Pricing options under the non-affine stochastic volatility models: An extension of the high-order compact numerical scheme

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A B S T R A C T

We consider an improvement of a high-order compact finite difference scheme for option pricing in non-affine stochastic volatility models. Upon applying a proper transformation to equate the different coefficients of second-order non-cross derivatives, a high-order compact finite difference scheme is developed to solve the partial differential equation with nonlinear coefficients that the option values satisfied. Based on the local von Neumann stability analysis, a theoretical stability result is obtained under certain restrictions. Numerical experiments are presented showing the convergence and validity of the expansion methods and the important effects of the non-affine coefficient and volatility of volatility on option values.

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1. Introduction

The correct specification of volatility dynamics is of crucial importance for option pricing. Most existing studies have analyzed the dynamics of volatility using an affine stochastic process, among which the Heston (1993) model is the most prominent work. Such models derive an analytically closed-form solution of the option price and have attracted greater attention for their advantage in explaining the time-varying, aggregation and leverage effect of the volatility of the asset return. Therefore, these models are also used in other areas, such as econometric estimation (Bates, 2006), term

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structure modeling (Dai and Singleton, 2000) and dynamic asset allocation (Liu, 2007). However, the affine stochastic volatility is relatively simple because of its square-root assumption, which fails in describing the nonlinear nature (Andersen et al., 2002) of a financial time series. Recently, many studies have focused on non-affine volatility models (Jones, 2003; Kaeck and Alexander, 2012), which have been found to capture the dynamics of major equity indices much better than the affine models.

Because the exact closed-form solution cannot be obtained in the non-affine stochastic volatility models or in the American option pricing problems, many numerical methods for option pricing have been developed using the finite difference method (Hout and Foulon, 2010), element-finite volume method (Zhang et al., 2015) or series expansion (Hu and Kanniainen, 2015). Most of the finite difference methods for option pricing addresses the one-dimensional case and use the standard, second-order finite difference, whereas Tangman et al. (2008) proposed a new high-order compact finite difference scheme for option pricing that is only suitable for the constant volatility case. Hout and Foulon (2010) considered the numerical analysis for option pricing in stochastic volatility; however, the final scheme there was of second-order solely due to the low order approximation of the cross diffusion term. Recently, a high-order compact finite difference scheme for option pricing in stochastic volatility models has been proposed by Düring and Fournié (2012), in which the stochastic volatility is considered to be affine models, which lead to an identical coefficient of second-order non-cross derivatives of the PDE. However, the non-affine stochastic volatility models complicate the problem, and we extend the approach proposed by Düring and Fournié to solve it.

An outline of this paper is as follows. In Section 2, we introduce the option pricing problems under non-affine models. In Section 3, we demonstrate the discretization of the pricing equation using a high-order compact scheme and analyze its stability. In Section 4, we present numerical experiments and results, and concluding remarks are provided in the last section.

2. Non-affine stochastic volatility models

According to the non-affine stochastic volatility models (Chourdakis and Dotsis, 2011), the dynamics of an asset’s price and stochastic volatility satisfy the following stochastic diffusions:

\[ dS_t = rS_t dt + \sqrt{\sigma_t} S_t dW^1_t, \]
\[ d\sigma_t = k(\theta - \sigma_t) dt + \nu \sigma^\gamma_t dW^2_t. \]

where \( r, k, \theta, \nu \) are the risk-free interest rate, the mean reversion speed, the long time mean of volatility, and the volatility of volatility, respectively, \( W^1_t, W^2_t \) are two standard Brownian motions with correlation \( \rho \in [-1, 1] \), and \( \gamma \) reveals the exponent in the diffusion function, which is named as a non-affine parameter in our paper. It is obvious that the Heston model is one special case for \( \gamma = 1 \). For brevity, we only consider the case of European put option pricing problems, however, the case of call options can be treated analogously. Let \( V(S, \sigma, t) \) denote the value of a European put option with strike price \( K \), application of Ito’s lemma and standard arbitrage arguments leads to:

