AN APPROACH FOR TENSION INSTABILITY IN SMOOTHED PARTICLE HYDRODYNAMICS (SPH)

C. T. Dyka†‡ and R. P. Ingel§
†GEO-Centers, Inc., 10903 Indian Head Highway, Fort Washington, MD 20744, U.S.A.
§Naval Research Laboratory, Code 6386, Washington, DC 20375-5343, U.S.A.

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Abstract—Smoothed particle hydrodynamics (SPH) has the potential to be an important method for structural analysis. However, the SPH method is currently plagued by tension instability. In this work, a new unconventional approach is introduced, in which the stresses are calculated at points other than the SPH nodes, to address this difficulty. This algorithm is implemented into a one-dimensional (1D) program called SPHID. In addition, objective stress rate calculations are discussed and specialized to 1D applications. The results from applying the unconventional approach to a simple bar are very encouraging. The bar holds together in tension and compares quite well to a finite element analysis.

1. INTRODUCTION

Smoothed particle hydrodynamics (SPH) is a Lagrangian method that requires no spatial mesh. SPH is really an interpolation method in which particles can be employed as part of the approximation [1]. (Note that in this paper, the words element and particle will be used interchangeably when referring to SPH methods.) The calculation of interactions among the particles is based upon their separation alone. This aspect, along with the absence of a grid, allows very large deformations to be computed in a straightforward fashion. In addition, arbitrary fracture surfaces can be opened without requiring special considerations such as an a priori knowledge of the fracture location, as in finite element methods (FEM). SPH is thus an appealing and valuable computational tool, especially for high deformation events such as impact.

However, SPH is a maturing method and still has a few technical barriers to overcome before becoming a widely used tool in computational mechanics. Its biggest drawback is the well known instability in tension—premature fragmentation of the SPH grid in tension. Swegle et al. [2] have done a formal stability analysis that clearly discusses the roots of tension instability. Hicks et al. [3] have applied a conservative smoothing approach with some success, but tension instability remains a serious obstacle in general. Other difficulties with SPH methods include the accurate calculation of finite strains, inhomogeneous media, stress oscillations or "sawtooth" behavior [4], and the application of natural boundary conditions.

In this paper, we present a new unconventional (UC) approach to address tension instability (and also oscillatory stresses) in which the stresses are calculated at points other than the SPH nodes. In addition, stress rate calculations are briefly addressed and specialized to one-dimensional (1D) applications.

2. THE SPH METHOD

The SPH method is non-local in nature and based upon interpolation—see Refs [1, 4–6]. Consider the function \( f(x) \) and its kernel average \( \langle f(x) \rangle \) expressed as:

\[
\langle f(x) \rangle = \int f(x') W(x - x', h) \, dV'.
\]

where \( V \) is the volume, \( x \) is the current position vector, and \( W \) is the kernel with a width measured by the parameter \( h \). Various possibilities exist for the choice of \( W(x - x', h) \) with the most popular being the cubic-b-spline [4]. The kernel \( W \) should satisfy the following condition:

\[
\lim_{h \to 0} W(x - x', h) = \delta(x - x')
\]

where \( \delta \) is Dirac's delta function. Normalization of \( W \) and compact support (\( W \) is zero outside a limited domain) is expressed as:

\[
\int W(x, h) \, dV = 1
\]

and

\[
W(x, h) = 0 \quad (|x| > 2h).
\]
Associating with a particle \( j \), a volume

\[
dV_i' = \frac{M'}{\rho'}
\]  

(4)

introduces the concept of particle mass, \( M' \), and density, \( \rho' \). The integral in eqn (1) can then be replaced by the approximation:

\[
\langle f(x) \rangle \approx \sum_{i=1}^{N} f(x') W(x - x', h) \frac{M'}{\rho'}
\]  

(5)

where \( N \) is the total number of particles.

