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ABSTRACT
We demonstrate the appearance of explosions in three quantities in interest rate models with log-normally distributed rates in discrete time. (1) The expectation of the money market account in the Black, Derman, Toy model, (2) the prices of Eurodollar futures contracts in a model with log-normally distributed rates in the terminal measure and (3) the prices of Eurodollar futures contracts in the one-factor log-normal Libor market model (LMM). We derive exact upper and lower bounds on the prices and on the standard deviation of the Monte Carlo pricing of Eurodollar futures in the one factor log-normal Libor market model. These bounds explode at a non-zero value of volatility, and thus imply a limitation on the applicability of the LMM and on its Monte Carlo simulation to sufficiently low volatilities.

1. Introduction
Interest rate models with log-normally distributed rates in continuous time are known to display singular behaviour. The simplest setting where this phenomenon appears is for log-normal short-rate models such as the Dothan model (Dothan 1978) and the Black–Karasinski model (Black and Karasinski 1991). It was shown by Hogan and Weintraub (1997) (see also Sandmann, Sondermann 1997; Andersen and Piterbarg 2007) that the Eurodollar futures prices in these models are divergent. Similar explosions appear in Heath, Jarrow, and Morton (HJM) model with log-normal volatility specification (Heath, Jarrow, and Morton 1992; Morton 1988), where the forward rates explode with unit probability. The case of these models is somewhat special, as the Eurodollar futures prices are well-behaved in other interest rate models of practical interest, such as the CIR (Cox, Ingersoll and Ross 1981), and the Hull-White model (Kirikos and Novak 1997; Henrard 2005).

It is widely believed that the same models when considered in discrete time are free of divergences see (Rebonato 2002, pp. 14–16) for an account of the historical development of the log-normal interest rate models. The discrete time version of the Dothan model is the Black, Derman, Toy (BDT) model (Black, Derman, and Toy 1990), while the Black–Karasinski model can be simulated both in discrete (Black and Karasinski 1991) and continuous time (Brigo and Mercurio 2006).
In this article, we demonstrate the appearance of numerical explosions for several quantities in interest rate models with log-normally distributed rates in discrete time. The explosions appear at a finite critical value of the rate volatility. This phenomenon is shown to appear for accrual quantities such as the money market account and the Eurodollar futures prices. The quantities considered remain finite but their numerical values grow very fast above the critical volatility such that they rapidly exceed machine precision. Thus for all practical purposes, they can be considered as real explosions, and their appearance introduces limitations on the use of the models for the particular application considered.

In Section 2, we consider the expectation of the money market account in a discrete time short rate model with rates following a geometric Brownian motion (BDT model). Using an exact solution one can show the appearance of a numerical explosion for this quantity, for sufficiently large number of time steps or volatility. This phenomenon and the conditions under which it appears have been studied in detail elsewhere (Pirjol and Zhu 2015). Here, we review the main conclusions of this study and point out its implications for the simulation of interest rate models with log-normally distributed rates in discrete time. Sections 3 and 4 consider the calculation of the Eurodollar futures prices in two interest rate models: a one-factor model with log-normally distributed rates in the terminal measure, and the one-factor log-normal Libor market model, respectively. The Eurodollar futures convexity adjustment is computed exactly in the former model, while for the latter we derive exact upper and lower bounds. Both the exact result and the bounds display numerical explosive behaviour for sufficiently large volatilities, which are of the order of typical market volatilities. These explosions limit the applicability of these models for the pricing of Eurodollar futures to sufficiently small values of the volatility.

2. The expectation of the money market account in the BDT model

We start by considering the BDT model (Black, Derman, and Toy 1990). The model is defined on a tenor of dates \( t_i \) with \( i = 0, 1, \ldots, n \), which are assumed to be uniformly spaced with time step \( \tau = t_{i+1} - t_i \). The model is defined in the spot measure \( \mathbb{Q} \) with numeraire the money market account with discrete time compounding:

\[
B_i = \prod_{k=0}^{i-1} (1 + L_k \tau).
\]  

(2.1)

We denoted here \( L_k := L_{k,k+1} \) the Libor rate for the period \( (t_k, t_{k+1}) \).

The BDT model is defined by the following distributional assumption for the Libors \( L_i \) in the spot measure:

\[
L_i = \hat{L}_i e^{\sigma_i W_i - \frac{1}{2} \sigma_i^2 \tau},
\]

(2.2)

where \( W_i \) is a standard Brownian motion in the spot measure sampled at the discrete times \( t_i \). \( \hat{L}_i \) are constants, which are determined by calibration to the initial yield curve (defined by the zero coupon bonds \( P_{0, i} := P_{0, t_i} \)), and \( \sigma_i \) are the rate volatilities, which are calibrated such that the model reproduces a given set of volatility instruments such as caplets or swaptions.
For given initial yield curve $P_{0,i}$ and rate volatilities $\sigma_i$, the calibration problem consists in finding $L_i$ such that $P_{0,i} = \mathbb{E}_Q[B_i^{\tau}]$ for all $1 < i \leq n$. The solution for $L_i$ exists provided that the following condition is satisfied $P_{0,i} > P_{0,i+1} > 0$ (Sandmann and Sondermann 1993). The solution to the calibration problem satisfies the inequality $L_i > L_i^{\text{wd}}$, where $L_i^{\text{wd}} = 1/\tau(P_{0,i}/P_{0,i+1-1})$ are the forward rates for the period $(t_i, t_{i+1})$.

In the zero volatility limit $\sigma = 0$, the money market account $B_n$ is given by a simple geometric progression $B_n|_{\sigma=0} = \prod_{k=0}^{n-1} (1 + L_k^{\text{wd}} \tau)^n$. For $\sigma > 0$, $B_n$ becomes a (positive defined) random variable. The average value and the higher positive integer moments of $B_n$ can be computed exactly using a recursion relation similar to that used in Section 5.1 of (Pirjol 2015) to compute a related quantity (denoted $\beta_i^{(p)}(W_i)$ in the proof of Proposition 2.1). The proof of this result is given in the Appendix.

Proposition 2.1. The money market account in the BDT model is

$$B_n = \prod_{k=0}^{n-1} \left( 1 + \bar{L}_k \tau e^{\sigma W_k - \frac{1}{2} \sigma^2 t_k} \right). \quad (2.3)$$

The $p$th moment of $B_n$ is given by

$$\mathbb{E}_Q[B_n^p] = (1 + \rho_0)^p \sum_{k=0}^p b_k^{(0,p)}, \quad (2.4)$$

where the coefficients $b_k^{(0,p)}$ are found by solving the backwards recursion:

$$b_k^{(i,p)} = b_k^{(i+1,p)} + \sum_{m=1}^p \binom{p}{m} b_{k-m}^{(i+1,p)} \rho_{i+1}^m e^{m(k-\frac{1}{2}m-\frac{1}{2})\sigma^2 t_{i+1}}, \quad (2.5)$$

with $\rho_k := \bar{L}_k \tau$ and initial conditions

$$b_0^{(n-1,p)} = 1, \quad b_k^{(n-1,p)} = 0, \quad k > 1. \quad (2.6)$$

The coefficients $b_k^{(i,p)}$ with negative indices $k < 0$ are zero.

### 2.1. Discrete time moment explosion of the money market account

Using the exact results from Proposition 2.1, one can study the dependence of the moments of the money market account $\mathbb{E}_Q[B_n^p]$ on $n$, $\sigma$ and $\tau$. We assume uniform parameters $\bar{L}_k = L_0$, $\sigma_k = \sigma$.