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \nu S \sigma^\gamma \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} \nu^2 \sigma^2 \frac{\partial^2 V}{\partial \sigma^2} + rS \frac{\partial V}{\partial S} + [k(\theta - \sigma) - \lambda \nu \sigma^\gamma] \frac{\partial V}{\partial \sigma} - rV = 0. \]

for \( S, \sigma \geq 0 \) and \( 0 \leq t \leq T \), where \( \lambda \) is the market price of risk. The terminal and boundary conditions to PDE (3) are provided as in Düring and Fournié (2012):

\[ V(S, \sigma, T) = \max(K - S, 0), \]
\[ \lim_{S \to \infty} V(S, \sigma, t) = 0, \lim_{S \to 0} V(S, \sigma, t) = Ke^{-r(T-t)}, \text{ for } T > t \geq 0, \sigma > 0, \]
\[ \lim_{\sigma \to 0} V(\sigma, \sigma, t) = 0, \lim_{\sigma \to \infty} V(\sigma, \sigma, t) = 0; \text{ for } T > t \geq 0, S > 0. \]
3. Extending the method of the high-order compact scheme

3.1. Derivation of the high-order compact scheme under non-affine models

In this section, we extend the high-order compact method to price options under non-affine volatility models. A proper variable substitution is introduced to convert the different coefficients of second-order non-cross derivatives in Eq. (3) into an identical one, which is the key step to structure the high-order compact scheme for PDE. For this, we apply the following changes in variables $x = \ln(S/K), \tau = T - t, u = e^{-\tau r} (V/K), y = 2 \sigma^2 (\gamma - 3) / \nu(3 - \gamma)$, and we obtain:

$$\frac{\partial u}{\partial \tau} + a(y) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + b(y) \frac{\partial^2 u}{\partial x \partial y} + c(y) \frac{\partial u}{\partial x} + d(y) \frac{\partial u}{\partial y} = 0,$$

(7)

for $x \in R, y > 0, 0 \leq \tau \leq T$, where

$$a(y) = -\frac{1}{2} \left( \frac{1}{2} \nu(3 - \gamma) y \right)^{-\frac{3}{4}}, \quad b(y) = -\rho \left( \frac{1}{2} \nu(3 - \gamma) y \right)^{-\frac{3}{4}}, \quad c(y) = -r + \frac{1}{2} \left( \frac{1}{2} \nu(3 - \gamma) y \right)^{-\frac{3}{4}},$$

$$d(y) = \frac{\nu(\gamma - 1)}{4} \left( \frac{1}{2} \nu(3 - \gamma) y \right)^{-\frac{3}{4}} - \frac{k \theta}{\nu} \left( \frac{1}{2} \nu(3 - \gamma) y \right)^{-\frac{3}{4}} + \frac{k}{\nu} \left( \frac{1}{2} \nu(3 - \gamma) y \right)^{-\frac{3}{4}}.$$  

Note that the above transformation, which can apply to a more general class of non-affine models, is an extension of the one used in During and Fournié (2012). The approach in our paper contrasts with that in the During and Fournié's work mainly because of the nonlinearity of the coefficients $a(y), b(y), c(y), d(y)$; it is obvious that by imposing restrictions $\gamma = 1$, our method reduces to During and Fournié's. The problem is completed by the new initial and boundary conditions:

$$u(x, y, 0) = \max(1 - e^y); \quad \text{for} \quad x \in R, y \geq 0,$$

(8)

$$\lim_{x \to \infty} u(x, y, \tau) = 0; \quad \lim_{x \to -\infty} u(x, y, \tau) = 1; \quad \text{for} \quad y > 0, 0 < \tau \leq T,$$

(9)

$$\lim_{y \to 0} u(x, y, \tau) = 0; \quad \lim_{y \to \infty} u(x, y, \tau) = 0; \quad \text{for} \quad x \in R, 0 < \tau \leq T.$$  

(10)

To solve the numerical problem (7)–(10) at the point $(x, y, \tau)$ in the unbounded domain $(-\infty, \infty) \times (0, \infty) \times (0, T)$, we introduce $x_{\max} = \ln 5$ and $\sigma_{\max} = 1$ according to Ikonen (2008), which are sufficiently large to eliminate the boundary effect; then, the corresponding max value of $y$ is $2/\nu(3-\gamma)$. With the step size in the $x, y, \tau$ directions being denoted by $h_1, h_2, \Delta \tau$, respectively, here, we set $h_1 = h_2 = h$ to conveniently apply a high-order compact scheme; the step numbers are correspondingly $N = \lfloor x_{\max}/h \rfloor + 1, M = \lfloor y_{\max}/h \rfloor + 1$, and $L = \lfloor T/\Delta \tau \rfloor + 1$. The approximate value of $u$ at the grid point is thus denoted by $u_{i,j} \approx u(x_i, y_j, \tau_n) = u(ih, jh, n\Delta \tau)$, where $x_i = ih, i = 1, 2, \ldots, N; y_j = jh, j = 1, 2, \ldots, M; \tau_n = n\Delta \tau, n = 1, 2, \ldots, L$. First, we introduce the high-order compact finite difference discretization for the two-dimensional equation after the reduction of the $\tau$ direction by introducing the notation $f(x, y) = -\partial u/\partial \tau$:

$$a(y)(u_{xx} + u_{yy}) + b(y)u_{xy} + c(y)u_x + d(y)u_y = f(x, y),$$

(11)

for $(x, y) \in (-x_{\max}, x_{\max}) \times (0, y_{\max})$. Using the first- and second-order central difference operator $\delta_x, \delta_y$ and $\delta_x^2, \delta_y^2$, the discrete approximation to Eq. (11) at point $(x_i, y_j)$ can be written as:

$$a(y)(\delta_x^2 u_{i,j} + \delta_y^2 u_{i,j}) + b(y)\delta_x \delta_y u_{i,j} + c(y)\delta_x u_{i,j} + d(y)\delta_y u_{i,j} + e_{i,j} = f_{i,j},$$

(12)

where the associated truncation error is given by:

$$e_{i,j} = \frac{1}{12} a(y)h^2(u_{xxxx} + u_{yyyy}) + \frac{1}{12} b(y)h^2(u_{xyyy} + u_{xxyy}) + \frac{1}{6} c(y)h^2u_{xxxy} + \frac{1}{6} d(y)h^2u_{yxyy} + o(h^4).$$

(13)

Note that there are four fourth-order derivatives in Eq. (13), but three relevant relations could be derived from auxiliary Eq. (11). However, the derivative becomes achievable by regarding $u_{xxxx} + u_{yyyy}$
and $u_{xxy} + u_{xyy}$ as a whole, respectively. Differentiating Eq. (11), we obtain:

$$u_{xxx} = -u_{xyy} - \frac{b(y)u_{xyy} + c(y)u_{xx} + d(y)u_{xy} - f_x}{a(y)},$$

$$u_{yyy} = -u_{xyy} - \left[ a'(y)u_{xx} + [a'(y) + d(y)]u_{yy} + [b'(y) + c(y)]u_{xy} + b(y)u_{xyy} + c'(y)u_x + d'(y)u_y - f_x \right]/a(y),$$

$$u_{xxyy} + u_{xyyy} = -\left[ a'(y)u_{xx} + [a'(y) + d(y)]u_{yy} + [b'(y) + c(y)]u_{xy} + b(y)u_{xyy} + c'(y)u_x + d'(y)u_y - f_x \right]/a(y).$$

(14)

(15)

(16)

$$a(y)(u_{xxx} + u_{xyy})$$

$$= -b(y)u_{xyy} - c(y)u_{xx} - 2a'(y) - d(y)u_{xy} - 2a(y)u_{xyy} - d(y) + 2a'(y)u_{xyy}$$

$$- [c(y) + 2b'(y)]u_{xyy} - a''(y)u_{xx} - [a''(y) + 2d'(y)]u_{xy} - c''(y)u_x - d''(y)u_y + f_x + f_y.$$  

(17)

where $a'(y), b'(y), c'(y), d'(y)$ and $a''(y), b''(y), c''(y), d''(y)$ are the first- and second-order derivatives of $a(y), b(y), c(y), d(y)$ with respect to $y$. For simplicity in notation, we denote the coefficients $a(y), b(y), c(y), d(y)$ as $a, b, c, d$ in the following. Substituting approximations (14)–(17) into (13), a new expression for the truncation error term can easily be deduced; inserting it into (12) yields the $o(h^4)$ approximation of PDE (11):

$$h^2 \left[ 2a + \frac{b^2}{a} \right] y_i y_j \partial_x^2 \partial_y^2 u_{ij,j} + \frac{h^2}{12} \left[ 2d + \frac{b}{a} \left( b' + 2c - \frac{a'd'}{a} \right) \right] \partial_x^2 \partial_y u_{ij} + \frac{h^2}{12} \left[ 2c + 2b' + \frac{a}{d-a'} \right] \delta_y \delta_x u_{ij} + \left[ \frac{h^2}{12} \left[ a'' + 2d' + \frac{a'}{d-2a'} \right] + b \right] \delta_y \delta_x u_{ij} + \left[ \frac{h^2}{12} \left[ a' + d' + \frac{a}{d-2a'} \right] + b \right] \delta_x \delta_y u_{ij} + \left[ \frac{h^2}{12} \left[ c'' + \frac{c}{d-2a'} \right] + c \right] \delta_x u_{ij}$$

$$= f_{ij} + \frac{h^2}{12} \left( \partial_x^2 + \partial_y^2 \right) f_{ij} + \frac{1}{12} \partial_y f_{ij} + \frac{1}{12} \partial_x f_{ij}.$$  

