallel. The equations used to model the membrane are a variation of the van der Pol equation. The famous Fitzhugh–Nagumo oscillator [3, 7, 13], used to model the action potential of a neuron is a two-variable simplification of the Hodgkin-Huxley equations [5] (see Chap. 20). The Fitzhugh–Nagumo model creates quite accurate action potentials and models the qualitative behavior of the neurons. The differential equations are given by

\[
\frac{du}{dt} = -u(u - \theta)(u - 1) - v + \omega, \quad \frac{dv}{dt} = \epsilon(u - \gamma v),
\]

where \( u \) is a voltage, \( v \) is the recovery of voltage, \( \theta \) is a threshold, \( \gamma \) is a shunting variable, and \( \omega \) is a constant voltage. For certain parameter values, the solution demonstrates a slow collection and fast release of voltage; this kind of behavior has been labeled integrate and fire. Note that, for biological systems, neurons cannot collect voltage immediately after firing and need to rest. Oscillatory behavior for the Fitzhugh-Nagumo system is shown in Fig. 11.3. MATLAB command lines for producing Fig. 11.3 are listed in Sect. 11.4.

Note that when \( \omega = \omega(t) \) is a periodic external input, the system becomes nonautonomous and can display chaotic behavior [13]. The reader can investigate these systems via the exercises in Chap. 15.

Perhaps the most famous class of differential equations that generalize (11.2) are those first investigated by Liénard in 1928 [6]:

\[
\ddot{x} + f(x)\dot{x} + g(x) = 0,
\]
or in the phase plane
\[
\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y. \tag{11.3}
\]

This system can be used to model mechanical systems, where \( f(x) \) is known as the damping term and \( g(x) \) is called the restoring force or stiffness. Equation (11.3) is also used to model resistor-inductor-capacitor circuits (see Chap. 8) with non-linear circuit elements. Limit cycles of Liénard systems will be discussed in some detail in Chaps. 16 and 17.

Possible physical interpretations for limit cycle behavior of certain dynamical systems are listed below:

- For an economic model, Bella [2] considers a Goodwin model of a class struggle and demonstrates emerging multiple limit cycles of different orientation.
- For predator-prey and epidemic models, the populations oscillate in phase with one another and the systems are robust (see Examples in Chap. 10).
- Periodic behavior is present in integrate and fire neurons (see Fig. 11.3).
- For mechanical systems, examples include the motion of simple nonlinear pendula (see Sect. 15.3), wing rock oscillations in aircraft flight dynamics [9], and surge oscillations in axial flow compressors [1].
For periodic chemical reactions, examples include the Landolt clock reaction and the Belousov–Zhabotinski reaction (see Chap. 14).

For electrical or electronic circuits, it is possible to construct simple electronic oscillators (e.g., Chua’s circuit, Chap. 14) using a nonlinear circuit element; a limit cycle can be observed if the circuit is connected to an oscilloscope.

Limit cycles are common solutions for all types of dynamical systems. Sometimes it becomes necessary to prove the existence and uniqueness of a limit cycle, as described in the next section.

11.2 Existence and Uniqueness of Limit Cycles in the Plane

To understand the existence and uniqueness theorem, it is necessary to define some features of phase plane portraits. Assume that the existence and uniqueness theorem from Chap. 8 holds for all solutions considered here.

Definitions 1 and 2 in Chap. 9 can be extended to nonlinear planar systems of the form \( \dot{x} = P(x, y), \dot{y} = Q(x, y) \); thus every solution, say, \( \phi(t) = (x(t), y(t)) \), can be represented as a curve in the plane and is called a trajectory. The phase portrait shows how the qualitative behavior is determined as \( x \) and \( y \) vary with \( t \). The trajectory can also be defined in terms of the spatial coordinates \( x \), as in Definition 3 below. A brief look at Example 1 will help the reader to understand Definitions 1–7 in this section.

**Definition 2.** flow on \( \mathbb{R}^2 \) is a mapping \( \pi : \mathbb{R}^2 \to \mathbb{R}^2 \) such that

1. \( \pi \) is continuous
2. \( \pi(x, 0) = x \) for all \( x \in \mathbb{R}^2 \)
3. \( \pi(\pi(x, t_1), t_2) = \pi(x, t_1 + t_2) \)

**Definition 3.** Suppose that \( I_x \) is the maximal interval of existence. The trajectory (or orbit) through \( x \) is defined as \( \gamma(x) = \{\pi(x, t) : t \in I_x \} \).

The positive semiorbit is defined as \( \gamma^+(x) = \{\pi(x, t) : t > 0\} \).

The negative semiorbit is defined as \( \gamma^-(x) = \{\pi(x, t) : t < 0\} \).

**Definition 4.** The positive limit set of a point \( x \) is defined as

\[ \Lambda^+(x) = \{y : \text{there exists a sequence } t_n \to \infty \text{ such that } \pi(x, t) \to y\} \].

The negative limit set of a point \( x \) is defined as

\[ \Lambda^-(x) = \{y : \text{there exists a sequence } t_n \to -\infty \text{ such that } \pi(x, t) \to y\} \].

In the phase plane, trajectories tend to a critical point, a closed orbit, or infinity.

**Definition 5.** A set \( S \) is invariant with respect to a flow if \( x \in S \) implies that \( \gamma(x) \subset S \).
A set $S$ is *positively invariant* with respect to a flow if $x \in S$ implies that $\gamma^+(x) \subset S$.

A set $S$ is *negatively invariant* with respect to a flow if $x \in S$ implies that $\gamma^-(x) \subset S$.