For the convenience of the reader, we give the explicit results for the first two moments of $B_n$, following from Proposition 2.1. They are given by

$$\mathbb{E}_Q[B_n] = (1 + \bar{L}_0 \tau) \sum_{j=0}^{n-1} c_j^{(0)}, \quad (2.7)$$

where

$$c_j^{(0)} = \frac{1}{(1 + \rho_0)^2} \sum_{k=0}^j \frac{b_k^{(0,2)}}{\rho_k} e^{\sigma W_k - \frac{1}{2} \sigma^2 t_k}.$$
\[ \mathbb{E}_Q[B_n^2] = (1 + L_0 \tau)^2 \sum_{j=0}^{2(n-1)} d_j^{(0)}, \]  

(2.8)

where \( c_j^{(i)} \) and \( d_j^{(i)} \) are the solutions to the backwards recursions:

\[ c_j^{(i)} = c_{j+1}^{(i+1)} + \bar{L}_{i+1} \tau c_{j-1}^{(i+1)} e^{\sigma^2(j-1)t_{i+1}}, \]

(2.9)

\[ d_j^{(i)} = d_{j+1}^{(i+1)} + 2\bar{L}_{i+1} \tau d_{j-1}^{(i+1)} e^{\sigma^2(j-1)t_{i+1}} + (\bar{L}_{i+1} \tau)^2 d_{j-2}^{(i+1)} e^{\sigma^2(2j-3)t_{i+1}}, \]

(2.10)

with initial conditions \( c_0^{(n-1)} = 1, d_0^{(n-1)} = 1 \) and all other coefficients \( c_k^{(n-1)} = d_k^{(n-1)} = 0 \).

The coefficients \( c_j^{(i)}, d_j^{(i)} \) with \( j < 0 \) are zero.

Figure 1. shows typical plots of the expectation \( \mathbb{E}_Q[B_n] \) as function of \( n \) at fixed \( \sigma, L_0, \tau \). The results of this numerical study show that the expectation of the money market account \( \mathbb{E}_Q[B_n] \) has an explosive behaviour at a certain time step \( n \). Although its numerical value remains finite, in the explosive phase this quantity grows very fast and can quickly exceed double precision in a finite time. The same phenomenon is observed by keeping fixed \( L_0, \tau, n \) and considering \( \mathbb{E}_Q[B_n] \) as function of the volatility \( \sigma \), and also at fixed \( L_0, n \tau, \sigma \) and making the time step \( \tau \) sufficiently small. A similar explosion is observed also for the higher integer moments \( p > 1 \). We show in Table 1. typical results for the average, the second and fourth moments of the money market account in a simulation with time step \( \tau = 1 \).

The conditions under which the explosion occurs have been derived in (Pirjol and Zhu 2015) under the assumption of constant coefficients \( \bar{L}_k = L_0 \) and uniform volatilities \( \sigma_k = \sigma \). This article considered the distributional properties of a random variable \( x_t \) satisfying the linear stochastic recursion \( x_{t+1} = (1 + \rho e^{\sigma W_{t+1}-\frac{1}{2} \sigma^2 t})x_t \), with \( \rho, \sigma > 0 \) real parameters and \( W_t \) is a standard Brownian motion sampled on the uniformly spaced times \( t_i \). This is identical to the discrete time process followed by the money market account \( B_t \), with the identification \( \rho = L_0 \tau \).

The properties of the moments \( \mathbb{E}_Q[B_n^q] \) with \( q \in \mathbb{N} \) simplify in the limit \( n \to \infty \) of a very large number of time steps. Using large deviations theory methods (Dembo and Zeitouni 1998; Varadhan 1984) one can prove the following result. Similar results are

<table>
<thead>
<tr>
<th>( n )</th>
<th>( (1 + L_0 \tau)^n )</th>
<th>( \mathbb{E}_Q[B_n] )</th>
<th>( \mathbb{E}_Q[B_n^2] )</th>
<th>( \mathbb{E}_Q[B_n^4] )</th>
<th>( \mathbb{E}<em>Q[B_n] )</em>{MC}</th>
<th>( \Sigma_{MC} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.05</td>
<td>1.05</td>
<td>1.10</td>
<td>1.22</td>
<td>1.05</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1.28</td>
<td>1.28</td>
<td>1.63</td>
<td>2.68</td>
<td>1.28</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>1.63</td>
<td>1.64</td>
<td>2.70</td>
<td>7.59</td>
<td>1.63</td>
<td>1.4 \times 10^{-4}</td>
</tr>
<tr>
<td>20</td>
<td>2.65</td>
<td>2.75</td>
<td>8.25</td>
<td>5.3 \times 10^3</td>
<td>2.73</td>
<td>7.6 \times 10^{-4}</td>
</tr>
<tr>
<td>30</td>
<td>4.32</td>
<td>4.95</td>
<td>2.93 \times 10^{11}</td>
<td>1.8 \times 10^{17}</td>
<td>4.88</td>
<td>0.004</td>
</tr>
<tr>
<td>40</td>
<td>7.04</td>
<td>184</td>
<td>3.96 \times 10^{50}</td>
<td>2.1 \times 10^{558}</td>
<td>10.48</td>
<td>0.048</td>
</tr>
<tr>
<td>41</td>
<td>7.39</td>
<td>11,871</td>
<td>1.61 \times 10^{102}</td>
<td>1.0 \times 10^{571}</td>
<td>11.59</td>
<td>0.07</td>
</tr>
<tr>
<td>42</td>
<td>7.76</td>
<td>1,424,670</td>
<td>3.54 \times 10^{114}</td>
<td>3.6 \times 10^{666}</td>
<td>12.91</td>
<td>0.10</td>
</tr>
<tr>
<td>43</td>
<td>8.15</td>
<td>2.89 \times 10^8</td>
<td>4.32 \times 10^{127}</td>
<td>1.2 \times 10^{725}</td>
<td>14.52</td>
<td>0.15</td>
</tr>
<tr>
<td>44</td>
<td>8.56</td>
<td>9.76 \times 10^9</td>
<td>3.05 \times 10^{141}</td>
<td>3.9 \times 10^{786}</td>
<td>16.51</td>
<td>0.23</td>
</tr>
<tr>
<td>45</td>
<td>8.99</td>
<td>5.45 \times 10^{10}</td>
<td>1.29 \times 10^{156}</td>
<td>1.6 \times 10^{851}</td>
<td>19.04</td>
<td>0.35</td>
</tr>
</tbody>
</table>

The last two columns show the Monte Carlo estimates of the average \( \mathbb{E}_Q[B_n]_{MC} \) and standard deviation \( \Sigma_{B_n, N} \) in a simulation with \( N = 10^6 \) paths.
obtained in other probability problems, such as the exponential random graph model (Aristoff and Zhu 2014; Radin and Yin, 2013).