(18)

The fourth-order compact finite difference scheme considered at the mesh point $(i, j)$ involves the nearest eight neighboring mesh points. Associated with the shape of the computational stencil, the indexes for each node from zero to nine are introduced:

$$\left(\begin{array}{cccc}
    u_{i-1,j+1} & u_{i,j+1} & u_{i+1,j+1} \\
    u_{i-1,j} & u_{i,j} & u_{i+1,j} \\
    u_{i-1,j-1} & u_{i,j-1} & u_{i+1,j-1}
\end{array}\right) \rightarrow \left(\begin{array}{cccc}
    u_0 & u_2 & u_5 \\
    u_3 & u_0 & u_1 \\
    u_7 & u_4 & u_6
\end{array}\right).$$

(19)

Hence, scheme (18) is defined by: $\sum_{l=0}^{8} \xi_{ij} u_l = \sum_{l=0}^{8} \psi_{ij} f_l, i = 0, \ldots, 8, j = 1, 2, \ldots, M,$ where

$$\xi_{0j} = -\frac{2}{h^2} \left[ \frac{h^2}{12} \left[ a'' + 2d' + \frac{a'}{d-2a'} \right] + a \right] \bigg|_{y=y_j} + \frac{1}{3h^2} \left( \frac{2a + b^2}{a} \right) \bigg|_{y=y_j},$$

$$\xi_{1,3j} = -\frac{1}{6h^2} \left( 2a + \frac{b^2}{a} \right) \bigg|_{y=y_j} + \frac{1}{12h} \left[ 2c + 2b' + \frac{2b}{a} (d-a') \right] \bigg|_{y=y_j}.$$
\[
+ \frac{1}{h^2} \left\{ \frac{h^2}{12} \left[ a'' + \frac{b'c'}{a} + \frac{c}{a} \left( c - \frac{a'b}{a} \right) + \frac{a'}{a} \left( d - 2a' \right) \right] + a \right\} \bigg|_{y=y_j} \\
\pm \frac{1}{2h} \left\{ \frac{h^2}{12} \left[ c' + \frac{c'}{a} \left( d - 2a' \right) \right] + c \right\} \bigg|_{y=y_j} 
\]
\[
\xi_{2j,4j} = -\frac{1}{6h^2} \left( 2a + \frac{b^2}{a} \right) \bigg|_{y=y_j} \\
\pm \frac{1}{12h} \left[ 2d + \frac{b}{a} \left( b' + 2c - \frac{a'd}{a} \right) \right] \bigg|_{y=y_j} \\
+ \frac{1}{h^2} \left\{ \frac{h^2}{12} \left[ a'' + 2d' + \frac{a' + d'}{a} \left( d - 2a' \right) \right] + a \right\} \bigg|_{y=y_j} \\
\pm \frac{1}{2h} \left\{ \frac{h^2}{12} \left[ d'' + \frac{d'+ d}{a} \left( d - 2a' \right) \right] + d \right\} \bigg|_{y=y_j}.
\]
\[
\xi_{5j,7j} = \frac{1}{12h^2} \left( 2a + \frac{b^2}{a} \right) \bigg|_{y=y_j} \\
\pm \frac{1}{24h} \left[ 2d + \frac{b}{a} \left( b' + 2c - \frac{a'd}{a} \right) \right] \bigg|_{y=y_j} \\
+ \frac{1}{4h^2} \left\{ \frac{h^2}{12} \left[ b'' + 2c' + \frac{bd'}{a} + \frac{d}{a} \left( c - \frac{a'b}{a} \right) + \frac{b'}{a} \left( d - 2a' \right) \right] + b \right\} \bigg|_{y=y_j}.
\]
\[
\xi_{6j,8j} = \frac{1}{12h^2} \left( 2a + \frac{b^2}{a} \right) \bigg|_{y=y_j} \\
\pm \frac{1}{24h} \left[ 2d + \frac{b}{a} \left( b' + 2c - \frac{a'd}{a} \right) \right] \bigg|_{y=y_j} \\
- \frac{1}{4h^2} \left\{ \frac{h^2}{12} \left[ b'' + 2c' + \frac{bd'}{a} + \frac{d}{a} \left( c - \frac{a'b}{a} \right) + \frac{b'}{a} \left( d - 2a' \right) \right] + b \right\} \bigg|_{y=y_j}.
\]
\[
\psi_{0j} = \frac{2}{3}, \psi_{1j,3j} = \frac{1}{12} \pm \frac{h}{24a} \left( c - \frac{a'b}{a} \right) \bigg|_{y=y_j}, \psi_{2j,4j} = \frac{1}{12} \pm \frac{h}{24a} \left( d - 2a' \right) \bigg|_{y=y_j}, \psi_{5j,7j} = \frac{b}{48a} \bigg|_{y=y_j}, \psi_{6j,8j} = -\frac{b}{48a} \bigg|_{y=y_j},
\]
where multiple indexes are used for simplicity with ± and ± signs, and the first index corresponds to the upper sign.

