Option Pricing with Threshold Mean Reversion

Zeyu Chi, Fangyuan Dong, and Hoi Ying Wong*

Mean reversion and regime switching are well-known features of commodity prices. Recent empirical research additionally documents the time variation of the mean reversion rate and volatility. This paper considers the option pricing framework for an underlying commodity price with mean reversion rate and volatility change according to a self-exciting regime switching model. We offer empirical evidence for the proposed model and derive analytic pricing formulas for the European and barrier options. Numerical examples demonstrate the application and the ability of the proposed model in capturing volatility smile and regime-switching in the mean reversion rate, simultaneously.


1. INTRODUCTION

Evidence on the mean reversion in commodities is abundant while derivatives pricing models with mean reversion have gained popularity in the past two decades. Most popular mean reversion models are inspired by the Ornstein–Uhlenbeck (OU) process, such as Cox-Ingersoll-Ross (1985) and Vasicek (1977). The OU process randomly oscillates around the long-term mean level with a constant mean reversion rate so that it permits serial dependence. Sorensen (1997) advocates mean reversion for currency exchange rate and find that the mean-reverting feature significantly affects the American currency options price. Ekvall, Jennergren, and Naslund (1997) give several reasons for mean-reverting exchange rates using an equilibrium model and derive the closed-form solution for pricing European currency options. Hilliard and Reis (1998) investigate how the price of commodity futures and future options are affected by stochastic convenient yield under the two-factor model of Schwartz (1997). It is also noticed that if the convenient yield is a deterministic function of asset price, two-factor model can be reduced to mean-reverting process. Hui and Lo (2006) and Wong and Lau (2008) employ a mean-reverting log-normal model (MRL) to value barrier options. Wong and Lo (2009) introduce a stochastic volatility to the MRL model. Fusai, Marena, and Roncoroni (2008) show the importance of mean reversion for commodity derivatives and derive an analytic solution to discrete Arithmetic Asian options. Chung and Wong (2014) further generalize it to include jumps.

The aforementioned research, however, concentrates on constant mean reversion rate and volatility, except for Wong and Lo (2009) who take stochastic volatility into account but still assume a constant mean reversion rate. When we empirically estimate the time

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series models for commodity prices, the mean reversion rate is found to be time varying and quite depends on the commodity price itself. In time series analysis, Tong (1983, 1990) describes this feature through the self-exciting threshold autoregressive model (SETAR). The SETAR model considers coefficients in an autoregression to take different values depending on whether the previous asset value is above or under a certain threshold, thus exhibiting regime switching dynamics.

The other model reflecting the regime-switching pattern is called Markov-modulated regime-switching (MMRS) model. Christoforidou and Ewald (2015a, 2015b) develop a class of analytical formulas for various option prices on commodities under multi-factor OU process with the volatilities controlled by a Markov process. It belongs to a different category in the reason that the SETAR model views regime-switching as an endogenous event determined by the asset value itself, whereas the MMRS model regards it exogenously governed by a Markov chain.

A natural extension of the discrete-time SETAR model is its diffusion limit which turns out to be a stochastic differential equation (SDE) with a piecewise linear drift term. Freidlin and Pfeiffer (1998) use the Brownian motion with a drift that switches between positive and negative to construct a diffusion threshold model with an unknown upper threshold and a zero lower threshold. Decamps, Goovaerts, and Schoutens (2006) study this class of threshold processes for interest rate and obtain semi-analytic expressions for the transition density of self-exciting threshold (SET) diffusion. When the mean reversion speed always stays positive, the SET diffusion is known as an ergodic threshold OU (TOU) process. Kutoyants (2012) renders parameter estimation scheme for several ergodic TOU processes.

This paper develops a new option pricing framework where the underlying asset price is modeled by a generalized TOU process. The generalized process allows both mean reversion rate and volatility to change once the asset price crosses certain threshold parameters, whereas classical TOU process assumes a constant volatility. Therefore, the associated SDE has piecewise linear drift and volatility terms. We call this generalized TOU process the threshold mean reversion (TMR).

We apply this model to the commodity derivatives market because our data analysis shows that commodities exhibit SETAR feature when they are fit to time series models. In addition, their volatilities show different values in different regimes separated by the estimated threshold. As commodities are not traded directly on exchange, the classical dynamic hedging using the underlying asset no longer works in this situation so that the drift term of the asset value process cannot be reduced to the risk-free interest rate under the pricing measure in general. In practice, investors often manage the commodity risk using futures contracts so that the drift term of the asset value process under the pricing measure should match or is calibrated to the observed futures term structure. Therefore, this paper allows the log-asset value to follow the TMR under the pricing measure as well. New pricing formulas are then needed for European and path-dependent options when the underlying commodity price follows a TMR process.

We derive the analytic option pricing formulas for European options and some popular path-dependent options, including barrier options and lookback options. The formulas enable us to investigate the impact of the mean reversion and regime-switching on these options. More delightfully, they are consistent with the existing pricing formulas proposed in Hilliard and Reis (1998) when there is only one regime and the two factors in the model of Hilliard and Reis (1998) are perfectly correlated.

The remainder of the paper is organized as follows. Section 2 introduces the TMR process using empirical data. Section 3 derives the analytic formulas of vanilla call option which will be useful for pricing barrier options. Section 4 presents solution to the valuation of exotic options such as barrier and lookback options. Section 5 gives several numerical
examples to show the accuracy of the solution. Our numerical experiments also show that the proposed model is able to generate a volatility smile. Section 6 concludes the paper.

2. THE MODEL

In this section, we propose a new family of tractable stochastic models for the log-asset value. The model is the continuous-time analog of the discrete-time SETAR time series model with a volatility depending on the regime as well.

2.1. Threshold Mean Reversion

Let \( X(t) \) be the log-asset value so that \( S(t) = e^{X(t)} \) is the observed asset price at time \( t \). We postulate that \( \{X(t)\}_{t \geq 0} \) satisfies the SDE:

\[
dX(t) = \sum_{i=0}^{n} \left( \kappa_i (\theta_i - X(t)) dt + \sigma_i dW_t \right) \mathbb{1}_{\{X(t) \in D_i\}}, \quad t \geq 0, \tag{1}
\]

where \( \{W(t)\}_{t \geq 0} \) is the Wiener process; \( h_1 > h_2 > \ldots > h_n \) are threshold parameters that characterize a sequence of interval domains: \( D_n = (-\infty, h_n), D_1 = [h_1, \infty), D_i = [h_{i+1}, h_i) \) for \( i = 2, \ldots, n - 1 \); and the positive real constants \( \kappa_i, \theta_i, \) and \( \sigma_i \) are the mean reversion rate, the mean reversion level, and volatility associated with the regime \( D_i \) for \( i = 1, 2, \ldots, n \), respectively.

When there are only two regimes, the SDE (1) reduces to

\[
dX(t) = \begin{cases} 
\kappa_1 (\theta_1 - X(t)) dt + \sigma_1 dW_t, & X(t) \geq h; \\
\kappa_2 (\theta_2 - X(t)) dt + \sigma_2 dW_t, & X(t) < h,
\end{cases} \tag{2}
\]

where seven parameters have to be estimated in practice, namely \( \kappa_i, \theta_i, \sigma_i, \) and \( h \) for \( i = 1, 2 \). Consider that the process starts in the first regime with parameters \( \kappa_1, \theta_1, \) and \( \sigma_1 \). It switches to the second regime with another set of parameters, \( \kappa_2, \theta_2, \) and \( \sigma_2 \), once \( X(t) \) falls below the threshold \( h \).

The TMR process in (1) can be thought of as a mixture of \( n + 1 \) mean reversion processes switching among regimes any number of times between successive observations. Although (1) is a nonlinear SDE, its coefficients are Lipschitz-continuous functions so that the SDE has a unique solution.