Theorem 2.2 ((Pirjol and Zhu 2015)). The limit

$$\lim_{n \to \infty} \frac{1}{\sigma^2 t_n n} \log \mathbb{E}_{Q}[B_{n}^q] = \lambda(\rho, \beta; q), \quad \text{with } \rho := L_0 \tau,$$

with $q \in \mathbb{N}$ exists, and depends only on $\rho$ and $\beta$. The function $\lambda(\rho, \beta; q)$ is the Lyapunov exponent of the positive integer $q$th moment, and is related to the Lyapunov exponent of the first moment $\lambda(\rho, \beta; 1) := \lambda(\rho, \beta)$ as $\lambda(\rho, \beta; q) = q \lambda(\rho, q \beta)$. The function $\lambda(\rho, \beta)$ is given by

$$\lambda(\rho, \beta) = \sup_{d\in(0,1)} \Lambda(d),$$

$$\Lambda(d) = \beta d^2 + \log(1 + \rho) - 2\beta(1 + \rho) d^3 \int_{0}^{1} dy \frac{y^2}{1 + \rho - e^{\beta d(y^2 - 1)}}.$$

We show in Figure 2. (left panel) typical plots of $\lambda(\rho, \beta)$ versus $\beta$ for several values of $\rho$. The function $\lambda(\rho, \beta)$ is everywhere continuous in its arguments $(\rho, \beta)$, but has a discontinuous derivative $\partial_\beta \lambda(\rho, \beta)$ at a certain point $\beta_{cr}(\rho)$ for $\rho$ below a critical value $\rho < \rho_c = 0.123$. Figure 2. (right panel) shows the critical curve $\beta_{cr}(\rho)$.

The critical curve $\beta_{cr}(\rho)$ is well approximated as

$$\tilde{\beta}_{cr}(\rho) = -3 \log \rho, \quad \rho < \rho_c^{MF} = e^{-2}.$$

This approximation of the critical curve ends at the critical point $(\rho_c, \tilde{\beta}_c) = (e^{-2}, 6)$, and it is shown in Figure 2. (right panel) as the dotted blue curve.

The function $\lambda(\rho, \beta)$ has a very rapid increase for $\beta > \beta_{cr}(\rho)$. This is observed as a rapid growth of the expectation $\mathbb{E}_{Q}[B_n]$. One has thus the following criterion for the numerical explosion of this quantity. The numerical explosion of $\mathbb{E}_{Q}[B_n]$ occurs when the scaling parameters:

$$\rho := L_0 \tau, \beta := \frac{1}{2} \sigma^2 t_n n$$

Figure 1. The expectation of the money market account $\mathbb{E}_{Q}[B_n]$ (left) and of the second moment $\mathbb{E}_{Q}[B_{n}^2]$ (right) vs. $n$ assuming yearly compounding $\tau = 1, L_0 = 2.5\%$ (red - solid lower curve) and $5.0\%$ (blue - solid upper curve), and the rate volatility is $\sigma = 10\%$. The dashed curves show the result in the zero rate volatility limit $B_n = (1 + L_0 \tau)^n$. 
cross the phase transition curve $\beta_{\text{cr}}(\rho)$ in Figure 2 (right) (the black curve) in the downward direction. The explosion appears only for sufficiently small values of the $\rho$ parameter, below the critical value $\rho < \rho_c = 0.123$.

A simple estimate of the explosion value of the volatility $\sigma_{\text{exp}}$ at fixed $L_0, \tau, t_n$ can be obtained from the simpler approximation of the critical curve given in Equation (2.13). As seen from Figure 2 (right panel), this approximation is an upper bound for $\beta_{\text{cr}}(\rho)$. This gives an upper bound on the explosion volatility:

$$\sigma_{\text{exp}}^2 t_n \leq -\frac{6}{n} \log(L_0 \tau), \quad L_0 \tau < e^{-2}. \quad (2.15)$$

The relation (2.15) can be used to find also the explosion time step $n_{\text{exp}}$ for given $L_0, \tau, \sigma$. For the example considered in Table 1, this gives $n_{\text{exp}} \leq 43$ which is close to the actual explosion time step ($n_{\text{exp}} = 40$).

Taking the time step to be very small $\tau \to 0$ at fixed maturity $t_n$, the relation (2.15) gives that the explosion value of the $\sigma$ parameter approaches zero. Alternatively, the time step $n_{\text{exp}}$ at which the explosion occurs goes to infinity as $\tau \to 0$, but in such a way that the explosion time $t_{\text{exp}} = \tau n_{\text{exp}}$ goes to zero. This recovers the known result that the expected value of the money market account explodes for arbitrarily small time in the continuous time for models with log-normal short rates (Andersen and Piterbarg 2010, 2007; Poulsen 1999; Sandmann and Sondermann 1997). For completeness we summarize in Section 2.3, the main known results for the distributional properties of the money market account in the Dothan model in continuous time.

### 2.2. Implications for Monte Carlo simulations

The moment explosion of the money market account has implications for the Monte Carlo simulation of the process $B_t$. Consider the MC estimate for the expectation

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**Figure 2.** (Left) Plots of the Lyapunov exponent $\lambda(\rho, \beta)$ vs. $\beta$ for $\rho = 0.025, 0.05, 0.125, 0.2$ (from bottom to top) (Pirjol and Zhu 2015). The dashed curves give upper and lower bounds $\frac{1}{2} \beta + \log(1 + \rho) \geq \lambda(\rho, \beta) \geq \frac{1}{2} \beta + \log \rho$, see Proposition 2 in (Pirjol and Zhu 2015). (Right) The transition curve $\beta_{\text{cr}}(\rho)$ (solid black), and its simpler approximation $\beta_{\text{cr}}(\rho) = -3 \log \rho$ (dashed blue) ending at $\beta_{\text{c}} = 6, \rho_{\text{c}} = e^2$ (blue dot).
\( \mathbb{E}_Q[B_n] \) obtained by averaging over \( N \) paths. The standard deviation of this estimate is related to the variance of \( B_n \) as, see e.g., (Glasserman 1999),

\[
\Sigma_{B_n,N} = \frac{1}{\sqrt{N}} \sqrt{\text{var} B_n}.
\]  

(2.16)

The explosion of the second moment \( \mathbb{E}_Q[B_n^2] \) implies that the variance of \( B_n \) grows very fast even as the average value \( \mathbb{E}_Q[B_n] \) is well behaved. This is seen in practical MC simulations as a rapid increase in the variance of the sample, but we will show next that a reliable estimate of this variance using the MC sample is problematic.

Consider as an example the numerical study in Table 1. Let us examine the numerical evaluation of the first moment \( \mathbb{E}_Q[B_n] \) at time step \( n = 40 \) using a Monte Carlo simulation with \( N \) samples. The standard deviation of such a determination can be estimated using the second moment \( \mathbb{E}_Q[B_n^2] \) in the table, and is

\[
\Sigma_{B_{40},N} = 2.0 \times 10^{45} \frac{1}{\sqrt{N}}.
\]

(2.17)

This implies that in order to compute the average of \( B_{40} \) to 10% relative error, one needs \( N = [2.0 \times 10^{45}/18.4]^2 \approx 1.2 \times 10^{88} \) MC paths. In the last two columns of Table 1, we show also the results of a MC calculation of the average \( \mathbb{E}_Q[B_n] \) with \( N = 10^6 \) paths. We note that the MC estimate of the standard deviation \( \Sigma_{B_{40},N} = 0.048 \) obtained from the sample of \( N = 10^6 \) paths is much smaller than the exact result (2.17). This is related in turn to the explosion of the fourth moment \( \mathbb{E}_Q[B_n^4] \), which determines the error of the determination of the second moment as \( \Delta \mathbb{E}_Q[B_n^2] = \sqrt{\text{var}(B_n^2)/\sqrt{N}} \approx 1.4 \times 10^{279}/\sqrt{N} \). This is a very large number for all realistic values of \( N \), which implies that the estimate of the variance of \( B_n \) from the sample considered is unreliable.