A general trajectory can be labeled $\gamma$ for simplicity.

**Definition 6.** A limit cycle, say, $\gamma_1$, is

- A *stable limit cycle* if $\Lambda^+(x) = \Gamma$ for all $x$ in some neighborhood; this implies that nearby trajectories are attracted to the limit cycle.

- An *unstable limit cycle* if $\Lambda^-(x) = \Gamma$ for all $x$ in some neighborhood; this implies that nearby trajectories are repelled away from the limit cycle.

- A *semistable limit cycle* if it is attracting on one side and repelling on the other.

The stability of limit cycles can also be deduced analytically using the Poincaré map (see Chap. 15). The following example will be used to illustrate each of the Definitions 1–6 above and 7 below.

**Definition 7.** The period, say, $T$, of a limit cycle is given by $x(t + T) = x(t)$, where $T$ is the minimum period. The period can be found by plotting a time series plot of the limit cycle (see the MATLAB command lines in Chap. 10).

**Example 1.** Describe some of the features for the following set of polar differential equations in terms of Definitions 1–7:

$$\dot{r} = r(1 - r)(2 - r)(3 - r), \quad \dot{\theta} = -1.$$  \hspace{1cm} (11.4)

**Solution.** A phase portrait is shown in Fig. 11.4. There is a unique critical point at the origin since $\dot{\theta}$ is nonzero. There are three limit cycles that may be determined from the equation $\dot{r} = 0$. They are the circles of radii one, two, and three, all centered at the origin. Let $\Gamma_i$ denote the limit cycle of radius $r = i$. 

![Fig. 11.4 Three limit cycles for system (11.4)]
There is one critical point at the origin. If a trajectory starts at this point, it remains there forever. A trajectory starting at \((1,0)\) will reach the point \((-1,0)\) when \(t_2 = \pi\), the orbit is clockwise. Continuing on this path for another time interval \(t_2 = \pi\), the motion is clockwise. Using part 3 of Definition 2, one can write \(\pi ((1,0), t_1), t_2) = \pi ((1,0), 2\pi)\) since the limit cycle is of period \(2\pi\) (see below). On the limit cycle \(\Gamma_1\), both the positive and negative semiorbits lie on \(\Gamma_1\).

Suppose that \(P = (1, 0)\) and \(Q = (4, 0)\) are two points in the plane. The limit sets are given by \(\Lambda^+(P) = \Gamma_1, \Lambda^-(P) = (0, 0), \Lambda^+(Q) = \Gamma_3, \) and \(\Lambda^-(Q) = \infty\).

The annulus \(A_1 = \{r \in \mathbb{R}^2 : 0 < r < 1\}\) is positively invariant, and the annulus \(A_2 = \{r \in \mathbb{R}^2 : 1 < r < 2\}\) is negatively invariant.

If \(0 < r < 1\), then \(\dot{r} > 0\) and the critical point at the origin is unstable. If \(1 < r < 2\), then \(\dot{r} < 0\) and \(\Gamma_1\) is a stable limit cycle. If \(2 < r < 3\), then \(\dot{r} > 0\) and \(\Gamma_2\) is an unstable limit cycle. Finally, if \(r > 3\), then \(\dot{r} < 0\) and \(\Gamma_3\) is a stable limit cycle.

Integrate both sides of \(\dot{\theta} = -1\) with respect to time to show that the period of all of the limit cycles is \(2\pi\).

**The Poincaré-Bendixson Theorem.** Suppose that \(\gamma^+\) is contained in a bounded region in which there are finitely many critical points. Then \(\Lambda^+(\gamma)\) is either

- A single critical point
- A single closed orbit
- A graphic—critical points joined by heteroclinic orbits.

A heteroclinic orbit connects two separate critical points and takes an infinite amount of time to make the connection; more detail is provided in Chap. 12.

**Corollary.** Let \(D\) be a bounded closed set containing no critical points and suppose that \(D\) is positively invariant. Then there exists a limit cycle contained in \(D\).

A proof to this theorem involves topological arguments and can be found in [11], for example.

**Example 2.** By considering the flow across the rectangle with corners at \((-1, 2),\) \((1, 2),\) \((1, -2),\) and \((-1, -2),\) prove that the following system has at least one limit cycle:

\[
\dot{x} = y - 8x^3, \quad \dot{y} = 2y - 4x - 2y^3.
\]  

**Solution.** The critical points are found by solving the equations \(\dot{x} = \dot{y} = 0\). Set \(y = 8x^3\). Then \(\dot{y} = 0\) if \(x(1 - 4x^2 + 256x^8) = 0\). The graph of the function \(y = 1 - 4x^2 + 256x^8\) is given in Fig. 11.5a. The graph has no roots and the origin is the only critical point.

Linearize at the origin in the usual way. It is not difficult to show that the origin is an unstable focus.

Consider the flow on the sides of the given rectangle:

- On \(y = 2, |x| \leq 1, \dot{y} = -4x - 12 < 0\).
- On \(y = -2, |x| \leq 1, \dot{y} = -4x + 12 > 0\).
The flow is depicted in Fig. 11.5b. The rectangle is positively invariant and there are no critical points other than the origin, which is unstable. Consider a small deleted neighborhood, say, $N_e$, around this critical point. For example, the boundary of $N_e$ could be a small ellipse. On this ellipse, all trajectories will cross outwards. Therefore, there exists a stable limit cycle lying inside the rectangular region and outside of $N_e$ by the corollary to the Poincaré–Bendixson theorem.