2.2. Discrete-Time Analog

A possible way to estimate the TMR model applies a discrete approximation. Suppose the log-spot prices \( X(t) \) are observed at time points \( t_j = j\delta, j = 1, \ldots, n - 1 \), where \( \delta = T/n \). For a small value of \( \delta \), a discrete approximation reads

\[
X(t_{j+1}) = (1 - \kappa_i \delta)X(t_j) + \kappa_i \theta_i \delta + \sigma_i [W(t_{j+1}) - W(t_j)], X_{t_j} \in D_i. \tag{3}
\]

Let \( X_{t_j} = X(t_j), a_{i,1} = (1 - \kappa_i \delta), b_i = \kappa_i \theta_i \delta, \) and \( e_t = W(t_{j+1}) - W(t_j) \sim \mathcal{N}(0, \delta) \). Then, the discrete-time model is essentially the SETAR(1) proposed by Tong (1983, 1990):

\[
X_t = \sum_{j=1}^{P} a_{i,j} X_{t-j} + b_i + \sigma_i e_t, \quad h_{i-1} \leq X_{t-d} < h_i, \tag{4}
\]

where \( t = 0, \pm 1, \pm 2, \ldots \), and \( i = 1, \ldots n \).
The threshold identification and parameter estimation of SETAR models are available, for example, in Chan and Kutoyants (2012) and Davies, Pemberton, and Petruccelli (1988). The R package has a subroutine for testing and fitting SETAR models up to three thresholds.

2.3. Empirical Threshold Identification for Commodities

To show that the model (1) is empirically relevant for option pricing in the commodity market, we empirically show the threshold effect with commodity price data. The data set is obtained from Bloomberg: LME 3M Aluminium, LME 3M Copper, and Soybeans.

Figures 1a–3a are the time series plots of log daily prices of aforementioned commodities. The estimation results suggest the existence of threshold effects. The horizontal line indicates the estimated threshold value. According to Stigler (2010), the threshold detection
problem can be formulated as the minimization problem on the sum of squares residuals (SSR). Specifically, the minimization problem is

\[ \hat{h} = \arg \min_{h} \text{SSR}(h). \]

The minimization can be numerically performed through a grid search. The SSR is calculated for each selected threshold value and the one minimizing the SSR is regarded as the estimator of the threshold. This method has received different names in the literature such as concentrated least squares and conditional least squares. Enders (2004) suggests that a strong threshold effect will generate a U-shaped pattern in the grid search graph.

Figures 1b–3b are grid search graphical results generated by the `selectSETAR` function in R with package `tsDyn`. The circled dots are the thresholds that provide the minimum SSR.
The figures show that there exists at least one threshold for all commodity under investigation. However, the grid search output for Copper appears in multiple local minimums, indicating the multiple thresholds effect. Estimation of the second threshold can be done in package `tsDyn` by setting the parameter `nthresh` to 2.

As two-regime case is easy for illustrating the estimation and the grid search result for a single threshold is graphically clear, we focus our model on one threshold (two regimes) case. Table I shows the estimated parameters and corresponding significant level based on SETAR(1) model (4). Most of the parameters pass t-test under certain confident level. Table II shows the estimated parameters in different regimes. $\kappa_i$ and $\theta_i$ can be simply transformed from $a_{i,1}$ and $b_i$ with $\delta = 1/250$, whereas $\sigma_i$ is evaluated by SSR. Estimated parameters significantly change with respect to different regimes in terms of at least one of $\kappa_i$, $\theta_i$, and $\sigma_i$. Therefore, our TMR model (1), which allows for a regime-dependent
volatility, seems to be more realistic compared to the standard SETAR with a constant volatility.

3. OPTION PRICING

This section derives the closed-form solution for vanilla and path-dependent options under the TMR model. Although the derivation can be applied to the general TMR of (1), we concentrate on the two-regime case to simplify the mathematical description. Under the pricing measure \( Q \), the log-asset value \( X(t) = \ln S(t) \) follows the TMR process

\[
dX(t) = \begin{cases} 
    \kappa_1(\theta_1 - X(t))dt + \sigma_1 dW^*(t), & h \leq X(t) < \infty \\
    \kappa_2(\theta_2 - X(t))dt + \sigma_2 dW^*(t), & -\infty < X(t) < h 
\end{cases}
\]

where \( W^*(t) \) is the \( Q \)-Wiener process, \( h \) is the threshold value, and \( \kappa_i, \theta_i, \) and \( \sigma_i \) are, respectively, the mean reversion rate, the long-term mean, and the volatility in the \( i \)-th regime for \( i = 1, 2 \). As \( \sum_{i=1}^{2} \kappa_i(\theta_i - X(t))1_{[X(t) \in D_i]} \) and \( \sum_{i=1}^{2} \sigma_i 1_{[X(t) \in D_i]} \) are piecewise linear functions, the Laplace transform (LT) technique becomes typically useful in the valuation procedure.

In general, the process under the physical probability measure \( P \) is different from that under the pricing measure \( Q \). However, the change of measure technique can be applied to obtain the \( Q \)-process from the \( P \)-process. Specifically, the Girsanov theorem shows that the processes under \( P \) and \( Q \) only differ by their long-term mean level. The long-term mean of the \( Q \)-process can then be calibrated to the observed futures price which shares the same maturity of the option.

### TABLE I

Summary of Parameters Estimation for Commodities in SETAR(1) Model

<table>
<thead>
<tr>
<th></th>
<th>Aluminum</th>
<th>Soybeans</th>
<th>Copper</th>
</tr>
</thead>
<tbody>
<tr>
<td>Threshold</td>
<td>7.851</td>
<td>7.128</td>
<td>8.578</td>
</tr>
<tr>
<td>( a_{1,1} )</td>
<td>0.965***</td>
<td>0.844***</td>
<td>0.987***</td>
</tr>
<tr>
<td>( a_{2,1} )</td>
<td>0.818***</td>
<td>0.916***</td>
<td>0.993***</td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>0.281*</td>
<td>1.113***</td>
<td>0.117***</td>
</tr>
<tr>
<td>( \theta_2 )</td>
<td>1.423**</td>
<td>0.593***</td>
<td>0.063#</td>
</tr>
<tr>
<td>Significance</td>
<td>0***</td>
<td>0.001**</td>
<td>0.01*</td>
</tr>
</tbody>
</table>

### TABLE II

Summary of Parameters Estimation for Commodities

<table>
<thead>
<tr>
<th></th>
<th>Aluminum</th>
<th>Soybeans</th>
<th>Copper</th>
</tr>
</thead>
<tbody>
<tr>
<td>Threshold</td>
<td>7.851</td>
<td>7.128</td>
<td>8.578</td>
</tr>
<tr>
<td>( \kappa_1 )</td>
<td>45.464</td>
<td>20.880</td>
<td>1.865</td>
</tr>
<tr>
<td>( \kappa_2 )</td>
<td>8.833</td>
<td>38.909</td>
<td>3.280</td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>7.823</td>
<td>7.097</td>
<td>8.420</td>
</tr>
<tr>
<td>( \theta_2 )</td>
<td>7.950</td>
<td>7.152</td>
<td>8.934</td>
</tr>
<tr>
<td>( \sigma_1 )</td>
<td>0.219</td>
<td>0.147</td>
<td>0.289</td>
</tr>
<tr>
<td>( \sigma_2 )</td>
<td>0.214</td>
<td>0.166</td>
<td>0.377</td>
</tr>
</tbody>
</table>
3.1. European Call Option

Consider the general SDE (1). For given $T > 0$, $t \in [0, T]$, denote the European option pricing function as

$$C(t, x) = E_{t,x}^Q[e^{-r(T-t)}\Phi(X(T))],$$

where $\Phi(X(T))$ is the terminal payoff function and $X(T)$ is the solution to (1) with initial condition $X(t) = x$.