This example shows that Monte Carlo simulation methods can not be used to compute precisely the expectation and higher moments of \( B_n \) in the explosive phase. The same phenomenon will be seen to appear in several other quantities in models with log-normally distributed rates, and introduces a limitation in the applicability of MC methods for computing these quantities.

### 2.3. Continuous time limit and relation to the Hogan–Weintraub singularity

In the continuous time limit, the BDT model with constant volatility \( \sigma_i = \sigma \) goes over into a short rate model with process for the short rate:

\[
dr_t = \sigma r_t dW_t + \mu(t)r_t dt.
\]

(2.18)

The money market account \( B_t \) is given by

\[
dB_t = r_t B_t dt,
\]

(2.19)

with initial condition \( B_0 = 1 \). The short rate \( r_t \) is given by

\[
r_t = r_0 e^{\sigma W_t} + \int_0^t ds \mu(s) - \frac{1}{2} \sigma^2 t.
\]

(2.20)
The solution of Equation (2.19) is given by the exponential of the time integral of the geometric Brownian motion

\[ B_t = \exp \left( r_0 \int_0^t ds \sigma W_s + \int_0^s \mu(u) du - \frac{1}{2} \sigma^2 s \right). \] (2.21)

The expectation of \( B_t \) (as well as all higher moments of \( B_t \)) is infinite, for any \( t > 0 \). This follows by noting that the time integral of the geometric Brownian motion is bounded from below by a log-normally distributed random variable, by the arithmetic-geometric means inequality

\[
\frac{1}{t} \int_0^t ds \sigma W_s + \int_0^t \mu(u) du - \frac{1}{2} \sigma^2 s \geq \exp \left( \frac{1}{t} \int_0^t ds (\sigma W_s + \int_0^s \mu(u) du - \frac{1}{2} \sigma^2 s) \right). \] (2.22)

The expectation of the exponential of the quantity on the right-hand side is infinite. This follows from the well-known result that the moment generating function \( E[e^{\theta X}] \) of a log-normally distributed random variable \( X \) is infinite for \( \theta > 0 \).

These results are well known in the financial mathematics literature (Andersen and Piterbarg 2010, 2007; Poulsen 1999; Sandmann and Sondermann 1997). Our results show that the approach of the discrete time model to the continuous time limit is not smooth, but proceeds through a discontinuity at some value of the time step size \( \tau \) where the rate of growth of \( E_Q[B_t] \) has a sudden increase. This is observed in simulations as numerical moment and path explosions.

The explosion of the expectation of \( B_t \) is related to (but does not imply) the Hogan–Weintraub singularity (Andersen and Piterbarg 2010, 2007; Hogan and Weintraub 1997). We discuss briefly this point, comparing the results for the discrete and continuous time settings. As mentioned, it is known (Hogan and Weintraub 1997) that the expectation of the inverse zero coupon bond in the risk-neutral measure is infinite in the Dothan and Black, Karasinski models in continuous time:

\[ E_Q[P^{-1}(T, T + \delta)|\mathcal{F}_t] = +\infty, \] (2.23)

for any \( T > t \). This implies that such models can not price Eurodollar futures.

The expectation in Equation (2.23) is bounded from above by the expectation of the money market account:

\[ E_Q[P^{-1}(T, T + \delta)|\mathcal{F}_t] \leq E_Q \left[ e^{\int_T^{T+\delta} r_s ds}|\mathcal{F}_T \right]. \] (2.24)

This follows by an application of the Jensen inequality (Sandmann and Sondermann 1997)

\[ P(T, T + \delta) = E_Q \left[ e^{-\int_T^{T+\delta} r_s ds}|\mathcal{F}_T \right] \geq E_Q^{-1} \left[ e^{\int_T^{T+\delta} r_s ds}|\mathcal{F}_T \right]. \] (2.25)

Taking the expectation of the inverse of both sides of this inequality, and using the tower property of conditional expectations, gives Equation (2.24). One can see that this gives an upper bound on the expectation in Equation (2.23), so its explosion is only a necessary but not a sufficient condition for the appearance of the Hogan–Weintraub
singularity. See Poulsen (1999) for a detailed and clear discussion. On the other hand, the inequality (2.24) can be used to prove the finiteness of the Eurodollar futures prices provided that the upper bound (Equation (2.24)) is finite. This approach was used in (Sandmann and Sondermann 1997) to prove such a result for the Sandmann–Sondermann modification of the Dothan model which assumes log-normality of the effective rate \( r_{e,t} = \log(1 + r_t) \).

One may ask if the Hogan–Weintraub divergence persists also in the discrete time version of the Dothan model, which is the BDT model. This question has been studied in detail in Pirjol (2015), and the main result is that the expectation \( \mathbb{E}_Q[P_{i,j}^{-1}] \) is finite, but explodes to very large and unphysical values at a certain critical value of the volatility \( \sigma \), or for sufficiently small simulation time step \( \tau \).

The analogue of the bound (2.24) for the discrete time BDT model is

\[
\mathbb{E}_Q[P_{i,j}^{-1}] \leq \mathbb{E}_Q \left[ \prod_{k=1}^{j-1} (1 + L_k \tau) \right] = \mathbb{E}_Q \left[ \frac{1}{B_i} B_j \right]. \tag{2.26}
\]

This follows by an application of the Jensen inequality to the expectation giving the zero coupon bond prices:

\[
P_{i,j}(W_i) = \mathbb{E}_Q \left[ \frac{1}{\prod_{k=1}^{j-1} (1 + L_k \tau)} |F_i| \right]. \tag{2.27}
\]

The bound (2.26) shows that the expectation \( \mathbb{E}_Q[P_{i,j}^{-1}] \) is clearly finite. The expectation appearing in the upper bound can be computed exactly, as shown in Section 5.1 of Pirjol (2015). The result has the same explosive behaviour as noted above for the average of the money market account. This explosion does not imply also an explosion of the expectation \( \mathbb{E}_Q[P_{i,j}^{-1}] \), as it is only a necessary, but not sufficient condition for the explosion of the latter quantity. The volatility \( \sigma \) at which the upper bound (2.26) has an explosion gives a lower bound on the explosion volatility for the Eurodollar futures prices in the BDT model. This bound was discussed in detail in Pirjol (2015).

3. Eurodollar futures in a model with log-normal rates in the terminal measure

Consider a one-factor short rate model defined on the tenor of dates \( \{t_0, t_1, \ldots, t_n\} \). The rate specification is

\[
L_{i,i+1} = \tilde{L}_i e^{\sigma_i W_i - \frac{\sigma_i^2}{2} t_i}, \tag{3.1}
\]

where \( W_i \) is a standard Brownian motion in the \( t_n - \) forward measure \( \mathbb{P}_n \) with numeraire the zero coupon bond \( P_{t, t_n} \). The coefficients \( \tilde{L}_i \) are determined by yield curve calibration such that the initial yield curve \( P_{0,i} \) is correctly reproduced.

This model is used in financial practice as a log-normal approximation to the log-normal Libor market model (Daniluk and Gatarek 2005; Kurbanmuradov, Sabelfeld, and Schoenmakers 2002) or as a parametric representation of the Markov functional model (Balland and Hughston 2000; Hunt and Kennedy 2005; Hunt, Kennedy, and
Consider the Eurodollar futures contract on the rate \( L_{i,i+1} \). Assuming discrete futures settlement at dates \( t_i \), the pricing of this instrument is related to the expectation of \( L_{i,i+1} \) in the spot measure \( \mathbb{Q} \) (Hunt and Kennedy 2005; Pozdnyakov and Steele, 2004). This can be expressed alternatively as an expectation in the terminal measure \( \mathbb{P}_n \):

\[
\mathbb{E}_Q[L_{i,i+1}] = P_{0,n} \mathbb{E}_n[B_i L_{i,i+1} P_{i,n}^{-1}] = P_{0,n} \mathbb{E}_n[B_i L_{i,i+1} \hat{P}_{i,i+1}(1 + L_{i,i+1})],
\]

(3.2)

where \( B_i = \Pi_{k=0}^{t_i-1}(1 + L_k \tau) \) is the money market account at time \( t_i \), and we denoted \( \hat{P}_{i,j} = P_{i,j}/P_{i,n} \) the numeraire-rebased zero coupon bonds.