Dropping the \( \langle \rangle \) sign, gradients of \( f(x) \) can be expressed as [4]:

\[
\nabla f(x) = -\sum_{i=1}^{N} \frac{M'}{\rho'} f(x') \frac{\partial W}{\partial x'_i}
\]  

(6)

in which the subscripts denote component and \( \ell \) is the \( \ell \)th component of the unit vector at \( j \). Another form for the gradient of \( f(x) \) that is widely used but which introduces additional approximation is [4]:

\[
\nabla f(x) = -\rho(x) \sum_{i=1}^{N} M' \left[ \frac{f(x)}{\rho'(x')} + \frac{f(x')}{\rho'(x)} \right] \frac{\partial W}{\partial x'_i}
\]  

(7)

3. THE SPH EQUATIONS (BRIEFLY)

In subscript notation, the conservation of linear momentum is expressed as:

\[
\frac{dV_m}{dt} = -\frac{1}{\rho} \frac{\partial \sigma_{mn}}{\partial x_n}
\]  

(8)

in which \( V_m \) is the velocity and \( \sigma_{mn} \) is the Cauchy stress tensor, and the summation convention is implied by the repeated index \( n \). Applying eqn (6), eqn (8) becomes at particle \( i \):

\[
\frac{dV_m}{dt} = -\frac{1}{\rho'} \sum_{j=1}^{N} \frac{M'}{\rho'} \frac{\partial W_{ij}}{\partial x'_i}
\]  

(9)

This form does not conserve momentum since the force on particle \( i \) due to \( j \) is not the same as \( j \) due to \( i \). A widely used alternative to eqn (9) that conserves momentum but introduces further approximation is obtained by substituting eqn (7) into eqn (8) to produce (no summation on \( i \)):

\[
\frac{dV_m}{dt} = -\frac{1}{\rho'} \sum_{j=1}^{N} M' \left[ \frac{\sigma_{mn}}{\rho' \rho'} + \frac{\sigma_{mn}}{\rho'n} \right] \frac{\partial W_{ij}}{\partial x'_i}
\]  

(10)

Artificial viscosity is usually included in the linear momentum as an artificial viscous pressure. Benz [1], Libersky and Petschek [5] and Libersky et al. [7] give details for including it into eqn (10).

Mass conservation is of the form:

\[
\frac{d\rho}{dt} = -\frac{\partial V_m}{\partial x_n}
\]  

(11)

Introducing eqn (7) into eqn (11) produces:

\[
\frac{d\rho'}{dt} = -\rho' \sum_{j=1}^{N} \frac{M'}{\rho'} \frac{\partial W_{ij}}{\partial x'_i}
\]  

(12)

A widely used alternative to eqn (12) is [4]:

\[
\frac{d\rho'}{dt} = -\rho' \sum_{j=1}^{N} M' \left( \frac{V'_m - V'_n}{\partial x'_i} \right) \frac{\partial W_{ij}}{\partial x'_i}
\]  

(13)

The density can also be directly determined from eqn (5) and is of the form:

\[
\rho' = \sum_{j=1}^{N} M' W_{ij}
\]  

(14)

Equation (14) will be employed in this paper.

In addition to momentum and mass conservation, energy conservation can also be included in the governing equations (see Refs [1] and [2] for instance). In this work, it has not been introduced since a linear elastic material will be assumed.

4. CONSTITUTIVE EQUATIONS AND STRESS RATES

The current position \( x(t) \) of the material point \( \bar{x} \) can be expressed as:

\[
x(t) = \bar{x} + U(t)
\]  

(15)

where \( U(t) \) is the current displacement as measured from \( \bar{x} \). In index notation, the rate of deformation \( d_{mn}(t) \) is of the form:

\[
d_{mn}(t) = \frac{1}{2} \left( \frac{\partial V_m}{\partial x_n} + \frac{\partial V_n}{\partial x_m} \right)
\]  

(16)

Note in eqn (16), that \( d_{mn} \) is Eulerian variable and the derivatives are with respect to \( x_\ell \). The rate of deformation can be put in SPH form by directly applying either eqn (6) or (7). In Ref. [2], further manipulations and approximations are introduced to produce the correct trace and to express \( d \) in terms of velocity differences. For 1D applications, the use of eqn (6) or (7) will suffice.

Next consider the velocity gradient \( L_{mn} \) which is defined by:

\[
L_{mn}(t) = \frac{\partial V_m}{\partial x_n}
\]  

(17)

Using eqn (16), \( L_{mn} \) can be divided into symmetric and skew-symmetric parts:

\[
L_{mn} = d_{mn} + \theta_{mn}
\]  

(18)
Tension instability in SPH

Tension instability is a problem that has long troubled conventional SPH methods and greatly limited its application. In Ref. [2], a stability analysis for 1D is presented. That analysis shows SPH calculations to be unstable when:

\[ \left( \frac{\partial^2 W}{\partial x^2} \right) \sigma_{11} > 0. \]  

This means that the standard SPH method, in which the node and the stress point are both located at the centroid of the element (particle), will be unstable in tension. Swegle et al. in [2] also demonstrate that this tension instability cannot be corrected by artificial viscosity.