If $X(t)$ is the solution of the SDE (1), then it is clear that $C(t, x)$ satisfies the partial differential equation (PDE):

$$C_t + \sum_{i=1}^n \kappa_i (\theta_i - x) 1_{\{x \in D_i\}} C_x + \frac{1}{2} \sum_{i=1}^n \sigma_i^2 1_{\{x \in D_i\}} C_{xx} = rC \quad (7)$$

This is a consequence of classical Feynman–Kac formula, see Shreve (2004) for technical details.

For vanilla call options, $\Phi(x) = (e^x - K)^+$ where $K$ is the strike price. Let $\tau = T - t$ be the time to maturity, $k = \ln K$, and $f(\tau, x) = C(t, x)$. To simplify matter, we apply the SDE (6) so that the PDE for $f(\tau, x)$ is given as follows:

$$-\frac{\partial f}{\partial \tau} + \kappa_i (\theta_i - x) \frac{\partial f}{\partial x} + \frac{1}{2} \sigma_i^2 \frac{\partial^2 f}{\partial x^2} = rf, \quad \text{for } x \in D_i, i = 1, 2, \quad (8)$$

$$f(0, x) = (e^x - e^k)^+,$$

where $D_1 = [h, \infty)$ and $D_2 = (-\infty, h)$. The first-order continuity of the value function yields the continuous and smooth pasting boundary conditions,

$$\lim_{x \to h^+} f(\tau, h^-) = \lim_{x \to h^+} f(\tau, h^+), \quad \lim_{x \to h^+} \frac{\partial f(\tau, x)}{\partial x} = \lim_{x \to h^-} \frac{\partial f(\tau, x)}{\partial x}.$$

Taking Laplace transform (LT) to the PDE (8) with respect to $\tau$ results in an ordinary differential equation (ODE). Let $\mathcal{F}(x, \gamma)$ be the Laplace transform of $f(\tau, x)$ so that

$$\mathcal{F}(x, \gamma) = \mathcal{L}(f(\tau, x)) = \int_0^\infty e^{-\gamma \tau} f(\tau, x)d\tau, \quad (9)$$

where $\gamma > \gamma_0$ for some real constant $\gamma_0$ making the integration converge. Then, the ODE governing $\mathcal{F}(x, \gamma)$ is given by

$$\frac{\sigma_i^2}{2} \frac{\partial^2 \mathcal{F}}{\partial x^2} + \kappa_i (\theta_i - x) \frac{\partial \mathcal{F}}{\partial x} - (\gamma + r) \mathcal{F} + (e^x - e^k)_+ = 0, \quad \text{for } x \in D_i, i = 1, 2, \quad (10)$$

with boundary conditions

$$\mathcal{F}(x, \gamma) = \mathcal{L}(e^x - e^{-rt} e^k) = \frac{e^x}{\gamma} - \frac{e^k}{r + \gamma} \quad \text{as } x \to \infty, \quad (11)$$

$$\mathcal{F}(x, \gamma) = \mathcal{L}(0) = 0 \quad \text{as } x \to -\infty. \quad (12)$$
Instead of solving the PDE (8), taking LT enables us to work on the second-order nonlinear ODE (10). The solution of the PDE can then be recovered from the solution of the ODE through inverting the LT. We derive the solution of the ODE for two possible situations: $k > h$ and $k \leq h$. For $k > h$, the ODE (10) can be separated into two parts:

$$
\frac{\sigma^2}{2} \frac{\partial^2 \hat{f}}{\partial x^2} + \kappa_1 (\theta_1 - x) \frac{\partial \hat{f}}{\partial x} - (\gamma + r) \hat{f} + (e^x - e^k)_+ = 0, \quad \text{for } x \in [h, \infty), \quad (13)
$$

$$
\frac{\sigma^2}{2} \frac{\partial^2 \hat{f}}{\partial x^2} + \kappa_2 (\theta_2 - x) \frac{\partial \hat{f}}{\partial x} - (\gamma + r) \hat{f} = 0, \quad \text{for } x \in [-\infty, h). \quad (14)
$$

Let $y_i(x) = \kappa_i/\sigma^2_i (\theta_i - x)^2$, $\alpha_i = (\gamma + r)/\kappa_i$, and $g_i(y_i(x), \gamma) = \hat{f}(x, \gamma)$. Then, simple transformation of variables shows that

$$
\frac{\partial^2 g_i}{\partial y_i^2} + \left( \frac{1}{2} - y_i \right) \frac{\partial g_i}{\partial y_i} - \frac{\alpha_i}{2} g_i = 0, \quad (15)
$$

which is essentially a hypergeometric differential equation. This result is due to the fact that an OU process has the same law as a Bessel process of integer order, see Maghsoodi (1996). Using (15), we are able to solve (13) and (14) to obtain the following theorem.

**Theorem 1.** For the two-regime case with a single threshold $h < k$, the European call option price has the analytic solution:

$$
C(t, x) = \mathcal{L}^{-1} \left[ \hat{f}(x, \gamma) \right]_{t=T-t},
$$

$$
\hat{f}(x, \gamma) = \hat{f}_{11}(x, \gamma)\mathbb{I}_{[k < x]} + \hat{f}_{12}(x, \gamma)\mathbb{I}_{[h < x \leq k]} + \hat{f}_2(x, \gamma)\mathbb{I}_{[x \leq h]},
$$

where

$$
\hat{f}_{11}(x, \gamma) = A_{11} M \left( \frac{\alpha_1}{2}, \frac{1}{2}; y_1(x) \right) + A_{12} U \left( \frac{\alpha_1}{2}, \frac{1}{2}; y_1(x) \right) + \hat{f}_0(x, \gamma), \quad (16)
$$

$$
\hat{f}_{12}(x, \gamma) = A_{21} M \left( \frac{\alpha_1}{2}, \frac{1}{2}; y_1(x) \right) + A_{22} U \left( \frac{\alpha_1}{2}, \frac{1}{2}; y_1(x) \right), \quad (17)
$$

$$
\hat{f}_2(x, \gamma) = B_1 M \left( \frac{\alpha_2}{2}, \frac{1}{2}; y_2(x) \right) + B_2 U \left( \frac{\alpha_2}{2}, \frac{1}{2}; y_2(x) \right), \quad (18)
$$

$$
\hat{f}_0(x, \gamma) = \mathcal{L}^{-1} \left[ F \mathbb{I}(d_1) - K \mathbb{I}(d_1) \right], \quad (19)
$$

with $N$ being the standard normal cumulative distribution function, and $M$ and $U$ being the confluent hypergeometric functions the first and second kinds, respectively. The constants appearing in the expressions are given as follows:

$$
\bar{F} = e^{x - x_1 \tau} \exp(\theta_1 (1 - e^{-x_1 \tau}) + \nu^2 / 2), \quad \nu^2 = \frac{\sigma_1^2 e^{-2k_1 \tau}}{2k_1} (e^{2k_1 \tau} - 1),
$$

$$
d_1 = \frac{\ln(\bar{F}/K) + \nu^2 / 2}{\nu}, \quad d_1 = d_2 - \nu.
Coefficients $A_{11}, A_{12}, A_{21}, A_{22}, B_1,$ and $B_2$ are solved uniquely from the system of linear equations:

\[
0 = A_{11} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{\alpha_1}{2})} + A_{12} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1+\alpha_1}{2})},
\]

\[
0 = B_1 \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{\alpha_2}{2})} + B_2 \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1+\alpha_2}{2})},
\]

\[
0 = A_{21} M\left(\frac{\alpha_1}{2}, \frac{1}{2}, y_1(h)\right) + A_{22} U\left(\frac{\alpha_1}{2}, \frac{1}{2}, y_1(h)\right) - B_1 M\left(\frac{\alpha_2}{2}, \frac{1}{2}, y_2(h)\right) - B_2 U\left(\frac{\alpha_2}{2}, \frac{1}{2}, y_2(h)\right),
\]