**Definition 8.** A planar simple closed curve is called a *Jordan curve*.

Consider the system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

(11.6)

where $P$ and $Q$ have continuous first-order partial derivatives. Let the vector field be denoted by $\mathbf{X}$ and let $\psi$ be a weighting factor that is continuously differentiable. Recall Green’s Theorem, which will be required to prove the following two theorems.
Green’s Theorem. Let $J$ be a Jordan curve of finite length. Suppose that $P$ and $Q$ are two continuously differentiable functions defined on the interior of $J$, say, $D$. Then

$$\iint_D \left[ \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right] \, dx \, dy = \oint_J P \, dy - Q \, dx.$$

Dulac’s Criteria. Consider an annular region, say, $A$, contained in an open set $E$. If

$$\nabla \cdot (\psi \mathbf{X}) = \text{div} (\psi \mathbf{X}) = \frac{\partial}{\partial x} (\psi P) + \frac{\partial}{\partial y} (\psi Q)$$

does not change sign in $A$, then there is at most one limit cycle entirely contained in $A$.

Proof. Suppose that $\Gamma_1$ and $\Gamma_2$ are limit cycles encircling $K$, as depicted in of periods $T_1$ and $T_2$, respectively. Apply Green’s Theorem to the region $R$ shown in Fig. 11.6.

$$\iint_R \left[ \frac{\partial (\psi \mathbf{P})}{\partial x} + \frac{\partial (\psi \mathbf{Q})}{\partial y} \right] \, dx \, dy = \oint_{\Gamma_2} \psi \mathbf{P} \, dy - \psi \mathbf{Q} \, dx + \int_L \psi \mathbf{P} \, dy - \psi \mathbf{Q} \, dx - \oint_{\Gamma_1} \psi \mathbf{P} \, dy - \psi \mathbf{Q} \, dx - \int_L \psi \mathbf{P} \, dy - \psi \mathbf{Q} \, dx.$$

Now on $\Gamma_1$ and $\Gamma_2$, $\dot{x} = P$ and $\dot{y} = Q$, so

$$\begin{align*}
\iint_R & \left[ \frac{\partial (\psi \mathbf{P})}{\partial x} + \frac{\partial (\psi \mathbf{Q})}{\partial y} \right] \, dx \, dy \\
& = \int_0^{T_2} (\psi \mathbf{P} \mathbf{Q} - \psi \mathbf{Q} \mathbf{P}) \, dt - \int_0^{T_1} (\psi \mathbf{P} \mathbf{Q} - \psi \mathbf{Q} \mathbf{P}) \, dt,
\end{align*}$$

Fig. 11.6 Two limit cycles encircling the region $K$
which is zero and contradicts the hypothesis that \( \text{div}(\psi X) \neq 0 \) in \( A \). Therefore, there is at most one limit cycle entirely contained in the annulus \( A \). \( \square \)

**Example 3.** Use Dulac’s criteria to prove that the system

\[
\dot{x} = -y + x(1 - 2x^2 - 3y^2), \quad \dot{y} = x + y(1 - 2x^2 - 3y^2)
\]

has a unique limit cycle in an annulus.

**Solution.** Convert to polar coordinates using the transformations

\[
r \dot{r} = x \dot{x} + y \dot{y}, \quad r^2 \dot{\theta} = x \dot{y} - y \dot{x}.
\]

Therefore, system (11.7) becomes

\[
\dot{r} = r(1 - 2r^2 - r^2 \sin^2 \theta), \quad \dot{\theta} = 1.
\]

Since \( \dot{\theta} = 1 \), the origin is the only critical point. On the circle \( r = \frac{1}{2} \), \( \dot{r} = \frac{1}{2}(\frac{3}{2} - \frac{1}{4} \sin^2 \theta) \). Hence \( \dot{r} > 0 \) on this circle. On the circle \( r = 1 \), \( \dot{r} = -1 - \sin^2 \theta \). Hence \( \dot{r} < 0 \) on this circle. If \( r \geq 1 \), then \( \dot{r} < 0 \), and if \( 0 < r \leq \frac{1}{2} \), then \( \dot{r} > 0 \).

Therefore, there exists a limit cycle in the annulus \( A = \{ r : \frac{1}{2} < r < 1 \} \) by the corollary to the Poincaré–Bendixson theorem.

Consider the annulus \( A \). Now \( \text{div}(X) = 2(1 - 4r^2 - 2r^2 \sin^2 \theta) \). If \( \frac{1}{2} < r < 1 \), then \( \text{div}(X) < 0 \). Since the divergence of the vector field does not change sign in the annulus \( A \), there is at most one limit cycle in \( A \) by Dulac’s criteria.

A phase portrait is given in Fig. 11.7.

**Example 4.** Plot a phase portrait for the Liénard system

\[
\dot{x} = y, \quad \dot{y} = -x - y(a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + a_{10}x^{10} + a_{12}x^{12} + a_{14}x^{14}),
\]

where \( a_2 = 90, a_4 = -882, a_6 = 2598.4, a_8 = -3359.997, a_{10} = 2133.34, a_{12} = -651.638 \), and \( a_{14} = 76.38 \).