The approach in this work to overcome tension instability is indicated in Fig. 1, in which the stress points (denoted by x) are computed at points away from the SPH nodes (denoted by 0 and located at the center), as measured by the distance R. The limits of R are:

\[ 0 \leq R \leq 0.5. \]

R = 0.5 corresponds to the conventional form in which both stress points are located at the centroid of the element, while R = 0 places the two stress points at the left and right edges of the element. In conventional SPH, because the stress points coincide with the SPH node, particle i does not enter into the calculation of \( \sigma_{11} \), since \( \frac{\partial W}{\partial x_i} = 0 \) for a symmetric kernel when \( i = j \). In this new approach, which we refer to as unconventional or UC, for \( 0 \leq R < 0.5 \) node i will be included in the stress calculations for SPH element i. Thus stress calculations from the unconventional approach should be more accurate since node i will be included at the stress points \( i_a \) and \( i_b \) (see Fig. 1). Also the UC method should help to lessen or eliminate oscillation or the “sawtooth” effect evident in standard SPH stress calculations [4].

**5. ADDRESSING TENSION INSTABILITY**

Because the displacement \( U_i \) is the time derivative of \( V_i \) and \( E \) is constant, eqn (26) can be directly integrated in time to yield:

\[ \sigma_{11} = E \varepsilon_{11} \]  

where through eqn (24)

\[ \varepsilon_{11}(t) = \frac{\partial}{\partial x_i} U_i(t). \]

Equations (6) or (7) are then introduced into eqn (28) to express \( \varepsilon_{11}(t) \) in an SPH form. We note that \( \varepsilon_{11} \), which is determined by integrating \( d_i \) in time, is not the well known linear strain. Rather it is a Eulerian variable because the derivative in eqn (28) is with respect to \( x_i(t) \) not \( \dot{x}_i \).
In addition and more importantly, however, the UC approach should also help to address the tension instability present in the conventional \((R = 0.5)\) SPH. The reason for this is as follows. Assume a uniform 1D grid, as shown in Fig. 1, with the smoothing length \(2h = 2L\), where \(L\) is the length of each SPH element. For the conventional approach, all nodes for the SPH gradient calculations are at least \(h (=L)\) distance away, and node \(i\) is not included in any of the calculations for stress and linear momentum at \(i\). In Ref. [2], it has been shown that instability in tension is governed by eqn (29), and for the standard cubic-b-spline kernel this is satisfied in tension whenever \(x_0 > 0.6h\) (approximately). Unfortunately for conventional SPH methods, eqn (29) is satisfied for all the SPH nodes included in the gradient calculations at node \(i\). As mentioned above for the UC approach, node \(i\) is included in the stress calculations at the stress points \(i_1\) and \(i_2\) \((R < 0.5)\), as shown in Fig. 1. Also, the stress points \(i_1\) and \(i_2\) will be included in the linear momentum calculated at node \(i\). Thus, eqn (29) will not be satisfied at these two stress points. In addition, the two SPH elements adjacent to \(i\), \(i - 1\) and \(i + 1\), may also contain stress points (depending on the value of \(R\)), which do not satisfy eqn (29). Therefore, the UC method should have a strong stabilizing effect on the SPH mesh in tension, not allowing it to prematurely fragment.

6. NUMERICAL IMPLEMENTATION SPH1D

For the unconventional (UC) formulation in our 1D code SPH1D, four options have been programmed for the calculation of stress and linear momentum. These options are specified by the parameter \(JSP\). Gradient calculations used to determine the stress \(\sigma_{ij}\) employ either eqn (6) or eqn (7) in this work. Linear momentum calculations are based on either eqns (9) or eqn (10). Recall that eqn (9) does not conserve momentum, but for the UC formulation eqns (6) and (9) are more accurate than eqns (7) and (10). Equations (7) and (10) represent the standard SPH approach in which additional approximations have been introduced [4, 5].

\(JSP = 1\) corresponds to the use of eqns (7) and (10) or the standard SPH particle equations, but with the UC formulation in which the stress points are not located at the SPH nodes \((0 \leq R < 0.5)\). For \(JSP = 2\), eqn (6) is employed for the stresses, while eqn (9) is used for linear momentum calculations. \(JSP = 3\) uses eqn (6) for the stresses and eqn (10) for linear momentum. Thus, options \(JSP = 1\) and \(3\) introduce additional approximations for the UC, but eqn (10) does enforce the conservation of momentum. Also it is noted that for the linear momentum calculations using either eqn (9) \((JSP = 2)\) or eqn (10) \((JSP = 1\) or 3), each of the two stress points within a particular SPH element \((\text{particle})\) is assumed to possess one half of the total mass of that element—this allows retention of the particle concept. In addition, when using eqn (10) \((JSP = 1\) or 3), the stress at the typical SPH node \(i\), since it is not calculated, is assumed to simply be the average of the two stress points \(i_1\) and \(i_2\) (see Fig. 1) in that element.