\[
0 = A_{11} M\left(\frac{\alpha_1}{2}, \frac{1}{2}, y_1(k)\right) + A_{12} U\left(\frac{\alpha_1}{2}, \frac{1}{2}, y_1(k)\right) + \hat{f}_0(k, \gamma) - A_{21} M\left(\frac{\alpha_1}{2}, \frac{1}{2}, y_1(k)\right) - A_{22} U\left(\frac{\alpha_1}{2}, \frac{1}{2}, y_1(k)\right),
\]

\[
0 = A_{11} M\left(\frac{\alpha_1 + 2}{2}, \frac{3}{2}, y_1(k)\right) - \frac{1}{2} A_{12} U\left(\frac{\alpha_1 + 2}{2}, \frac{3}{2}, y_1(k)\right) + \frac{\hat{f}_0(k, \gamma)}{y_1(k)\alpha_1} - A_{21} M\left(\frac{\alpha_1 + 2}{2}, \frac{3}{2}, y_1(k)\right) + \frac{1}{2} A_{22} U\left(\frac{\alpha_1 + 2}{2}, \frac{3}{2}, y_1(k)\right),
\]

\[
0 = \alpha_1 A_{21} M\left(\frac{\alpha_1 + 2}{2}, \frac{3}{2}, y_1(h)\right) y_1'(h) - \alpha_1 \frac{1}{2} A_{22} U\left(\frac{\alpha_1 + 2}{2}, \frac{3}{2}, y_1(h)\right) y_1'(h) - \alpha_2 B_1 M\left(\frac{\alpha_2}{2} + 1, \frac{3}{2}, y_2(h)\right) y_2'(h) + \frac{\alpha_2}{2} B_2 U\left(\frac{\alpha_2}{2} + 1, \frac{3}{2}, y_2(h)\right) y_2'(h),
\]

where $\Gamma(\cdot)$ is the classical gamma function.

Proof. Please refer to Appendix A. \qed

The analytic solution in Theorem 1 seems complicated as it involves the confluent hypergeometric functions and a Laplace inversion. However, many standard software have already had built-in routines for confluent hypergeometric functions, such as Mathematica, MATLAB, and R packages. The key is to invert the LT. Appendix B shows the connection between Laplace inversion and Fourier transform. Therefore, standard fast Fourier transform (FFT) technique can be applied to efficiently compute the option price. We refer interested audience to Wong and Guan (2011) for the implementation of FFT. For $x > k$, the Laplace inverse of $\hat{f}_0$ is just the Black–Scholes formula appearing in (19).

### 3.2. Greeks

The analytic solution also facilitates Greek computation and risk management. For instance, the computation of value-at-risk often requires the option delta. As $S = e^x$, the delta is defined
as
\[ \Delta = \frac{\partial F(S, t, T)}{\partial S} = \frac{\partial f(x, \tau)}{\partial x} S. \] (20)

By Theorem 1, we deduce that
\[
\Delta = \begin{cases} 
L^{-1}\{A_{11}M_x(\frac{a_1}{2}, \frac{1}{2}; y_1(x)) + A_{12}U_x(\frac{a_1}{2}, \frac{1}{2}; y_1(x)) + \frac{\partial \hat{g}_0(x, y)}{\partial x}\} & , x \in [k, \infty) \\
L^{-1}\{A_{21}M_x(\frac{a_1}{2}, \frac{1}{2}; y_1(x)) + A_{22}U_x(\frac{a_1}{2}, \frac{1}{2}; y_1(x))\} & , x \in [h, k) \\
L^{-1}\{B_1M_x(\frac{a_2}{2}, \frac{1}{2}; y_2(x)) + B_2U_x(\frac{a_2}{2}, \frac{1}{2}; y_2(x))\} & , x \in (-\infty, h). 
\end{cases}
\]

where
\[ L^{-1}\left\{ \frac{\partial \hat{g}_0(x, y)}{\partial x} \right\} = e^{-(\kappa+r)\tau} \left[ e^{x-d^2_i/2} \frac{e^{x-d^2_i/2}}{2\sqrt{\pi}(r-\sigma^2/2)} - e^{k-d^2_i/2} \frac{e^{k-d^2_i/2}}{2\sqrt{\pi}(r-\sigma^2/2)} \right]. \]

Similar calculation can be applied to the option gamma.

3.3. Futures

As a more frequent product in commodity market, the futures price is defined as
\[ F(t, x) = E_{t,x}^Q [S(T)] = E_{t,x}^Q [e^{X(T)}]. \]

Compared with \( C(t, x) \), \( F(t, x) \) is the case when \( r = 0 \) and \( \Phi(x) = e^x \). Therefore, ODE (10) is replaced by
\[ \frac{\sigma^2}{2} \frac{\partial^2 \hat{g}}{\partial x^2} + \kappa_l(\theta_l - x) \frac{\partial \hat{g}}{\partial x} - \gamma \hat{g} + e^x = 0, \text{ for } x \in D_i, i = 1, 2, \] (21)

where \( g(\tau, x) = F(t, x) \) and \( \hat{g}(x, \gamma) = \mathcal{L}[g(\tau, x)] \) with boundary conditions
\[ \hat{g}(x, \gamma) = \frac{e^x}{\gamma} \text{ as } x \to \infty, \] (22)
\[ \hat{g}(x, \gamma) = 0 \text{ as } x \to -\infty. \] (23)

Slightly different from the call option, the second-order non-linear ODEs (21) are identical and inhomogeneous in both regimes, which means the solution for \( \hat{g}(x, \gamma) \) can be expressed as a combination of two functions with the same form.

Theorem 2. For the two-regime case with a single threshold \( h \), the futures price has the analytic solution:
\[ F(t, x) = \mathcal{L}^{-1}\left[ \hat{g}(x, \gamma) \right] |_{\tau = T-t}, \]
\[ \hat{g}(x, \gamma) = \hat{g}_1(x, \gamma) 1_{[x > h]} + \hat{g}_2(x, \gamma) 1_{[x \leq h]}, \]
\[ \hat{g}_i(x, \gamma) = C_i M \left( \frac{\alpha_i}{2}, \frac{1}{2}; y_i(x) \right) + D_i U \left( \frac{\alpha_i}{2}, \frac{1}{2}; y_i(x) \right) + \hat{g}_0(x, \gamma), \quad (24) \]

\[ \hat{g}_0(x, \gamma) = L \{ \exp(xe^{-\kappa_i \tau} + \theta_i(1 - e^{-\kappa_i \tau}) + \frac{\sigma_i^2}{4\kappa_i}(1 - e^{-2\kappa_i \tau})) \}, \quad (25) \]

where for \( i = 1, 2 \),

\[ \hat{g}_i(x, \gamma) = C_i M \left( \frac{\alpha_i}{2}, \frac{1}{2}; y_i(x) \right) + D_i U \left( \frac{\alpha_i}{2}, \frac{1}{2}; y_i(x) \right) + \hat{g}_0(x, \gamma), \quad (24) \]

\[ \hat{g}_0(x, \gamma) = L \{ \exp(xe^{-\kappa_i \tau} + \theta_i(1 - e^{-\kappa_i \tau}) + \frac{\sigma_i^2}{4\kappa_i}(1 - e^{-2\kappa_i \tau})) \}, \quad (25) \]

with \( M \) and \( U \) being the confluent hypergeometric functions the first and second kind, respectively. Coefficients \( C_1, C_2, D_1, \) and \( D_2 \) are solved uniquely from the system of linear equations:

\[ 0 = C_1 \frac{\Gamma\left( \frac{3}{2} \right)}{\Gamma\left( \frac{1+\alpha_1}{2} \right)} + D_1 \frac{\Gamma\left( \frac{2}{2} \right)}{\Gamma\left( \frac{1+\alpha_2}{2} \right)}, \]

\[ 0 = C_2 \frac{\Gamma\left( \frac{3}{2} \right)}{\Gamma\left( \frac{1+\alpha_2}{2} \right)} + D_2 \frac{\Gamma\left( \frac{2}{2} \right)}{\Gamma\left( \frac{1+\alpha_1}{2} \right)}. \]

4. PATH-DEPENDENT OPTIONS

This section develops a path-dependent option pricing framework under the TMR. Analytic solutions are derived for barrier and lookback options.