**Solution.** Not all limit cycles are convex closed curves as Fig. 11.8 demonstrates.

### 11.3 Nonexistence of Limit Cycles in the Plane

**Bendixson’s Criteria.** Consider system (11.6) and suppose that \( D \) is a simply connected domain (no holes in \( D \)) and that

\[
\nabla.(\psi X) = \text{div}(\psi X) = \frac{\partial}{\partial x}(\psi P) + \frac{\partial}{\partial y}(\psi Q) \neq 0
\]

in \( D \). Then there are no limit cycles entirely contained in \( D \).
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Fig. 11.7 A phase portrait for system (11.7) showing the unique limit cycle

Fig. 11.8 [MATLAB] A phase portrait for Example 4. The limit cycle is a nonconvex closed curve
Proof. Suppose that \( D \) contains a limit cycle \( \Gamma \) of period \( T \). Then from Green’s Theorem

\[
\iint_D \left[ \frac{\partial (\psi P)}{\partial x} + \frac{\partial (\psi Q)}{\partial y} \right] \, dx \, dy = \oint_{\Gamma} (\psi P \, dy - \psi Q \, dx)
\]

\[
= \int_0^T \left( \psi P \frac{dy}{dt} - \psi Q \frac{dx}{dt} \right) \, dt = 0
\]

since on \( \Gamma \), \( \dot{x} = P \) and \( \dot{y} = Q \). This contradicts the hypothesis that \( \text{div}(\psi X) \neq 0 \), and therefore \( D \) contains no limit cycles entirely. \( \square \)

Definition 9. Suppose there is a compass on a Jordan curve \( C \) and that the needle points in the direction of the vector field. The compass is moved in a counterclockwise direction around the Jordan curve by \( 2\pi \) radians. When it returns to its initial position, the needle will have moved through an angle, say, \( \Theta \). The index, say, \( I_X(C) \), is defined as

\[
I_X(C) = \frac{\Delta \Theta}{2\pi},
\]

where \( \Delta \Theta \) is the overall change in the angle \( \Theta \).

The above definition can be applied to isolated critical points. For example, the index of a node, focus, or center is \( +1 \) and the index of a col is \( -1 \). The following result is clear.

**Theorem 1.** The sum of the indices of the critical points contained entirely within a limit cycle is \( +1 \).

The next theorem then follows.

**Theorem 2.** A limit cycle contains at least one critical point.

When proving that a system has no limit cycles, the following items should be considered:

1. Bendixson’s criteria
2. Indices
3. Invariant lines
4. Critical points

**Example 5.** Prove that none of the following systems have any limit cycles:

(a) \( \dot{x} = 1 + y^2 - e^{xy}, \quad \dot{y} = xy + \cos^2 y \).
(b) \( \dot{x} = y^2 - x, \quad \dot{y} = y + x^2 + yx^3 \).
(c) \( \dot{x} = y + x^3, \quad \dot{y} = x + y + y^3 \).
(d) \( \dot{x} = 2xy - 2y^4, \quad \dot{y} = x^2 - y^2 - xy^3 \).
(e) \( \dot{x} = x(2 - y - x), \quad \dot{y} = y(4x - x^2 - 3), \) given \( \psi = \frac{1}{xy} \).
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Solutions.

(a) The system has no critical points and hence no limit cycles by Theorem 2.
(b) The origin is the only critical point and it is a saddle point or col. Since the index of a col is $-1$, there are no limit cycles from Theorem 1.
(c) Find the divergence, $\text{div} \mathbf{X} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 3x^2 + 3y^2 + 1 \neq 0$. Hence there are no limit cycles by Bendixson’s criteria.
(d) Find the divergence, $\text{div} \mathbf{X} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = -3x^2 y$. Now $\text{div} \mathbf{X} = 0$ if either $x = 0$ or $y = 0$. However, on the line $x = 0$, $\dot{x} = -2y^4 \leq 0$, and on the line $y = 0$, $\dot{y} = x^2 \geq 0$. Therefore, a limit cycle must lie wholly in one of the four quadrants. This is not possible since $\text{div} \mathbf{X}$ is nonzero here. Hence there are no limit cycles by Bendixson’s criteria. Draw a small diagram to help you understand the solution.
(e) The axes are invariant since $\dot{x} = 0$ if $x = 0$ and $\dot{y} = 0$ if $y = 0$. The weighted divergence is given by $\text{div}(\psi \mathbf{X}) = \frac{\partial}{\partial x}(\psi P) + \frac{\partial}{\partial y}(\psi Q) = -\frac{1}{y}$. Therefore, there are no limit cycles contained entirely in any of the quadrants, and since the axes are invariant, there are no limit cycles in the whole plane.

Example 6. Prove that the system

$$\dot{x} = x(1 - 4x + y), \quad \dot{y} = y(2 + 3x - 2y)$$

has no limit cycles by applying Bendixson’s criteria with $\psi = x^m y^n$.

Solution. The axes are invariant since $\dot{x} = 0$ on $x = 0$ and $\dot{y} = 0$ on $y = 0$. Now

$$\text{div}(\psi \mathbf{X}) = \frac{\partial}{\partial x} \left(x^{m+1} y^n - 4x^{m+2} y^n + x^{m+1} y^{n+1}\right) + \frac{\partial}{\partial y} \left(2x^m y^{n+1} + 3x^{m+1} y^{n+1} - 2x^m y^{n+2}\right),$$

which simplifies to

$$\text{div}(\psi \mathbf{X}) = (m + 2n + 2)x^m y^n + (-4m + 3n - 5)x^{m+1} y^n + (m - 2n - 3)x^m y^{n+1}. $$

Select $m = \frac{1}{2}$ and $n = -\frac{5}{4}$. Then

$$\text{div}(\psi \mathbf{X}) = -\frac{43}{4} x^{\frac{1}{2}} y^{-\frac{5}{4}}.$$ 

Therefore, there are no limit cycles contained entirely in any of the four quadrants, and since the axes are invariant, there are no limit cycles at all.
11.4 Perturbation Methods

This section introduces the reader to some basic perturbation methods by means of example. The theory involves mathematical methods for finding series expansion approximations for perturbed systems. Perturbation theory can be applied to algebraic equations, boundary value problems, difference equations, Hamiltonian systems, ODEs, and PDEs, and in modern times the theory underlies almost all of quantum field theory and quantum chemistry. There are whole books devoted to the study of perturbation methods and the reader is directed to the references [4, 8, 14], for more detailed theory and more in-depth explanations.