The high-order compact scheme of the original problem (7) can be obtained by extending the two-dimensional Eq. (11) considering \( f(x,y) = -\partial u / \partial \tau \). Applying the implicit difference to the time derivative \( \partial u / \partial \tau \) (\( u^{n+1}_i, u^n_j \)) yields: \( \sum_{i=0}^{8} \left[ (u^{n+1}_i, u^n_j) + (1 - \mu) \xi_{ij} u^n_j \right] = -\sum_{i=0}^{8} \phi_{ij} (u^{n+1}_i, u^n_j - u^n_i / \Delta \tau) \), where \( u^n_j \) denotes the nine point \( u^n_j \) in the computational stencil, and \( \mu = 0, 1, 2, 1 \) represents the forward Euler, Crank-Nicolson and backward Euler schemes, respectively. Thus, the resulting discrete scheme for node \((i, j)\) at the \( n \)th time can be rewritten as:
\[
\sum_{i=0}^{8} \beta_{ij} u^{n+1}_i = \phi_{ij} u^n_j,
\]
where the coefficients \( \beta_{ij} \) and \( \phi_{ij} \) are given by: \( \beta_{ij} = \mu \xi_{ij} + \frac{\phi_{ij}}{\Delta \tau} \), \( \phi_{ij} = -(1 - \mu) \xi_{ij} + \frac{\phi_{ij}}{\Delta \tau} \).

3.2. Stability analysis

The proof of the stability is not trivial because our scheme is applied to a problem with nonlinear variable coefficients, which is much more complex than the linear one in During and Fournié (2012). However, the principle of “frozen coefficients” and the von Neumann stability analysis are also used to discuss the stability of our scheme. If each frozen coefficient problem is stable, the variable coefficient problem is also stable. We now rewrite \( u^n_{i,j} \) as:
\[
u^n_{i,j} = g^{i} e^{i Hz_{1} + i j z_{2}}, \]
where \( I \) is the imaginary unit, \( z_1, z_2 \in [0, \pi/2] \). Then, the scheme is stable if, for all \( z_1 \) and \( z_2 \), the amplification factor \( G = g^{n+1}/g^n \) satisfies the relation

\[
|G|^2 - 1 \leq 0. \tag{22}
\]

As During and Fournié (2012) pointed out, the complete stability analysis is currently unachievable, however, we can show the following result.

**Theorem 1.** For \( r = \rho = \theta = 0 \) and \( \mu = 1/2 \), scheme (20) satisfies the stability condition (22) under the restrictions that \( \gamma \geq 1 \) and the step size \( h \) is sufficiently small.