4.1. Moment-Generating Function of the First Passage Time

Both barrier and lookback options depend on the distributional property of the first passage time (FPT). Consider

\[ \tau_B = \inf \{ t > 0 | S_t \geq B \} = \inf \{ t > 0 | X_t \geq b \}, \quad (26) \]

where \( b = \ln B \). Then, \( \tau_B \) is known as the FPT to hit the barrier level \( B \). The distribution of \( \tau_B \) can be inferred by its moment-generating function (MGF). Denote \( M_{\tau_B}(x, \alpha) \) as the MGF.

Proof. The proof is similar with Appendix A. The particular solution for the inhomogeneous ODE can also be found using Lemma 1 with \( \xi = 1 \) in Wong and Lau (2008).
of FPT that the process \(\{X_t\}\) hits the upward barrier level \(b\). Mathematically,

\[
M_{\tau_B}(x, \alpha) = E\left[ e^{-\alpha \tau_B} \middle| X_0 = x \leq b \right].
\]  

(27)

**Lemma 1.** If \(\{X_t\}\) satisfies (6), then \(M_{\tau_B}(x, \alpha) := M_{\tau_B}(x)\), for \(\alpha\) being a fixed constant, is the unique continuous solution of the following ODE

\[
-aM_{\tau_B} + \kappa_i(\theta_i - x) \frac{\partial M_{\tau_B}}{\partial x} + \frac{1}{2} \sigma_i^2 \frac{\partial^2 M_{\tau_B}}{\partial x^2} = 0, \quad x \in D_i,
\]

with the boundary conditions, \(M_{\tau_B}(-\infty) = 0\) and \(M_{\tau_B}(b) = 1\).

**Proof.** For \(t < \tau_B\), applying Itô’s lemma to \(e^{-\alpha t}M_{\tau_B}(x)\) yields

\[
d[e^{-\alpha t}M_{\tau_B}(X_t)] = \sum_{i=1}^{2} \mathbb{1}_{\{x \in D_i\}} \left[ \left( -ae^{-\alpha t}M_{\tau_B} + \kappa_i(\theta_i - x)e^{-\alpha t} \frac{\partial M_{\tau_B}}{\partial x} \right. \right. \\
+ \frac{\sigma_i^2}{2} e^{-\alpha t} \frac{\partial^2 M_{\tau_B}}{\partial x^2} \left. \right) dt + \sigma_i e^{-\alpha t} \frac{\partial M_{\tau_B}}{\partial x} dW^*_t \bigg].
\]

As \(\{X_t\}\) has a continuous sample path, \(X_{\tau_B} = b\) and \(M_{\tau_B}(X_{\tau_B}) = M_{\tau_B}(b)\). However, \(M_{\tau_B}(b) = 1\) by the definition in (27). Taking integration from 0 to \(\tau_B\) to the above expression, we have

\[
e^{-\alpha \tau_B} - M_{\tau_B}(x) = \int_0^{\tau_B} \sum_{i=1}^{2} \mathbb{1}_{\{x \in D_i\}} \left[ \left( -ae^{-\alpha t}M_{\tau_B} + \kappa_i(\theta_i - x)e^{-\alpha t} \frac{\partial M_{\tau_B}}{\partial x} \right. \right. \\
+ \frac{\sigma_i^2}{2} e^{-\alpha t} \frac{\partial^2 M_{\tau_B}}{\partial x^2} \left. \right) dt + \sigma_i e^{-\alpha t} \frac{\partial M_{\tau_B}}{\partial x} dW^*_t \bigg].
\]

Taking conditional expectation to both sides implies that

\[
M_{\tau_B}(x) = E\left[ e^{-\alpha \tau_B} \middle| x \leq b \right] - E\left[ \int_0^{\tau_B} \sum_{i=1}^{2} \mathbb{1}_{\{x \in D_i\}} e^{-\alpha t} \left( -aM_{\tau_B} + \kappa_i(\theta_i - x) \frac{\partial M_{\tau_B}}{\partial x} \right. \right. \\
+ \frac{\sigma_i^2}{2} \frac{\partial^2 M_{\tau_B}}{\partial x^2} \bigg) dt \bigg| x \leq b \bigg].
\]

In order to have \(M_{\tau_B}(x) = E\left[ e^{-\alpha \tau_B} \middle| x \leq b \right]\) matching the definition in (27), we must have (28), which is a second-order ODE that requires two boundary conditions. The first boundary condition in the lemma is obtained by setting \(\tau_B\) to infinity in (27). When \(x\) tends to negative infinity, it needs infinite long time to reach the upper finite barrier. The second boundary condition has been discussed in the early part of this proof. 

Solving the nonlinear equation is very similar to solving (10), except for that it contains no inhomogeneous term and obeys a different set of boundary conditions.
4.2. Barrier Options

Our first interest is the valuation of barrier options. Knock-out is a popular barrier feature. A knock-out option is similar to a vanilla option except for the existence of a barrier, called the out-strike, when hit would cause the option to extinguish at any time during the option’s life. Another popular barrier feature is the knock-in feature for which an option becomes alive when the underlying asset breaches the barrier. We focus on up-and-in (UI) option in this paper, although other barrier options can be valued using the same framework.
A UI option has a payoff function given by \( \varphi(x) \mathbb{1}_{[\tau_B < T-t]} \), where \( T \) is the time to maturity; \( \tau_B \) is the first passage time that \( x_{\tau_B} \geq b \). The present value of a UI option is then given by

\[
\text{UI}(x, B) = e^{-rT} E[\varphi(X_T)\mathbb{1}_{[\tau_B < T-t]} | X_t = x].
\]

(29)

where \( \varphi(x) \) is the payoff function of a corresponding vanilla option.

Consider the Laplace transform of a UI option with respect to the time to maturity, \( /FS = T - t \). The following proposition characterizes the general pricing formula of an UI option.

**Proposition 1.** Under the TMR model, the pricing formula of the UI option with the corresponding vanilla payoff, \( \varphi(x) \), satisfies

\[
\mathcal{L}[\text{UI}(S, K, B)] = E[e^{-(\gamma + r)\tau_B}] \int_{0}^{\infty} e^{-\gamma t^* V(k, t^*)} dt^*,
\]

where \( b = \ln B \) and \( V(x, k, t^*) \) is the pricing formula of the corresponding European option with payoff \( \varphi(x) \) with \( x = \ln S \).

**Proof.** Let \( t^* = \tau - \tau_B \). By the definition of Laplace transform,

\[
\mathcal{L}[\text{UI}(S, K, B)] = \int_{0}^{\infty} e^{-(\gamma + r)t} E[\varphi(X_t)\mathbb{1}_{[\tau_B < t]}] dt
\]

\[
= E \left[ \int_{-\tau_B}^{\infty} e^{-(\gamma + r)(t^* + \tau_B)} \varphi(X_{t^* + \tau_B}) \mathbb{1}_{[\tau_B < t^* + \tau_B]} dt^* \right]
\]

\[
= E \left[ \int_{-\tau_B}^{0} e^{-(\gamma + r)(t^* + \tau_B)} \varphi(X_{t^* + \tau_B}) \mathbb{1}_{[0 < t^*]} dt^* \right]
\]

\[
+ E \left[ \int_{0}^{\infty} e^{-(\gamma + r)(t^* + \tau_B)} \varphi(X_{t^* + \tau_B}) \mathbb{1}_{[t < t^*]} dt^* \right]
\]

\[
= 0 + \int_{0}^{\infty} E \left[ e^{-(\gamma + r)(t^* + \tau_B)} \varphi(X_{t^* + \tau_B}) \right] dt^*
\]

\[
= \int_{0}^{\tau_B} E[ e^{-(\gamma + r)t_B} \varphi(X_{\tau_B}) ] dt^*
\]

\[
= E[ e^{-(\gamma + r)\tau_B} ] \int_{0}^{\tau_B} e^{-\gamma t^* V(k, t^*)} dt^*,
\]

where \( V(x, k, t) = E[e^{-rt} \varphi(X_t)|X_0 = x] \) is the pricing formula of the corresponding European option with payoff \( \varphi(x) \) under TMR. Note that the second last line of the above calculation is true because of the Markov property of the model. As the TMR model assumes a continuous sample path, \( X_{\tau_B} = b \).