To keep the theory simple and in relation to other material in this chapter, the author has decided to focus on perturbed ODEs of the form

$$\ddot{x} + x = \epsilon f(x, \dot{x}),$$  \hspace{1cm} (11.8)

where $0 \leq \epsilon \ll 1$ and $f(x, \dot{x})$ is an arbitrary smooth function. The unperturbed system represents a linear oscillator and when $0 < \epsilon \ll 1$, system (11.8) becomes a weakly nonlinear oscillator. Systems of this form include the Duffing equation

$$\ddot{x} + x = \epsilon x^3,$$  \hspace{1cm} (11.9)

and the van der Pol equation

$$\ddot{x} + x = \epsilon (x^2 - 1) \dot{x}.$$  \hspace{1cm} (11.10)

The main idea begins with the assumption that the solution to the perturbed system can be expressed as an asymptotic expansion of the form

$$x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \ldots.$$  \hspace{1cm} (11.11)

**Definition 10.** The sequence $f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \phi_n(\epsilon)$ is an asymptotic (or Poincaré) expansion of the continuous function $f(\epsilon)$ if and only if, for all $n \geq 0$,

$$f(\epsilon) = \sum_{n=0}^{N} a_n \phi_n(\epsilon) + O(\phi_{N+1}(\epsilon)) \quad \text{as} \quad \epsilon \to 0,$$  \hspace{1cm} (11.12)

where the sequence constitutes an asymptotic scale such that for every $n \geq 0$,

$$\phi_{n+1}(\epsilon) = o(\phi_n(\epsilon)) \quad \text{as} \quad \epsilon \to 0.$$

**Definition 11.** An asymptotic expansion (11.12) is said to be uniform if in addition

$$|R_N(x, \epsilon)| \leq K|\phi_{N+1}(\epsilon)|,$$
for $\epsilon$ in a neighborhood of 0, where the Nth remainder $R_N(x, \epsilon) = O(\phi_{N+1}(\epsilon))$ as $\epsilon \to 0$, and $K$ is a constant.

In this particular case, we will be looking for asymptotic expansions of the form

$$x(t, \epsilon) \sim \sum_k x_k(t) \delta_k(\epsilon),$$

where $\delta_k(\epsilon) = \epsilon^k$ is an asymptotic scale. It is important to note that the asymptotic expansions often do not converge; however, one-term and two-term approximations provide an analytical expression that is dependent on the parameter, $\epsilon$, and some initial conditions. The major advantage that the perturbation analysis has over numerical analysis is that a general solution is available through perturbation methods where numerical methods only lead to a single solution.

Example 7. Use perturbation theory to find a one-term and two-term asymptotic expansion of Duffing’s equation (11.9) with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$.

Solution. Substitute (11.11) into (11.9) to get

$$\frac{d^2}{dt^2} (x_0 + \epsilon x_1 + \ldots) + (x_0 + \epsilon x_1 + \ldots) = \epsilon (x_0 + \epsilon x_1 + \ldots)^3.$$

Use the `collect` command in M to group terms according to powers of $\epsilon$; thus

$$[\ddot{x}_0 + x_0] + \epsilon [\ddot{x}_1 + x_1 - x_0^3] + O(\epsilon^2) = 0.$$

The order equations are

$$O(1): \quad \ddot{x}_0 + x_0 = 0, \quad x_0(0) = 1, \quad \dot{x}_0(0) = 0,$$

$$O(\epsilon): \quad \ddot{x}_1 + x_1 = x_0^3, \quad x_1(0) = 0, \quad \dot{x}_1(0) = 0.$$

The $O(1)$ solution is $x_0 = \cos(t)$. Let us compare this solution with the numerical solution, say, $x_N$, when $\epsilon = 0.01$. Figure 11.9 shows the time against the error, $x_N - x_0$, for $0 \leq t \leq 100$.

Using MATLAB, the $O(\epsilon)$ solution is computed to be

$$x_1 = \frac{1}{32} (-8 \cos(t) + 8 \cos^5(t) + 12 \sin(t) + 8 \sin(t) \sin(2t) + \sin(t) \sin(4t))$$

which simplifies to
Fig. 11.9 The error between the numerical solution \( x_N \) and the one-term expansion \( x_0 \) for the Duffing system (11.9) when \( \epsilon = 0.01 \)

Fig. 11.10 The error between the numerical solution \( x_N \) and the two-term expansion \( x_P \) for the Duffing system (11.9) when \( \epsilon = 0.01 \)

Thus,

\[ x = x_P = \cos(t) + \epsilon \left( \frac{3}{8} t \sin(t) + \frac{1}{16} \sin(2t) \right). \]

where \( x_P \) represents the Poincaré expansion up to the second term. The term \( t \sin(t) \) is called a secular term and is an oscillatory term of growing amplitude. Unfortunately, the secular term leads to a nonuniformity for large \( t \). Figure 11.10 shows the error for the two-term Poincaré expansion, \( x_N - x_P \), when \( \epsilon = 0.01 \).