The expectation (3.2) can be computed exactly in the particular case of uniform volatility \( \sigma_i = \sigma \). This can be done using a simple modification of the recursion relation in Proposition 2.1, and is given by the following result.

Proposition 3.1. Consider the expectation

\[
M_n^{(q)} = \mathbb{E}[\Pi_{k=0}^{t_i-1}(1 + r_k e^{\sigma W_k - \frac{1}{2} \sigma^2 t_k}) e^{q \sigma W_n - \frac{1}{2} (q \sigma)^2 t_n}],
\]

(3.3)

where \( r_k, \sigma \) are real positive numbers, and \( W_i \) is a standard Brownian motion started at zero \( W_0 = 0 \) and sampled at times \( t_k \). This expectation is given exactly by

\[
M_n^{(q)} = \sum_{p=q}^{n-1+q} c_p^{(0)},
\]

(3.4)

where \( c_p^{(0)} \) are given by the solution of the recursion relation

\[
c_p^{(i)} = c_p^{(i+1)} + r_{i+1} c_{p-1}^{(i+1)} e^{\sigma^2 (p-1) t_{i+1}},
\]

(3.5)

with the initial condition at \( i = n - 1 \)

\[
c_q^{(n-1)} = 1, \quad c_p^{(n-1)} = 0 \text{ for all } p \neq q.
\]

Proof. The proof is similar to that of Proposition 2.1 and is omitted.

3.1. Results

We consider the Eurodollar futures on the rate \( L_{n-1,n} \) spanned by the last time step \((t_{n-1}, t_n)\). The expectation of this rate in the terminal measure \( \mathbb{P}_n \) is simply the forward rate \( \hat{L}_{n-1,n} = L_{n-1,n}^{\text{twd}} \), since \( \mathbb{P}_n \) coincides with the forward measure for this rate. Also, we have \( \hat{P}_{n-1,n} = 1 \). The expression (3.2) simplifies to

\[
\mathbb{E}_Q[L_{n-1,n}] = P_{0,n} \mathbb{E}_n[B_{n-1} L_{n-1,n}(1 + L_{n-1,n} \tau)] = P_{0,n} \tau_{n-1,n}^{\text{twd}} \left( M_{n-1}^{(1)} + L_{n-1,n}^{\text{twd}} \tau M_{n-1}^{(2)} \right),
\]

(3.7)
where $M_{n-1}^{(1)}, M_{n-1}^{(2)}$ are given by Proposition 3.1 with the substitutions $r_k \rightarrow \tilde{L}_k \tau$. The multipliers $\tilde{L}_k$ are obtained from the yield curve calibration of the model to the forward Libors $L_{k}^{\text{fwd}}$. We computed them exactly using the analytical solution of the model given in (Pirjol 2013).

The Eurodollar futures convexity adjustment will be parameterized in terms of the ratio

$$\kappa_{\text{ED}} = \frac{M_{n-1}^{(1)} + \tau_{n-1} L_{n-1}^{(2)}}{(1 + L_0 \tau)^{n-1} (1 + L_{n-1}^{\text{fwd}} \tau)}. \quad (3.8)$$

This quantity is defined such that it is equal to one in the zero volatility $\sigma \rightarrow 0$ limit, and is a multiplicative measure of the convexity adjustment for the Eurodollar futures contract on $L_{n-1,n}$.

We show in Figure 3. plots of $\log \kappa_{\text{ED}}$ versus $\sigma$, for several values of the forward Libors $L_{fwd}$ and total tenor $n$. For the numerical simulation we assume for simplicity uniform forward Libors $L_{i}^{\text{fwd}} = L_0$, for $i = 0, 1, \ldots, n-1$. The numerical results for $\log \kappa_{\text{ED}}$ in Figure 3. show an explosive behaviour at a certain value of the volatility $\sigma$. Although the form of the expectation (3.7) is very similar to that of the money market account in the BDT model, there is also a difference as the multipliers $\tilde{L}_i$ are not equal. They are determined by the calibration to the initial yield curve $P_{0,i}$ and depend on the volatility $\sigma$. The multipliers $\tilde{L}_i$ satisfy the inequality $\tilde{L}_i < L_0$. For sufficiently large volatility they decrease rapidly and become very small above a certain critical volatility. This is due to an explosion in the expectations $N_i = E_n[P_{i,i+1} e^{\sigma W_i - \frac{1}{2} \sigma^2 t_i}]$ which appear in the calibration of the model, for sufficiently large volatility (Pirjol 2013). The explosive behaviour of the expectation (3.7) occurs despite this suppression of the coefficients $\tilde{L}_i$ due to calibration to the initial yield curve.

![Figure 3. The Eurodollar futures convexity adjustment $\log \kappa_{\text{ED}}$ for the rate $L_{n-1,n}$ in the model with log-normal rates in the terminal measure. The rate tenor and time discretization step is $\tau = 0.25$, and the forward rate is $L_0 = 1.0\%$ (black), $5.0\%$ (blue), $10.0\%$ (red) (from bottom to top). The total number of time steps is $n = 20$ (left) and $n = 40$ (right).](image)
4. Log-normal Libor market model

We consider in this section the pricing of Eurodollar futures in the one-factor log-normal Libor market model (Brace, Gatarek, and Musiela 1997; Miltersen, Sandmann, and Sondermann 1997; Jamshidian 1997, 2010). We assume the same tenor of dates \( \{ t_i \} , i = 0, 1, \ldots , n \) as in the previous section. We consider a market with given forward Libors \( L_i^{fwd} \) for the non-overlapping tenors \( ( t_i , t_{i+1} ) \) and log-normal caplet volatilities \( \sigma_j \).

The log-normal Libor model (Brace, Gatarek, and Musiela 1997; Miltersen, Sandmann, and Sondermann 1997; Jamshidian 1997) gives a possible solution for the dynamics of the forward Libor rates \( F_k(t) := F(t; t_k , t_{k+1}) , k = 1, 2, \ldots , n - 1 \) which is compatible with this market. Under the \( t_n \)-forward measure, with numeraire \( P_{t,n} \), the dynamics of the forward Libors \( F_k(t) , k = 1, 2, \ldots , n - 1 \) are

\[
\begin{align*}
\frac{dF_{n-1}(t)}{F_{n-1}(t)} &= \sigma_{n-1} dW_t, \\
\frac{dF_{n-2}(t)}{F_{n-2}(t)} &= \sigma_{n-2} dW_t - \sigma_{n-2} \frac{\tau \sigma_{n-1} F_{n-1}(t)}{1 + F_{n-1}(t) \tau} dt, \\
\vdots \\
\frac{dF_k(t)}{F_k(t)} &= \sigma_k dW_t - \sigma_k \sum_{j=k+1}^{n-1} \frac{\tau \sigma_j F_j(t)}{1 + F_j(t) \tau} dt,
\end{align*}
\]

with initial conditions \( F_i(0) = L_i^{fwd} \). Here \( W_t \) is a standard Brownian motion in the \( t_n \)-forward measure \( P_n \). We assumed here a one-factor version of the log-normal LMM, where all forward Libors are driven by a common Brownian motion \( W_t \). The model can be formulated in a more general form, which can accomodate an arbitrary correlation structure between the \( n \) Libor rates. Also, we assumed for simplicity time-independent volatilities \( \sigma_k \). Model (4.1) is the simplest dynamics of the forward Libors compatible with the given market of forward Libors and caplet volatilities.