**Corollary 1.** Under the TMR model, the pricing formula of a UI option satisfies

\[
\mathcal{L}[\text{UI}(S, K, B)] = M_{\tau_B}(r + \gamma) \mathcal{L}[V(x, b, \tau)].
\]

(30)
If \( \varphi(x) = (e^x - e^k)_+ \), then the UI call (UIC) option having the payoff function \( (e^x - e^k)_+1_{\{tau < t\}} \) can be evaluated as,

\[
L[UIC(S, K, B)] = \frac{M_{TB}(x, r + \gamma)}{FSB} \hat{f}(x, \gamma),
\]

where \( \hat{f}(x, \gamma) \) is the Laplace transform of European call option price obtained in Theorem 1 and \( M_{TB}(r + \gamma) \) is the MGF obtained in Theorem 3.

4.2.1. Knock-out options and rebate

Knock-out barrier options can then be implied by the parity relation among barrier options. For instance, the up-and-out call (UOC) option can be valued as the vanilla call price minus the UIC price, as we have UIC + UOC = C. However, this argument only works for European options without rebate. Usually, a knock-out option pays a constant cash rebate to the option holder when the underlying asset price hits the barrier. Consider a UOC that pays $1 to the holder on breaching the barrier. The present value of this rebate is \( E[e^{-r/FSB}1_{\{TB < T - t\}}] \). The Laplace transform of this present value with respect to \( \tau = T - t \) is

\[
L_{\tau, \gamma}E[e^{-r\tau}1_{\{TB < T - t\}}] = E \left[ \int_0^\infty e^{-\alpha \tau} e^{-r\tau}1_{\{TB < T - t\}} d\tau \right] = \frac{MB(\gamma + r)}{\gamma}.
\]

When present value of rebate \( R \) would be refunded to the option holder, we invert (32) and then multiply it to \( R \).

4.3. Lookback Options

Lookback options are popular path-dependent options in the financial market, especially exchange rate market. The options provide the holders a chance to realize large gains in the event of huge movements of the underlying asset value. The floating strike of the lookback option allows the holder to purchase or sell the underlying asset with the strike price as extreme value over a pre-agreed period. Investors, who speculate on the volatility, may be interested in the lookback spread option, in which the payoff depends on the difference between the maximum and minimum of asset value over a time horizon. More exotic forms of lookback payoffs are discussed by He, Keirstead, and Rebholz (1998). Wong and Chan (2007) show that lookback option features are embedded in dynamic fund protection of insurance products. Our objective here is to derive analytic solutions for lookback options under the TMR model.

For instance, the floating strike lookback put option allows the option holder to sell the asset at the realized maximum asset price during the life of the contract. Consider the payoff function of the lookback option as follows.

\[
\varphi_{LB}(S) = M_0^T - S_T,
\]

where \( M_0^T = \sup\{S_{\tau}|0 \leq \tau \leq T\} \). It is clear that \( M_0^T \geq S_T \). The lookback option price is expressed as,

\[
P_{LB} = e^{-r(T-t)}E[M_0^T - S_T|S_t = S].
\]
Proposition 2. The pricing formula for floating strike lookback put option with threshold mean reverting is given by

\[ P_{LB} = e^{-r(T-t)} \int_{M_0^t}^{\infty} L^{-1} \left[ \frac{M_y(y)}{\gamma} \right] dy - e^{-r(T-t)} M_0^t - V(S_t, 0, t), \]  

(35)

where \( V(S_t, 0, t) \) is the European call option value under the TMR model pricing framework with zero strike at time \( t \).

Proof. This proposition is actually a special case in Wong and Lau (2008), whose Proposition 3 is not restricted to TMR model but applicable in general model. Here, we emphasize the important steps of derivation. \( P_{LB} \) can be expressed as follows:

\[ P_{LB} = e^{-r(T-t)} E_t[M_0^T - S_T | \mathcal{F}_t] \]

\[ = e^{-r(T-t)} \{ E_t[M_0^T] - E_t[S_T] \} \]

\[ = e^{-r(T-t)} \{ E_t[\max(M_0^t, M_t^T)] - E_t[S_T] \} \]

\[ = e^{-r(T-t)} \{ E_t[\max(M_t^T - M_0^t, 0)] + M_0^t - E_t[S_T] \}. \]  

(36)

Consider

\[ \max(M_t^T - M_0^t, 0) = \begin{cases} M_t^T - M_0^t, & M_t^T > M_0^t \\ 0, & M_t^T \leq M_0^t \end{cases} \]  

(37)

\[ = \int_{M_0^t}^{\infty} \Phi(M_t^T > y) dy. \]  

(38)

Then,

\[ E_t[\max(M_t^T - M_0^t, 0)] = \int_{M_0^t}^{\infty} Pr(M_t^T > y) dy. \]  

(39)

As a result \( E_t[M_0^T] \) will become the following:

\[ E_t[M_0^T] = E_t[\max(M_0^t, M_t^T)] \]

\[ = E_t[\max(M_t^T - M_0^t, 0)] + M_0^t \]

\[ = \int_{M_0^t}^{\infty} Pr(M_t^T > y) dy + M_0^t. \]  

(40)

\[ (41) \]

\[ (42) \]

The lookback option value can then be obtained as follows:

\[ P_{LB} = e^{-r(T-t)} \left\{ \int_{M_0^t}^{\infty} Pr(M_t^T > y) dy + M_0^t \right\} - e^{-r(T-t)} E_t[S_T]. \]  

(43)

As the process of \( \{S_t\} \) is not necessarily a martingale but consistent with the futures price, the last term in (6) is not equal to the \( S_T \). We get its value through the European call option by setting the strike to zero. In fact, one can replace \( E_t[S_T] \) by the market-observed futures price with maturity \( T \). \( \square \)
The result can be extended to value turbo warrants. Wong and Chan (2008) show that a turbo put consists of two parts: the first part resembles an up-and-out put (UOP) option, with a zero rebate, and the second part is an up-and-in lookback (UIL) option. For a model with continuous sample path, including the TMR model proposed in this paper, the UIL option can be further decomposed into the product of a constant rebate and a simple lookback option. Therefore, putting all the results for barrier, rebate, and loopback options together helps calculating the turbo warrant.

5. NUMERICAL EXAMPLES

In this section, we examine the computational efficiency of our analytic solution by a number of numerical examples. In addition, we also discuss the pricing behavior of options under TMR. To show that our solution is correct, accurate, and efficient, we compare the numerical results computed by our solution against the Monte Carlo (MC) simulation.

5.1. European Call Option

We begin our numerical experiment with the European call option. To simplify matters, consider a single threshold which separates two regimes. The MC simulation is based on 10,000 sample paths with a time step of 1/500 although we assume 250 trading days a year.

In addition to the pricing error, we take a closer look at how the option price changes with the spot price, $S_0$ and maturity, $T$. All remaining constant parameters are listed in Table III.