By introducing a strained coordinate, the nonuniformity may be overcome and this is the idea behind the Lindstedt–Poincaré technique for periodic systems. The idea is to introduce a straining transformation of the form

\[ \frac{\tau}{t} = 1 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \ldots, \quad (11.13) \]

and seek values \( \omega_1, \omega_2, \ldots \), that avoid secular terms appearing in the expansion.

Example 8. Use the Lindstedt–Poincaré technique to determine a two-term uniform asymptotic expansion of Duffing’s equation (11.9) with initial conditions \( x(0) = 1 \) and \( \dot{x}(0) = 0 \).
11.4 Perturbation Methods

Solution. Using the transformation given in (11.13)

\[
\frac{d}{dt} = \frac{d\tau}{dt} \frac{d}{d\tau} = (1 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \cdots) \frac{d}{d\tau},
\]

\[
\frac{d^2}{dt^2} = \left(1 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \cdots\right)^2 \frac{d^2}{d\tau^2}.
\]

Applying the transformation to (11.9) leads to

\[
\left(1 + 2\epsilon \omega_1 + \epsilon^2 \left(\omega_1^2 + 2\omega_2\right) + \cdots\right) \frac{d^2x}{d\tau^2} + x = \epsilon x^3,
\]

where \(x\) is now a function of the strained variable \(\tau\). Assume that

\[
x(\tau, \epsilon) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \ldots.
\]  

(11.14)

Substituting (11.14) into (11.9) using MATLAB gives the following order equations:

\begin{align*}
&O(1) : \quad \frac{d^2x_0}{d\tau^2} + x_0 = 0, \\
&\quad \quad x_0(\tau = 0) = 1, \quad \frac{dx_0}{d\tau}(\tau = 0) = 0, \\
&O(\epsilon) : \quad \frac{d^2x_1}{d\tau^2} + x_1 = x_0^3 - 2\omega_1 \frac{d^2x_0}{d\tau^2}, \\
&\quad \quad x_1(0) = 0, \quad \frac{dx_1}{d\tau}(0) = 0, \\
&O(\epsilon^2) : \quad \frac{d^2x_2}{d\tau^2} + x_2 = 3x_0^2 x_1 - 2\omega_1 \frac{d^2x_1}{d\tau^2} - (\omega_1^2 + 2\omega_2) \frac{d^2x_0}{d\tau^2}, \\
&\quad \quad x_2(0) = 0, \quad \frac{dx_2}{d\tau}(0) = 0.
\end{align*}

The \(O(1)\) solution is \(x_0(\tau) = \cos(\tau)\). Using MATLAB, the solution to the \(O(\epsilon)\) equation is

\[
x_1(\tau) = \frac{1}{8} \sin(\tau) \left(3\tau + 8\omega_1 \tau + \cos(\tau) \sin(\tau)\right).
\]

To avoid secular terms, select \(\omega_1 = -\frac{3}{8}\), then the \(O(\epsilon)\) solution is

\[
x_1(\tau) = \frac{1}{8} \sin^2(\tau) \cos(\tau).
\]
Using MATLAB, the $O(\epsilon^2)$ solution is

$$x_2(\tau) = \frac{1}{512} \sin(\tau) \left( 42\tau + 512\omega_2 \tau + 23 \sin(2\tau) - \sin(4\tau) \right),$$

and selecting $\omega_2 = -\frac{21}{256}$ avoids secular terms.

The two-term uniformly valid expansion of (11.9) is

$$x(\tau, \epsilon) \sim x_{LP} = \cos(\tau) + \frac{\epsilon}{8} \sin^2(\tau) \cos(\tau),$$

where

$$\tau = t \left(1 - \frac{3}{8} \epsilon - \frac{21}{256} \epsilon^2 + O(\epsilon^3)\right).$$

as $\epsilon \to 0$. Note that the straining transformation is given to a higher order than the expansion of the solution. The difference between the two-term uniform asymptotic expansion and the numerical solution is depicted in Fig. 11.11.

Unfortunately, the Lindstedt-Poincaré technique does not always work for oscillatory systems. An example of its failure is provided by the van der Pol equation (11.10).

**Example 9.** Show that the Lindstedt–Poincaré technique fails for the ODE (11.10) with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$.

**Solution.** Substituting (11.14) into (11.10) using MATLAB gives the following order equations:

$$O(1): \quad \frac{d^2x_0}{d\tau^2} + x_0 = 0,$$

$$x_0(\tau = 0) = 1, \quad \frac{dx_0}{d\tau}(\tau = 0) = 0.$$
The $O(\epsilon)$ solution is $x_0(\tau) = \cos(\tau)$. Using MATLAB, the solution to the $O(\epsilon)$ equation can be simplified to

$$x_1(\tau) = \frac{1}{16} \left( (6\tau \cos(\tau) - (5 - 16\tau\omega_1 + \cos(2\tau)) \sin(\tau) \right)$$

or

$$x_1(\tau) = \frac{1}{16} \left( [6\tau \cos(\tau) + 16\tau\omega_1 \sin(\tau)] - (5 + \cos(2\tau)) \sin(\tau) \right).$$

To remove secular terms set $\omega_1 = -\frac{3}{8} \cot(\tau)$, then

$$x(\tau, \epsilon) = \cos(\tau) + O(\epsilon),$$

where

$$\tau = t - \frac{3}{8} \epsilon t \cot(t) + O(\epsilon^2).$$

This is invalid since the cotangent function is singular when $t = n\pi$, where $n$ is an integer. Unfortunately, the Lindstedt–Poincaré technique does not work for all ODEs of the form (11.8); it cannot be used to obtain approximations that evolve aperiodically on a slow time scale.