The positivity of the forward rates \( F_k(t) > 0 \) implies the inequalities

\[
0 < \frac{\tau F_k(t)}{1 + F_k(t) \tau} < 1, \quad k = 1, 2, \ldots , n - 1,
\]

which gives corresponding inequalities for the drift terms in Equation (4.1). By the comparison theorem (Karatzas and Shreve 1991), Theorem 5.2.18, (Yamada 1973), the following inequalities hold with probability one (see also Remark 2.3 in (Brace, Gatarek, and Musiela 1997)):

\[
F_k^{\text{down}}(t) < F_k(t) < F_k^{\text{up}}(t),
\]

where

\[
F_k^{\text{down}}(t) = F_k(0) \exp(-\sigma_k \sum_{p=k+1}^{n-1} \sigma_p t) \exp(\sigma_k W_t - \frac{1}{2} \sigma_k^2 t),
\]

(4.6)
These bounds imply that the probability distributions of the forward Libors $F_k(t)$ in the terminal measure $\mathbb{P}_n$ have log-normal tails. An alternative proof of this result was given in (Gerhold 2011).

The pricing of Eurodollar futures on the Libor rate $L_{i,i+1} = F_i(t_i)$ reduces to the evaluation of the expectation:

$$
\mathbb{E}_Q[L_{i,i+1}] = P_{0,n}\mathbb{E}_n[B_iF_i(t_i)P_{i,i+1}^{-1}] = P_{0,n}\mathbb{E}_n[B_iF_i(t_i)\hat{P}_{i,i+1}(1 + F_i(t)\tau)].
$$

(4.8)

This is identical to the expression (3.2) in the model considered in the previous section. We will derive upper and lower bounds on this expectation for the last Libor rate $i = n - 1$, assuming uniform forward rates and caplet volatilities $L_i^{\text{fwd}} = L_0$ and $\sigma_i = \sigma$. The relevant expectations can be evaluated exactly using Proposition 3.1 with the substitutions

$$
r_k \rightarrow F_k(0)\tau
$$

(4.9)

for the upper bound, and

$$
r_k \rightarrow F_k(0)\tau e^{-(n-k-1)\sigma^2\tau_k}
$$

(4.10)

for the lower bound.

We start by computing the upper bound on the multiplicative convexity adjustment factor $\kappa_{\text{ED}}$ defined in Equation (3.8). This is clearly a finite value, and the finiteness of the Eurodollar futures prices noted in (Brace, Gatarek, and Musiela 1997) was one of the reasons for the acceptance and widespread use of the Libor market models. We show in Figure 4. plots of $\log\kappa_{\text{ED}}^{\text{up}}$ (red curves) with $\kappa_{\text{ED}}^{\text{up}}$ the upper bound on the convexity adjustment, defined as in Equation (3.8). The upper bound has an explosion at a critical value of the volatility, which is relatively small (about $\sigma = 50\%$ for quarterly time step, $L_0 = 5\%$ and $t_{n-1} = 4.75$ years). The plots in Figure 4. show also the lower

![Figure 4](image_url)

**Figure 4.** Upper (solid red) and lower (solid blue) bounds on the Eurodollar futures convexity adjustment $\log\kappa_{\text{ED}}$ vs. $\sigma$ for the rate $L_{n-1,n}$ in the one-factor log-normal Libor market model. The dashed blue curves show the lower bound on $\log\kappa_{\text{ED}}$ defined in (4.13). The rate tenor is $\tau = 0.25$, and the forward Libor rate is flat with $L_0 = 5.0\%$ (left), $10.0\%$ (right). The total number of time steps is $n = 20$ with time step $\tau = 0.25$. 
bound $\log \kappa_{\text{ED}}^{\text{down}}$ (blue curves), which display also an explosion at a higher value of the volatility.

These results show that the Eurodollar futures convexity adjustment in the Libor market model explodes to unphysical values for sufficiently large volatilities. For maturity $T = 5$ years and quarterly simulation time step $\tau = 0.25$, the explosion volatility of the lower bound $\log \kappa_{\text{ED}}^{\text{down}}$ is $\sigma_{\text{exp}} \approx 110\%$ for $L_0 = 5\%$ and $\sigma_{\text{exp}} \approx 100\%$ for $L_0 = 10\%$. This explosion introduces a limitation of the applicability of this model for pricing Eurodollar futures to volatilities below a maximum allowed level, which depends on the rate tenor, maximum maturity and simulation time step. We give next an analytical upper bound on the explosion volatility of the lower bound which makes explicit its dependence on the model parameters.

**Proposition 4.1.** The explosion volatility of the lower bound on the price of the Eurodollar futures on $L_{n-1,n}$ in the LMM with uniform parameters $L_0, \sigma$ is bounded from above as

$$
\sigma_{\text{exp}}^2 t_n \leq -\frac{2n}{n-1} \log(L_0 \tau).
$$

(4.11)

This bound on the explosion volatility $\sigma_{\text{exp}}$ becomes smaller as the rate $L_0$ increases (for $L_0 \tau < 1$), and as the maturity $t_n$ increases. For the two cases shown in Figure 4, the bound on $\sigma_{\text{exp}}$ is 132.4% and 121.5%, respectively.

These bounds divide the range of the volatility parameter $\sigma$ into three regions: (a) the low-volatility region, below the explosion volatility of the upper bound $\kappa_{\text{ED}}^{\text{up}}$. In this region the model is well behaved. (b) An intermediate volatility region, between the explosion volatilities of the upper and lower bounds. In this region an explosive behaviour of Eurodollar futures prices is possible, but is not required by the bounds. (c) The large volatility region, above the explosion volatility of the lower bound $\kappa_{\text{ED}}^{\text{down}}$. In this region the Eurodollar futures prices explode to unphysical values.

Although we assumed in this calculation uniform model parameters $L_i^{\text{fwd}} = L_0, \sigma_i = \sigma$, these bounds can be extended to the general case of arbitrary bounded parameters $(L_i^{\text{fwd}}, \sigma_i)$, by using $L_i^{\text{fwd}} = \sup_i L_i^{\text{fwd}}, \sigma = \sup_i \sigma_i$ for the upper bound, and $L_i^{\text{fwd}} = \inf_i L_i^{\text{fwd}}, \sigma = \inf_i \sigma_i$ for the lower bound.

The Eurodollar futures convexity adjustment has been computed in the log-normal Libor market model in (Jäckel and Kawai 2005), using an analytical approximation based on the Itô-Taylor expansion, and checked by Monte Carlo simulation. The adjustment was found to be well-behaved and no singularity was observed for maturities up to $T = 5$ years. Two scenarios have been considered: (i) normal vols, moderate rates scenario: $\sigma = 40\%$ and $L_0 = 5\%$, (ii) higher vols, low rates scenario: $\sigma = 60\%$ and $L_0 = 1\%$. Both scenarios lie in region b), where an explosion may occur, but is not required by the bounds considered.