Table IV presents the numerical results for the European call option, where MC stands for MC simulation and ILT for our analytic solution using inverse Laplace transform. It can be seen that numerical values from the analytic solution are very close to the corresponding

<table>
<thead>
<tr>
<th>Spot</th>
<th>MC</th>
<th>ILT</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>2.864</td>
<td>2.9089</td>
</tr>
<tr>
<td>95</td>
<td>4.1099</td>
<td>4.0355</td>
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<td>100</td>
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<td>6.0153</td>
</tr>
<tr>
<td>105</td>
<td>8.2455</td>
<td>8.1882</td>
</tr>
</tbody>
</table>
simulated values. It provides us with some evidence that our analytic solution is correct even though the entire expression seems complicated. However, the MC simulation may have a large variance/standard error and the analytic solution implemented through inverse Laplace transform takes much shorter computational time.

Figure 4 shows that the TMR model is able to generate the implied volatility smirk which matches with the market observation. Although the volatility smile is only a downward sloping curve for the one-threshold TMR model, the two-thresholds model is able to produce a U-shaped curve when the low-price and high-price regimes have the volatility larger than that of the middle-price regime.

We also investigate the impact of the threshold effect described by the two-regime (one-threshold) TMR. Figure 5 shows that the option prices under TMR are placed in between those under Black–Scholes model with high and low volatilities. The existence of the threshold provides an additional flexibility to fit the volatility surface.

5.2. Barrier Option

We conduct a similar experiment for the UIC option. When all parameters are the same as the associated vanilla call option in Table III, we set the barrier level to 115.

Table V shows the numerical results for the UIC option. The option prices obtained from inverse Laplace transform is generally very close to the price obtained from the simulation.

6. CONCLUSION

Our empirical study shows that there is a strong threshold mean-reverting effect in some financial assets, especially in the commodity market. We propose a novel option pricing framework under threshold mean-reversion for the case of one threshold, although the approach can be easily generalized to multiple thresholds. We derive partial differential equations for various kinds of options and solve them through the Laplace Transform technique.
Numerical studies show that the solution implemented with a numerical Laplace inversion is accurate. The examples also show that TMR is able to capture threshold mean reversion and implied volatility smile.

APPENDIX A

PROOF OF THEOREM 3.1

Let \( \tilde{f}_1 \) and \( \tilde{f}_2 \) be the solutions of (13) and (14), respectively. By the solution of (15), we have

\[
\tilde{f}_1(x, \gamma) = A_1 M \left( \frac{\alpha_1}{2}, \frac{1}{2}; y_1(x) \right) + A_2 U \left( \frac{\alpha_1}{2}, \frac{1}{2}; y_1(x) \right) + \tilde{f}_p(x, \gamma)
\]

\[
\tilde{f}_2(x, \gamma) = B_1 M \left( \frac{\alpha_2}{2}, \frac{1}{2}; y_2(x) \right) + B_2 U \left( \frac{\alpha_2}{2}, \frac{1}{2}; y_2(x) \right).
\]

As (13) is an inhomogeneous ODE, there is an additional particular solution \( \tilde{f}_p(x, \gamma) \).
The constants $A_i, B_i, i = 1, 2$ are determined by boundary conditions. In general, the particular function $\hat{f}_p(x, \gamma)$ appearing in the function $\hat{f}_1(x, \gamma)$ is determined using the Wronskian of the homogeneous equation:

$$W(x) = W(M(\eta, \beta, x), U(\eta, \beta, x)) = -\frac{\Gamma(\beta)}{\Gamma(\eta)}x^{-\beta}e^x$$

for $|\arg x| \leq \pi, \eta, \beta \neq \pm 0, \pm 1, \pm 2, \ldots$.

By the method of variation of parameters, we have

$$\hat{f}_p(x, \gamma) = M\left(\frac{\alpha_1}{2}, \frac{1}{2}; y_1(x)\right) \int_k^x U\left(\frac{\alpha_1}{2}, \frac{1}{2}; y_1(x)\right) \frac{e^\tau - e^k}{W(y_1)} d\tau$$

$$+ U\left(\frac{\alpha_1}{2}, \frac{1}{2}; y_1(x)\right) \int_k^x M\left(\frac{\alpha_1}{2}, \frac{1}{2}; y_1(\tau)\right) \frac{e^\tau - e^k}{W(y_1(\tau))} d\tau.$$

$A_i, B_i, i = 1, 2$ in (16) and (18) are determined according to the boundary conditions (11) and (12). The solution also requires that $\hat{f}(x, \gamma)$ is continuous and differentiable at threshold, $h$. As a result, we need the far field boundary conditions:

$$\hat{f}_1(x, \gamma) \to e^x - e^k / \gamma + r \text{ as } x \to \infty; \quad (A.1)$$

$$\hat{f}_2(x, \gamma) \to 0 \text{ as } x \to -\infty, \quad (A.2)$$

and smooth-pasting conditions:

$$\lim_{x \to h^+} \hat{f}_1(x, \gamma) = \lim_{x \to h^-} \hat{f}_2(x, \gamma); \quad (A.3)$$

$$\left. \frac{d\hat{f}_1(\tau, x)}{dx} \right|_{x=h^+} = \left. \frac{d\hat{f}_2(\tau, x)}{dx} \right|_{x=h^-}. \quad (A.4)$$

To get rid of the improper integration at infinity deduced from the far field boundary condition (A.1), we solve the general solution in $D_1$ by decomposing $\hat{f}_1(x, \gamma)$ into two parts separated by the log strike price, $k$, once $k$ falls into $D_1$. We express the general solution $\hat{f}_1$ as follows:

$$\hat{f}_1(x, \gamma) = \begin{cases} \hat{f}_{11}(x, \gamma), & k \leq x < \infty \\ \hat{f}_{12}(x, \gamma), & h < x < k \end{cases}. \quad (A.5)$$

Therefore, Equation (13) is homogeneous ODE for $x < k$ and inhomogeneous for $x \geq k$. There is a trick to get rid of the complicated Wronskian for the particular solution. We observe that it resembles the no-threshold case for $x > k > h$ so that we select the Laplace transform of the European call price under the ordinary mean-reverting process as the particular solution.

- For $x \in [k, \infty)$,

$$R_{11} : \hat{f}_{11}(x, \gamma) = A_{11} M\left(\frac{\alpha_1}{2}, \frac{1}{2}; y_1\right) + A_{12} U\left(\frac{\alpha_1}{2}, \frac{1}{2}; y_1\right) + \hat{f}_0(x, \gamma). \quad (A.6)$$
For \( x \in [h, k) \),

\[
R_{12} : \hat{f}_{21}(x, \gamma) = A_{12} M \left( \frac{\alpha_1}{2}, 1; \gamma \right) + A_{22} U \left( \frac{\alpha_1}{2}, 1; \gamma \right). \tag{A.7}
\]

The particular function \( \hat{f}_0(x, \gamma) \) is the Laplace transform with respect to maturity time of the value function associated with mean-reversion without thresholds. The closed-form solution for the European option with ordinary mean-reversion can be found in Wong and Lau (2008) and is shown in the theorem.