Consider the van der Pol equation (11.10); Fig. 11.12 shows a trajectory starting at $x(0) = 0.1, \dot{x}(0) = 0$ for $\epsilon = 0.05$ and $0 \leq t \leq 800$. The trajectory spirals around the origin and it takes many cycles for the amplitude to grow substantially. As $t \to \infty$, the trajectory asymptotes to a limit cycle of approximate radius two. This is an example of a system whose solutions depend simultaneously on widely different scales. In this case there are two time scales: a fast time scale for the sinusoidal oscillations $\sim O(1)$ and a slow time scale over which the amplitude grows $\sim O(\frac{1}{\epsilon})$. The method of multiple scales introduces new slow-time variables for each time scale of interest in the problem.

**The Method of Multiple Scales**

Introduce new time scales, say, $\tau_0 = t$ and $\tau_1 = \epsilon t$, and seek approximate solutions of the form

$$O(\epsilon) : \frac{d^2 x_1}{d \tau_1^2} + x_1 = \frac{d x_0}{d \tau} - x_0 \frac{d x_0}{d \tau} - 2\omega_1 \frac{d^2 x_0}{d \tau^2}.$$
Substitute into the ODE and solve the resulting PDEs. An example is given below.

**Example 10.** Use the method of multiple scales to determine a uniformly valid one-term expansion for the van der Pol equation (11.10) with initial conditions \( x(0) = a \) and \( \dot{x}(0) = 0 \).

**Solution.** Substitute (11.15) into (11.10) using MATLAB gives the following order equations:

\[
O(1) : \quad \frac{\partial^2 x_0}{\partial \tau_0^2} + x_0 = 0,
\]

\[
O(\epsilon) : \quad \frac{\partial^2 x_1}{\partial \tau_0^2} + x_1 = -2 \frac{\partial x_0}{\partial \tau_0 \tau_1} - (x_0^2 - 1) \frac{\partial x_0}{\partial \tau_0}.
\]

The general solution to the \( O(1) \) PDE may be found using MATLAB,

\[
x_0(\tau_0, \tau_1) = c_1(\tau_1) \cos(\tau_0) + c_2(\tau_1) \sin(\tau_0)
\]

which using trigonometric identities can be expressed as

\[
x_0(\tau_0, \tau_1) = R(\tau_1) \cos(\tau_0 + \theta(\tau_1)),
\]

where \( R(\tau_1) \) and \( \theta(\tau_1) \) are the slowly varying amplitude and phase of \( x_0 \), respectively. Substituting (11.16), the \( O(\epsilon) \) equation becomes
\[
\frac{\partial^2 x_1}{\partial \tau_0^2} + x_1 = -2 \left( \frac{dR}{d\tau_1} \sin(\tau_0 + \theta(\tau_1)) + R(\tau_1) \frac{d\theta}{d\tau_1} \cos(\tau_0 + \theta(\tau_1)) \right) \\
- R(\tau_1) \sin(\tau_0 + \theta(\tau_1)) \left( R^2(\tau_1) \cos^2(\tau_0 + \theta(\tau_1)) - 1 \right). 
\] (11.17)

In order to avoid resonant terms on the right-hand side which lead to secular terms in the solution, it is necessary to remove the linear terms \( \cos(\tau_0 + \theta(\tau_1)) \) and \( \sin(\tau_0 + \theta(\tau_1)) \) from the equation. Equation (11.17) then becomes

\[
\frac{\partial^2 x_1}{\partial \tau_0^2} + x_1 = \left\{ -2 \frac{dR}{d\tau_1} + R - \frac{R^3}{4} \right\} \sin(\tau_0 + \theta(\tau_1)) \\
\left\{ -2R \frac{d\theta}{d\tau_1} \right\} \cos(\tau_0 + \theta(\tau_1)) - \frac{R^3}{4} \sin(3\tau_0 + 3\theta(\tau_1)).
\]

To avoid secular terms, set

\[
-2 \frac{dR}{d\tau_1} + R - \frac{R^3}{4} = 0 \quad \text{and} \quad \frac{d\theta}{d\tau_1} = 0. 
\] (11.18)

The initial conditions are \( x_0(0,0) = a \) and \( \frac{dx_0}{dx_0} = 0 \) leading to \( \theta(0) = 0 \) and \( R(0) = \frac{q}{\pi} \). The solutions to system (11.18) with these initial conditions are easily computed with MATLAB; thus

\[
R(\tau_1) = \frac{2}{\sqrt{1 + \left( \frac{4}{a^2} - 1 \right) e^{-\tau_1}}}, \quad \text{and} \quad \theta(\tau_1) = 0.
\]

Therefore, the uniformly valid one-term solution is

\[
x_0(\tau_0, \tau_1) = \frac{2 \cos(\tau_0)}{\sqrt{1 + \left( \frac{4}{a^2} - 1 \right) e^{-\tau_1}}} + O(\epsilon)
\]
or
\[
x(t) = \frac{2 \cos(t)}{\sqrt{1 + \left( \frac{4}{a^2} - 1 \right) e^{-\epsilon t}}} + O(\epsilon).
\]

As \( t \to \infty \), the solution tends asymptotically to the limit cycle \( x = 2 \cos(t) + O(\epsilon) \), for all initial conditions. Notice that only the initial condition \( a = 2 \) gives a periodic solution.