Proof. Using (21) in (20), an expression for \( G \) can be obtained, \( G = \frac{(A-B)+(C-D)I}{(A+D)+(C+D)I} \), where

\[
A = [\varphi_0 + (\varphi_{1} + \varphi_{3}) \cos z_1 + (\varphi_{2} + \varphi_{4}) \cos z_2 + (\varphi_{5} + \varphi_{7}) \cos z_1 + z_2] \\
+ (\varphi_{6} + \varphi_{8}) \cos (z_2 - z_1)]/\Delta \tau.
\]

\[
B = [\xi_0 + (\xi_{1} + \xi_{3}) \cos z_1 + (\xi_{2} + \xi_{4}) \cos z_2 + (\xi_{5} + \xi_{7}) \cos z_1 + z_2] \\
+ (\xi_{6} + \xi_{8}) \cos (z_2 - z_1)]/2.
\]

\[
C = [((\varphi_{1} - \varphi_{3}) \sin z_1 + (\varphi_{2} - \varphi_{4}) \sin z_2 + (\varphi_{5} - \varphi_{7}) \sin z_1 + z_2 + (\varphi_{6} - \varphi_{8}) \sin (z_2 - z_1)]/\Delta \tau.
\]

\[
D = [(\xi_{1} - \xi_{3}) \sin z_1 + (\xi_{2} - \xi_{4}) \sin z_2 + (\xi_{5} - \xi_{7}) \sin z_1 + z_2 + (\xi_{6} - \xi_{8}) \sin (z_2 - z_1)]/2.
\]

Moreover, (22) can be written as \(-4(AB + CD)(A - B)^2 + (C - D)^2)/[(A + B)^2 + (C + D)^2] \leq 0\), which is equivalent to \( AB + CD \geq 0 \). After calculation and simplification, we found

\[
A = \frac{1}{6\Delta \tau} (4 + \cos z_1 + \cos z_2), \quad C = -\frac{h}{12\Delta \tau} \sin z_1 + \frac{h}{12a\Delta \tau} (d - a') \sin z_2.
\]

\[
B = \frac{1}{48} \left[ \frac{\nu^2(\gamma - 1)}{4a^2} - \frac{a}{3} \right] (1 - \cos z_1) + \frac{\nu^2(\gamma - 1)}{192a^2} (1 - \cos z_2)
\]

\[+ 2 \sin^2 z_1 \frac{a}{2} \left[ -\frac{a}{h^2} - \frac{\nu}{48} (d - 2a') + \frac{2a}{3h^2} \sin^2 z_2 \right] \]

\[+ 2 \sin^2 z_2 \frac{a}{2} \left[ -\frac{a}{h^2} - \frac{1}{4} \left( 2d' + \frac{a + d'}{a} (d - 2a') \right) - \frac{d}{3} \sin z_1 \frac{1}{2} \sin z_1 \cot z_2 \right],
\]

\[
D = \left( -\frac{a}{3h} - \frac{h}{24} d'' - \frac{hv}{96} (d - 2a') \right) \sin z_1 + \left( \frac{d}{3h} + \frac{h}{24} d'' + \frac{hd'}{24} (d - 2a') \right) \sin z_2 + \frac{d - a}{3h} \cos z_2 \sin z_1.
\]

It is obvious that \( A > 0, |A| \geq |C| \). For \( B \), note that the first two items are not less than zero when \( \gamma \geq 1 \); and that the latter two items are nonnegative when the step size \( h \) is sufficiently small. Hence, \( B \geq 0 \) when \( \gamma \geq 1 \) and \( h \) is sufficiently small. Moreover, we verify that \( |B| \geq \nu^2(\gamma - 1)/96a^2, |D| \leq h(d'' + a')/24 + h(d'' - 2a')(a' - 4d')/96a^2 \); it is easy to observe that \( |B| \geq |D| \) when \( \gamma \geq 1 \) and \( h \) is sufficiently small. Under the specific conditions in Theorem 1, we have \( AB \geq 0 \) and \( |AB| \geq |CD| \); therefore, \( AB + CD \geq 0 \), which completes the proof. \( \square \)