The solution also requires that \( \hat{f}_1(x, \gamma) \) is continuous and differentiable at the strike, \( k \). This implies smooth pasting conditions at \( k \):

\[
\lim_{x \to k^+} \hat{f}_{11}(x, \gamma) = \lim_{x \to k^-} \hat{f}_{12}(x, \gamma), \tag{A.8}
\]

\[
\lim_{x \to k^+} \frac{\partial \hat{f}_{11}(\tau, x)}{\partial x} = \lim_{x \to k^-} \frac{\partial \hat{f}_{12}(\tau, x)}{\partial x}. \tag{A.9}
\]

The following properties of the hypergeometric functions are useful, (Lebedev, 1972):

\[
\frac{d^m M(\eta, \beta; x)}{dx^m} = \frac{(\eta)_m}{(\beta)_m} M(\eta + m, \beta + m; x) \tag{A.10}
\]

\[
\frac{d^m U(\eta, \beta; x)}{dx^m} = (-1)^m (\eta)_m U(\eta + m, \beta + m; x). \tag{A.11}
\]

The following asymptotic representation is also clear for large \( |x| \) and if \( \arg x \leq \pi/2 - \epsilon \), where \( \epsilon > 0 \) and arbitrarily small:

\[
M(\eta, \beta; x) = \frac{\beta}{\eta} e^{x^{-\beta-\eta}} \left[ \sum_{j=0}^{n} \frac{(\beta - \eta)_j (1 - \eta)_j}{j!} x^{-j} + o(|x|^{-n-1}) \right], \tag{A.10}
\]

\[
U(\eta, \beta; x) = x^{-\eta} \left[ \sum_{j=0}^{n} (-1)^j (\eta)_j (1 + \eta - \beta)_j x^{-j} + o(|x|^{-n-1}) \right]. \tag{A.11}
\]

With the boundary conditions in (A.1) and (A.2), the smooth pasting conditions in (A.3), (A.4), (A.8), and (A.9) and the asymptotic property of \( M(\eta, \beta, x) \), we deduce the system of linear equations for solving \( A_{11}, A_{12}, A_{21}, A_{22}, B_1, \) and \( B_2 \) uniquely as shown in the theorem.

**Remark 1.** Specifically, if the set of parameters coincides to be the same in both regimes, the expression \( C(t, x) \) in Theorem 3.1 is consistent with the pricing formula \((22)\) in Hilliard and Reis (1998) under two-factor model. Note that if the convenient yield \( \delta_t \) in Hilliard and Reis (1998) is a deterministic function of underlying asset \( S_t \), \( \delta_t = \kappa c \ln S_t \), the pricing formula \((22)\) is reduced to pricing formula \((4)\) in Wong and Lau (2008) under one-factor model. Therefore, the Laplace transformation \( \hat{f}(x, \gamma) \) should become \( \hat{f}_0(x, \gamma) \) theoretically.

To verify this property, we denote infinitesimal generators \( \mathcal{G}_i \) and \( \mathcal{G} \) to be:

\[
\mathcal{G}_i(x) := \frac{1}{2} \sigma_i^2 \frac{\partial^2}{\partial x^2} + \kappa_i (\theta_i - x) \frac{\partial}{\partial x} + (\gamma + r), \tag{A.12}
\]
for $i = 1, 2$ and

$$G(\cdot) := \frac{1}{2} \sum_{i=1}^{2} \sigma_i^2 1_{\{x \in D_i\}} \frac{\partial^2}{\partial x^2} + \sum_{i=1}^{2} \kappa_i (\theta_i - x) 1_{\{x \in D_i\}} \frac{\partial}{\partial x} + (\gamma + r),$$  \hspace{1cm} \text{(A.13)}

Therefore, if it is the two-regime case with $\kappa_1 = \kappa_2, \theta_1 = \theta_2, \sigma_1 = \sigma_2$, $\hat{f}(x, \gamma) = \sum_{i=1}^{2} \hat{f}_i(x, \gamma) 1_{\{x \in D_i\}}$ satisfies

$$\mathcal{G}\hat{f}(x, \gamma) = -(e^x - e^k)_+,$$  \hspace{1cm} \text{(A.14)}

which can be reduced to

$$\mathcal{G}_1\hat{f}(x, \gamma) = -(e^x - e^k)_+. \hspace{1cm} \text{(A.15)}$$

On the other hand, the pricing formula $f_0(x, \tau) = E^Q_{\mathcal{F}_\tau}[e^{-rT} (e^{X_T} - e^k)_+]$ under single-regime mean-reverting model with parameters $\{\kappa_1, \theta_1, \sigma_1\}$ satisfies a certain PDE with terminal condition by applying Feynman–Kac formula. Similarly, the solution of $\hat{f}_0(x, \gamma)$ defined as the Laplace transform of $f_0(x, \tau)$ with respect to $\tau$ can be solved from the following ODE:

$$\mathcal{G}_1\hat{f}_0(x, \gamma) = -(e^x - e^k)_+. \hspace{1cm} \text{(A.16)}$$

By observing (A.15) and (A.16), we have $\hat{f}(x, \gamma)$ coincidently becoming $\hat{f}_0(x, \gamma)$.

**APPENDIX B**

**FOURIER TRANSFORM AND NUMERICAL LAPLACE INVERSION**

The European call option price, $f(x, \tau)$, takes the form,

$$f(\tau, x) = \mathcal{L}_x^{-1}[\hat{f}(x, \gamma) ]|_{x=\tau}.$$  \hspace{1cm} \text{(B.1)}

One of the most well-known Fourier series methods is the one proposed by Dubner and Abate (1968) and then improved by Abate and Whitt (1992b). It is essentially a trapezoidal rule approximation to discretize the Bromwich integral:

$$F(\tau) = \frac{2e^{\alpha \tau}}{\pi} \int_0^{+\infty} \text{Re}(\hat{F}(a + i\mu)) \cos(\mu \tau) d\nu,$$  \hspace{1cm} \text{(B.2)}

and

$$F(\tau) = -\frac{2e^{\alpha \tau}}{\pi} \int_0^{+\infty} \text{Im}(\hat{F}(a + i\mu)) \sin(\mu \tau) d\nu.$$  \hspace{1cm} \text{(B.3)}

An important feature of the Fourier series method is that there is a closed-form expression for the error in the computed inverse transform. Thus, the maximum error in this technique is controllable. Trapezoidal rule turns out to be surprisingly effective in this context with periodic and oscillating integrands, because the errors tend to zero.
Applying the trapezoidal rule with step size $\delta$ to the expression in (B.2), we have

$$F(\tau) \simeq F^D_A(\tau) = \frac{\delta e^{a\tau}}{\pi} \Re(\hat{F}(a)) + \frac{2\delta e^{a\tau}}{\pi} \sum_{j=1}^{\infty} \Re(\hat{F}(a + j\delta)) \cos(j\delta\tau). \quad (B.4)$$

Setting $\delta = \pi/(2\tau)$ and $a = A/(2\tau)$ eliminates the cosine terms to produce an alternating series

$$F^D_A(\tau) = \frac{e^{A/2}}{2\tau} \Re\left(\widehat{F}\left(\frac{A}{2\tau}\right)\right) + \frac{e^{A/2}}{\tau} \sum_{j=1}^{\infty} (-1)^k \Re\left(\widehat{F}\left(\frac{A + 2j\pi i}{2\tau}\right)\right). \quad (B.5)$$

The choice of $A$ is made in such a way that $a$ falls at the left of the real part of all the singularities of the function $\widehat{F}(\gamma)$. For $|F(\tau)| < M$, Abate and Whitt (1992b) show that the discretization error is bounded by

$$|F(\tau) - F^D_A(\tau)| < M \frac{e^{-A}}{1 - e^{-A}} \simeq Me^{-A}. \quad (B.6)$$

Abate and Whitt (1992b) suggest to set $A$ equal to 18.4 and propose the use of the Euler algorithm in order to compute the infinite sum in (B.5). This algorithm consists of summing explicitly the first $n$ terms of the series and then taking a weighted average of additional $m$ terms. We apply the Euler algorithm to estimate the series so that

$$F^D_A(\tau) \approx E(\tau; n, m) = \sum_{j=0}^{m} \binom{m}{j} 2^{-m} s_{n+j}(\tau), \quad (B.7)$$

where $s_n(\tau)$ is the $n$th partial sum:

$$s_n(\tau) = \frac{e^{A/2}}{2\tau} \Re\left[\widehat{F}\left(\frac{A}{2\tau}\right)\right] + \frac{e^{A/2}}{\tau} \sum_{j=1}^{n} (-1)^j \Re\left[\widehat{F}\left(\frac{A + 2j\pi i}{2\tau}\right)\right]. \quad (B.8)$$

REFERENCES