Figure 11.13 shows the error between the numerical solution and the one-term multiple scale approximation, say, \( x_{MS} \), when \( \epsilon = 0.01 \) and \( x(0) = 1, \dot{x}(0) = 0 \).
11.5 MATLAB Commands

See Chap. 9 for help with plotting phase portraits.

% Programs 11a - Phase portrait (Fig. 11.2).
% Limit cycle of a van der Pol system.
% IMPORTANT - vectorfield.m must be in the same
% folder.
clear
hold on
sys=@(t,x) [x(2);-x(1)-5*x(2)*((x(1))^2-1)];
vectorfield(sys,-3:.3:3,-10:1.3:10);
[t,xs] = ode45(sys,[0 30],[2 1]);
plot(xs(:,1),xs(:,2))
hold off
axis([-3 3 -10 10])
fsi=15;
set(gca,‘xtick’,-3:1:3,’FontSize’,fsi)
set(gca,‘ytick’,-10:5:10,’FontSize’,fsi)
xlabel(‘x(t)’,‘FontSize’,fsi)
ylabel(‘y(t)’,‘FontSize’,fsi)
hold off

% Programs 11b - Phase portrait (Fig. 11.8).
% Non-convex limit cycle of a Lienard system.
clear
hold on
sys1=@(t,x) [x(2);-x(1)-x(2)*(90*(x(1))^2-882*(x(1))^4+2598.4*(x(1))^6
-3359.997*(x(1))^8+2133.34*(x(1))^10-651.638*(x(1))^12+76.38*(x(1))^14)];
[t,xt] = ode45(sys1,[0 30],[1.4 0]);
plot(xt(:,1),xt(:,2))
hold off
11.6 Exercises

1. Prove that the system
\[ \dot{x} = y + x \left( \frac{1}{2} - x^2 - y^2 \right), \quad \dot{y} = -x + y \left( 1 - x^2 - y^2 \right) \]
has a stable limit cycle. Plot the limit cycle.

2. By considering the flow across the square with coordinates \((1,1), (1,-1), (-1,-1), (-1,1)\), centered at the origin, prove that the system
\[ \dot{x} = -y + x \cos(\pi x), \quad \dot{y} = x - y^3 \]
has a stable limit cycle. Plot the vector field, limit cycle, and square.

3. Prove that the system
\[ \dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3 \]
has a unique limit cycle.

4. Prove that the system.
\[ \dot{x} = y + x(\alpha - x^2 - y^2), \quad \dot{y} = -x + y(1 - x^2 - y^2), \]
where \(0 < \alpha < 1\), has a limit cycle and determine its stability.

5. For which parameter values does the Holling–Tanner model
\[ \dot{x} = x \beta \left( 1 - \frac{x}{k} \right) - \frac{rxy}{(a+ax)}, \quad \dot{y} = by \left( 1 - \frac{Ny}{x} \right) \]
have a limit cycle?

6. Plot phase portraits for the Liénard system
\[ \dot{x} = y - \mu(-x + x^3), \quad \dot{y} = -x, \]
when (a) \(\mu = 0.01\) and (b) \(\mu = 10\).
7. Prove that none of the following systems have limit cycles:

(a) \( \dot{x} = y, \quad \dot{y} = -x - (1 + x^2 + x^4)y \)
(b) \( \dot{x} = x - x^2 + 2y^2, \quad \dot{y} = y(x + 1) \)
(c) \( \dot{x} = y^2 - 2x, \quad \dot{y} = 3 - 4y - 2x^2y \)
(d) \( \dot{x} = -x + y^3 - y^4, \quad \dot{y} = 1 - 2y - x^2y + x^4 \)
(e) \( \dot{x} = x^2 - y - 1, \quad \dot{y} = y(x - 2) \)
(f) \( \dot{x} = x - y^2(1 + x^3), \quad \dot{y} = x^5 - y \)
(g) \( \dot{x} = 4x - 2x^2 - y^2, \quad \dot{y} = x(1 + xy) \)

8. Prove that neither of the following systems have limit cycles using the given multipliers:

(a) \( \dot{x} = x(4 + 5x + 2y), \quad \dot{y} = y(-2 + 7x + 3y), \quad \psi = \frac{1}{xy^2} \)
(b) \( \dot{x} = x(\beta - \delta x - \gamma y), \quad \dot{y} = y(b - dy - cx), \quad \psi = \frac{1}{xy} \)

In case (b), prove that there are no limit cycles in the first quadrant only. These differential equations were used as a general model for competing species in Chap. 10.

9. Use the Lindstedt–Poincaré technique to obtain a one-term uniform expansion for the ODE

\[
\frac{d^2 x}{dt^2} + x = \epsilon x \left(1 - \left(\frac{dx}{dt}\right)^2\right),
\]

with initial conditions \( x(0) = a \) and \( \dot{x}(0) = 0 \).

10. Using the method of multiple scales, show that the one-term uniform valid expansion of the ODE

\[
\frac{d^2 x}{dt^2} + x = -\epsilon \frac{dx}{dt},
\]

with initial conditions \( x(0) = b, \dot{x}(0) = 0 \) is

\[
x(t, \epsilon) \sim x_{MS} = be^{-\frac{t}{\epsilon^2}} \cos(t),
\]

as \( \epsilon \to 0 \).

References

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