Another potential limitation on the applicability of the model comes from the explosion of the variance of the payoff, which leads to uncontrollably large errors in the Monte Carlo simulation of the model. We will derive next an exact lower bound on the variance of the payoff of the Eurodollar futures in the log-normal LMM, which will give a lower bound on the error of a Monte Carlo calculation of this quantity. Denote the random variable whose expectation has to be evaluated $X = \Pi_{k=0}^{T-2}(1 + L_k \tau) L_{n-1}(1 + L_{n-1} \tau)$. A lower bound on
the variance of this random variable \( \text{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \) can be obtained by combining a lower bound for \( \mathbb{E}[X^2] \) with an upper bound on \( \mathbb{E}[X] \). We have

\[
\text{var}(X) \geq (1 + L_0 \tau)^{2(n-1)} L_0^2 (1 + L_0 \tau)^2 (\kappa_{ED}^{\text{low}} - (\kappa_{ED}^{\text{up}})^2) = L_0^2 (\Sigma_X^{\text{low}})^2 ,
\]

where \( \kappa_{ED}^{\text{up}} \) is the upper bound on the coefficient \( \kappa_{ED} \) defined in Equation (3.8). This can be computed as before by an application of Proposition 3.1 with the substitution of Equation (4.9). We introduced also \( \kappa_{2,ED} \) which is the multiplicative convexity adjustment of the second moment of \( X \) and is defined as

\[
\kappa_{2,ED} = \frac{N_{n-1}^{(2)} + 2L_0 \tau N_{n-1}^{(3)} + (L_0 \tau)^2 N_{n-1}^{(4)}}{(1 + L_0 \tau)^{2(n-1)} (1 + L_0 \tau)^2} , \tag{4.13}
\]

where \( N_{n}^{(k)} \) is defined as

\[
N_{n}^{(k)} = \mathbb{E}[\prod_{q=1}^{n-1} (1 + r_q e^{\sigma W_q - \frac{1}{2} \sigma^2 t_q})^2 e^{q \sigma^2 (\frac{1}{2} q^2 - \frac{1}{2} \sigma^2 t_q)}] . \tag{4.14}
\]

A lower bound on these expectations can be computed exactly using the following result, which is analogous to Proposition 3.1, followed by the substitution of Equation (4.10).

Proposition 4.2. Consider the expectation

\[
N_{n}^{(q)} = \mathbb{E}\left[ \prod_{k=1}^{n-1} (1 + r_k e^{\sigma W_k - \frac{1}{2} \sigma^2 t_k})^2 e^{q \sigma^2 W_n - \frac{1}{2} q^2 t_n} \right] , \tag{4.15}
\]

where \( r_k, \sigma \) are real positive numbers, and \( W_t \) is a standard Brownian motion started at zero and sampled at times \( t_k \). This is given by

\[
N_{n}^{(q)} = \sum_{p=q}^{2(n-1)+q} d_p^{(0)} , \tag{4.16}
\]

where \( d_p^{(0)} \) are given by the solution of the recursion relation

\[
d_p^{(i)} = d_p^{(i+1)} + 2r_{i+1} d_{p-1}^{(i+1)} e^{\sigma^2 (p-1) r_{i+1}} + r_{i+1}^2 d_{p-2}^{(i+1)} e^{\sigma^2 (2p-3) r_{i+1}} , \tag{4.17}
\]

with the initial condition at \( i = n-1 \)

\[
d_k^{(n-1)} = 1 , \quad d_p^{(n-1)} = 0 \text{ for all } p \neq q . \tag{4.18}
\]

Proof. The proof is similar to that of Proposition 2.1 and is omitted.

We show in Figure 5. plots of the lower bound on the variance of the payoff of the Eurodollar futures on the rate \( L_{n-1,n} \) versus the volatility \( \sigma \), for the same model parameters as in the left plot of Figure 4. The curves show \( \log_{10}(\Sigma_X^{\text{low}}/\sqrt{N}) \) with \( N = 10^6 \). These plots show that the payoff variance explodes at some critical value of the volatility, which implies that the error of a MC calculation of these products in the terminal measure \( \mathbb{P}_n \) becomes very large above a certain critical volatility. In this region the MC simulation method becomes unreliable for this particular application. For \( L_0 = 5\% \) the critical volatility is about 45\%, which decreases to about 40\% for \( L_0 = 10\% \).
We have shown that certain expectations related to the pricing of financial instruments have explosive behaviour at large volatility in several widely used log-normal interest rate models simulated in discrete time. Although the existence of such explosions has been known for a long time in the continuous-time version of such models, experience with the discrete time version of these models appears to suggest that no divergences are present. While this statement is strictly true mathematically, in the sense that the expectations are finite in the discrete time case, the actual numerical values can become unrealistically large, such that they are clearly unphysical.

We discussed the appearance of such numerical explosions in three interest rate models with log-normal rates in discrete time. The first quantity is the expectation of the money market account in the BDT model. The discretely compounded money market account plays a central role in the simulation of interest rate models in the spot measure, where it represents the numeraire (Jamshidian 1997). A good understanding of its distributional properties is clearly of great practical importance. Due to an autocorrelation effect between successive compounding factors, the expectation and the higher positive integer moments of the money market account under stochastic interest rates following a geometric Brownian motion have a numerical explosion (Pirjol 2015; Pirjol and Zhu 2015). The criteria for the appearance of this explosion have been derived in (Pirjol and Zhu 2015). The explosion time decreases with the rate volatility and with the time step size, and approaches zero in the continuous time limit, as expected from the continuous time theory (Andersen and Piterbarg 2007). This explosion implies that the distribution of the money market account has heavy tails, and the explosive paths appear when sampling from the tails of this distribution.

Figure 5. Lower bounds on the standard deviation of a MC calculation of the Eurodollar futures convexity adjustment $\kappa_{ED}$ for the rate $L_{n-1,n}$ in the one-factor Libor market model. The plots show $\log_{10}(\frac{\sum_{X}^{\text{low}}}{\sqrt{N}})$ with $N = 10^6$ vs. $\sigma$. The rate tenor is $\tau = 0.25$, and the forward Libor rate is flat with $L_{\text{twd}} = 5.0\%$ (blue - lower curve), $10.0\%$ (black - upper curve). The total number of time steps is $n = 20$ with time step $\tau = 0.25$.

5. Summary and discussion

We have shown that certain expectations related to the pricing of financial instruments have explosive behaviour at large volatility in several widely used log-normal interest rate models simulated in discrete time. Although the existence of such explosions has been known for a long time in the continuous-time version of such models, experience with the discrete time version of these models appears to suggest that no divergences are present. While this statement is strictly true mathematically, in the sense that the expectations are finite in the discrete time case, the actual numerical values can become unrealistically large, such that they are clearly unphysical.