For the more involved situations that the correlation \( \rho \) and the long time mean of volatility \( \theta \) are non-zero, we reference the Lemma 2 in During and Fournié (2012), which is not repeated here. We found that one of the sufficient conditions in Lemma 2 that \( |G|^2 - 1 = 0 \) for \( \gamma = 0 \) is true; consequently, we conjecture that the stability condition (22) is also satisfied for non-zero \( \rho \) and \( \theta \), although it will be difficult to provide an analytical proof. Therefore, we perform additional numerical tests to validate the stability property of scheme (20) in the next section.
In this section, numerical experiments are performed to test the effectiveness of the scheme (20). According to During and Fournié (2012) and Zhu and Chen (2011), the parameters are set as follows: \( \mu = 1/2, r = 0.1, \rho = -0.5, k = 2.5, \theta = 0.16, \nu = 0.45, T = 0.5, K = $10 \). We set \( \lambda \) to zero to streamline, although our scheme applies to the case that \( \lambda \) is non-zero by extending the coefficient straightforward. Moreover, for the non-affine parameter \( \gamma \), the commonly used values in previous literature are \( \gamma = 1, 2, 3 \), and others (such as in Ballestra and Cecere, 2015) that are no larger than 3. Furthermore, Ignatieva (2009) and Kaeck and Alexander (2012) indicated that moving from the square root diffusion towards \( 3/2 \) diffusion allows to capture the behavior of outliers at tails. Therefore, it is reasonable to set \( \gamma < 3 \), which makes \( y \) positive. For comparison, we calculate four sets of European put options with different non-affine parameters \( \gamma = 0.5, 1, 1.5, 2.5 \). For the step size, \( h \) needs to be sufficiently small to meet stability conditions without restrictions on the time step size, and \( h \) and \( \Delta \tau \) are set to 0.1 and 0.01, respectively. Fig. 1 shows that \( h = 0.1 \) is sufficiently small to guarantee the stability of our scheme. However, the monotonicity of the solution is not granted in Fig. 1(a) because \( \gamma < 1 \), whereas the stability appears suitable in Fig. 1(b), (c) and (d) with \( \gamma \geq 1 \), which is exactly consistent with the theoretical result. From Fig. 1, we can also find that the option prices increase with \( \gamma \) but not obviously, the specific option prices are shown in Table 1. This table shows that the European option prices increase with the non-affine parameter, which indicates the importance of the non-affine parameter for option pricing and could not be ignored.
Furthermore, denote $error^n = u^n_{i,j}(2h, \tau) - u^n_{2i,2j}(h, \tau), E_\infty(h, \tau) = \max_{1 \leq n \leq T/\Delta \tau} |error^n|. order = \log_2 \frac{E_\infty(2h, \tau)}{E_\infty(h, \tau)}$, we should exam the ratios of the convergence errors with grid steps along spatial directions being successively decreased, whereas the grid step along the time direction is fixed. For this target, we fix the time grid step to be $\Delta \tau = 0.01$ and vary the sizes of the spatial step from 0.4 to 0.05. Table 2 shows that the convergence orders of the scheme in the spatial direction are approaching to 4, hence approximately $o(h^4)$, and errors approach 0 as $h \to 0$.

Numerical analysis is also performed to obtain insight into the influences of the non-affine parameter and the volatility of volatility on European put option prices. Fig. 2 shows that the European option price is an increasing function of the volatility $\sigma$, and it increases faster at the low volatility level than at the high volatility level. Furthermore, Fig. 2 displays that the non-affine parameter influences the option price by impacting the volatility process, when $\gamma$ takes a larger value (for example, $\gamma = 2$), the influence of volatility on the option price is much more obvious than when $\gamma$ has a smaller value (for example, $\gamma = 1$). Fig. 3 illustrates the European option prices under different volatility of volatility $\nu$ and different volatility level $\sigma$. Every subgraph of Fig. 3 displays that the European option price is a decreasing function of $\nu$. One more result from the comparison of the two subgraphs in Fig. 3 is that the option prices changes more with regard to the volatility of volatility difference when the
Fig. 2. The impact of volatility and the non-affine coefficient on the European put options prices, (a) option values with different $S$ at $\gamma = 1$, (b) option values with different $S$ at $\gamma = 2$.

Fig. 3. European put options prices vs. underlying asset price with different volatilities of volatility, (a) option values with different $\nu$ at $\sigma = 0.2304$, (b) option values with different $\nu$ at $\sigma = 0.7056$.

volatility takes a higher value, as observed in Fig. 3 (b); however, the changes is less when the value of volatility is lower, as observed in Fig. 3 (a).

5. Conclusion

This paper presents an improvement of a high-order compact finite difference scheme for the European option pricing problem under the non-affine stochastic volatility model. The key features of the current scheme are its high efficiency with fourth-order accuracy in space and second-order accuracy in time for the PDE with nonlinear coefficients and inequality coefficients of the second-order non-cross derivatives. Numerical examples demonstrate the efficiency of the proposed scheme. This scheme could be extended to other stochastic volatility models and the American option pricing problem. For the first extension, an appropriate transformation would be introduced to equal the different coefficients of second-order non-cross derivatives. In the second case, one must take a front-fixing transformation and combine the high-order discretization with a high-order resolution of the optimal exercise price. We leave extensions for future research.
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