We discussed the appearance of such numerical explosions in three interest rate models with log-normal rates in discrete time. The first quantity is the expectation of the money market account in the BDT model. The discretely compounded money market account plays a central role in the simulation of interest rate models in the spot measure, where it represents the numeraire (Jamshidian 1997). A good understanding of its distributional properties is clearly of great practical importance. Due to an autocorrelation effect between successive compounding factors, the expectation and the higher positive integer moments of the money market account under stochastic interest rates following a geometric Brownian motion have a numerical explosion (Pirjol 2015; Pirjol and Zhu 2015). The criteria for the appearance of this explosion have been derived in (Pirjol and Zhu 2015). The explosion time decreases with the rate volatility and with the time step size, and approaches zero in the continuous time limit, as expected from the continuous time theory (Andersen and Piterbarg 2007). This explosion implies that the distribution of the money market account has heavy tails, and the explosive paths appear when sampling from the tails of this distribution.
We showed in this article that similar explosive phenomena appear in expectations and variances of certain accrual-type payoffs, which have the compounding structure of the money market account, such as the Eurodollar futures prices. We illustrated this phenomenon on the case of two interest rate models: i) a one-factor short rate model with log-normal rates in the terminal measure, and ii) the one-factor log-normal Libor market model. The Eurodollar futures can be priced exactly in the former model, using the exact solution of this model presented in (Pirjol 2013). The result shows explosive behaviour at a critical value of the volatility. While no similar exact result is available in the log-normal Libor market model, we derive exact upper and lower bounds on the Eurodollar futures prices in the log-normal Libor market model with uniform volatility, or more generally with bounded parameters \( (L_i^{\text{fwd}}, \sigma_i) \). Both bounds display the same explosive behaviour at sufficiently large volatility. We also derive an exact lower bound on the error of a Monte Carlo calculation of this quantity, which has a similar explosive behaviour. This introduces a limitation on the applicability of this simulation method to sufficiently low volatilities.

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Disclosure statement

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Appendix

A. Proofs

Proof. [Proof of Proposition 2.1] The $p$th moment of the money market account $B_n$ is given by the expectation

$$
E_Q[B_n^p] = E_Q\left[\prod_{k=0}^{n-1} (1 + \rho_k e^{\sigma W_k - \frac{1}{2} \sigma^2 t_k})^p\right],
$$

(A.1)

with $\rho_k := \bar{L}_k \tau$. Define the conditional expectations

$$
\beta_i^{(p)}(W_i) := E_Q\left[\prod_{k=i+1}^{n-1} (1 + \rho_k e^{\sigma W_k - \frac{1}{2} \sigma^2 t_k})^p | \mathcal{F}_i\right].
$$

(A.2)

The $p$th moment of the bank account is expressed in terms of this expectation as

$$
E_Q[B_n^p] = (1 + \rho_0)^p \beta_0^{(p)}(0).
$$

(A.3)

Let us show that the function $\beta_i^{(p)}(W_i)$ has the form

$$
\beta_i^{(p)}(W_i) = \sum_{k=0}^{p(n-i-1)} b_k^{i(p)} \bar{L}_k e^{\sigma W_i - \frac{1}{2} \sigma^2 t_i}.
$$

(A.4)

This follows by backward induction in $i$. It holds for $i = n - 2$, as we have by explicit computation

$$
\beta_{n-2}^{(p)}(W_{n-2}) = E_Q\left[\prod_{k=0}^{n-2} (1 + \rho_{n-1} e^{\sigma W_{n-1} - \frac{1}{2} \sigma^2 t_{n-1}})^p | \mathcal{F}_{n-2}\right]
$$

$$
= \sum_{m=0}^{p} \binom{p}{m} \rho_{n-1}^m \bar{L}_{n-1} \beta_{n-1}^{m(n-1-\frac{1}{2} \sigma^2 t_{n-1})} | \mathcal{F}_{n-2}\right]
$$

(A.5)

$$
= \sum_{m=0}^{p} \binom{p}{m} \rho_{n-1}^m \frac{1}{m!} e^{\frac{1}{2} \sigma^2 t_{n-1}} \beta_{n-1}^{m(n-1-\frac{1}{2} \sigma^2 t_{n-1})},
$$

where $\binom{p}{m} = \frac{p!}{m!(p-m)!}$ are the binomial coefficients. Second, note that the functions $\beta_i^{(p)}(W_i)$ satisfy the backwards recursion

$$
\beta_i^{(p)}(W_i) = \sum_{k=0}^{p(n-i-1)} b_k^{i(p)} \bar{L}_k e^{\sigma W_i - \frac{1}{2} \sigma^2 t_i}.
$$
\[ \beta_t^{(p)}(W_t) = \mathbb{E}_t \left[ (1 + \rho_{t+1} e^{\theta W_{t+1} - \theta^2 t_{t+1}}) \beta_{t+1}^{(p)}(W_{t+1}) | \mathcal{F}_i \right]. \]  \hspace{1cm} (A.6)

Substituting here the expansion (A.4) assumed to hold at \( i+1 \), one finds that it holds also at \( i \), with coefficients given by Equation (2.5).

In conclusion, the coefficients \( b_k^{(i,p)} \) can be computed exactly by backwards recursion from the relation (2.5) with the initial conditions at \( i = n - 1 \)

\[ b_0^{(n-1,p)} = 1, \quad b_k^{(n-1,p)} = 0, \quad k \geq 1. \]  \hspace{1cm} (A.7)

The coefficients \( b_k^{(i,p)} \) with negative indices \( k<0 \) are zero.

The final result for the moments (A.1) follows from (A.3) and is given in Equation (2.4).

**Proof.** [Proof of Proposition 4.1]

The result (4.11) follows from a lower bound on the expectation (4.11) with \( i = n - 1 \). This is obtained by keeping only the terms of highest order in \( L_0 \)

\[ \mathbb{E}_n [B_{n-1} F_{n-1}(t_{n-1}) (1 + F_{n-1}(t_{n-1}) \tau)] \geq T_1 + T_2. \]  \hspace{1cm} (A.8)

The two terms are

\[ T_1 := \mathbb{E}_n \left[ \prod_{k=0}^{n-2} (F_k(t_k) \tau) F_{n-1}(t_{n-1}) \right] \]  \hspace{1cm} (A.9)

\[ = (L_0 \tau)^{n-1} \mathbb{E}_n \left[ \sum_{k=0}^{n-3} \sigma W_k - \frac{1}{2} \sigma^2 t_k + \frac{1}{2} (\sigma W_{n-1} - \frac{1}{2} \sigma^2 t_{n-1}) - \sum_{k=0}^{n-3} (n-k-1) \sigma^2 t_k \right] \]

\[ = (L_0 \tau)^{n-1} L_0, \]

and

\[ T_2 := \mathbb{E}_n \left[ \prod_{k=0}^{n-1} (F_k(t_k) \tau) F_{n-1}(t_{n-1}) \right] \]  \hspace{1cm} (A.10)

\[ = (L_0 \tau)^n \mathbb{E}_n \left[ \sum_{k=0}^{n-2} \sigma W_k - \frac{1}{2} \sigma^2 t_k + \frac{1}{2} (\sigma W_{n-1} - \frac{1}{2} \sigma^2 t_{n-1}) - \sum_{k=0}^{n-2} (n-k-1) \sigma^2 t_k \right] \]

\[ = (L_0 \tau)^n L_0 e^{\frac{1}{2} \sigma^2 t n(n-1)}. \]

\( T_1 \) is constant, but \( T_2 \) grows with \( \sigma \). The dependence of \( T_2 \) of \( \sigma \) can be studied by writing it as

\[ T_2 = L_0 \exp \left( n \log(L_0 \tau) + \frac{1}{2} \sigma^2 t n(n-1) \right) \]  \hspace{1cm} (A.11)

For small \( \sigma \approx 0 \) the exponent is negative and \( T_2 \) is small. As \( \sigma \) increases, the sum in the exponent becomes zero at some value \( \sigma_0 \). For \( \sigma > \sigma_0 \) the exponent is positive and \( T_2 \) has an explosive increase. The value \( \sigma_0 \) gives the upper bound (Equation (4.11)) on the explosion of the expectation (A.8).