The fourth option \((JSP = 4)\) in SPH1D uses the same equations as \(JSP = 2\), or eqn (6) for stress calculations and eqn (9) for linear momentum. However, with \(JSP = 4\) the stress \(\sigma_{ij}\) is calculated through eqns (6) and (27) and (28) only at the element centroid \((\text{as if } R = 0.5\) in Fig. 1\) and it is assumed that:

\[\sigma_{ij} = \sigma_{ij}^* = \sigma_{ij}^0.\] (31)

So \(\sigma_{ij}\) is assumed to be constant in SPH element \(i\). Linear momentum is then calculated using eqn (9), but with the stresses applied not at \(i\) (the centroid) but rather at \(i_1\) and \(i_2\), as specified in the UC approach when \(0 \leq R < 0.5\). The option \(JSP = 4\) thus represents a compromise between the conventional and unconventional SPH approaches, that may at least help to stabilize the mesh in tension, although not addressing stress oscillation or "sawtooth" behavior.

Ghost particles are employed for the application of essential boundary conditions. See Ref. [5] for the enforcement of free and fixed end conditions. Also, explicit time integration in the form of a modified central difference (MCD) method is applied to the linear momentum equations. Taylor and Flanagan [9] briefly discuss the MCD method which consists of a forward difference to compute the velocities followed by a backward difference to calculate the displacements. In equation form the MCD is:

\[V(t) = V(t - dt) + dr \cdot A(t - dt)\] (32a)

\[U(t) = U(t - dt) + dr \cdot V(t - dt)\] (32b)

where \(A\) is the acceleration and \(dt\) is the time increment.

Overall, the SPH1D program is a relatively simple 1D code, but it will be a useful platform to demonstrate our unconventional approach. Finally, in SPH1D only a linear elastic material model has been implemented at this point.

7. APPLICATION

In this section as a test of our unconventional approach, the SPH1D code will be applied to the elastic 1D bar described in Fig. 2. The bar is fixed at the right end \(B\) and the left quarter of the bar is given an initial velocity of \(V_0 = -5\) \(\text{m s}^{-1}\), thus putting the bar in tension initially. Standard SPH methods \((R = 0.5)\) cannot solve this problem due to the tension instability that will immediately develop. As indicated in Fig. 2a, the SPH grid is very coarse with only 40 uniform SPH elements (particles) used in the model. A comparable finite element model using the ABAQUS [10, 11] program that consists of 40, 2D solid elements is described in Fig. 2b.
Tension instability in SPH

Young’s Modulus, \( E = 200 \text{E9} \) Pascals
Density, \( \rho = 7833 \) kg/m\(^3\)
Wave Speed, \( c = 5033 \) m/sec

Figure 3 presents the displacement time history of the left end A (SPH node 1 actually) for the UC results with \( R = 0.25 \) (so the stress points are at the quarter points of the SPH elements). Also included are the finite element method (FEM) results using ABAQUS with implicit time integration, which is unconditionally stable. The time step used for the explicit SPH calculations as well as for the implicit FEM model is \( \Delta t = 0.4 \times 10^{-6} \) s. This time step is based upon an estimate of the Courant stability limit of approximately \( 0.66 \times 10^{-6} \) s, which is determined by dividing the length of the typical SPH element by the wave speed \( c \).

As indicated in Fig. 3, the SPH1D results (solid line) with \( R = 0.25 \) are very close to the ABAQUS predictions (dashed line) for the displacement history at the left end of the bar, point A. Some slight phase difference between the two analyses develops later, but this is probably due to the use of an explicit solver for SPH1D and an implicit solver for ABAQUS, as well as basic differences between the SPH and FEM forms of discretization.

The \( JSP = 2 \) option was used to generate Fig. 3. This option in SPH1D employs eqn (6) for stress calculations and eqn (9) for linear momentum. The use of the other options, \( JSP = 1, 3 \) which employ

Fig. 2. A simple bar given an initial velocity: (a) SPH1D grid, (b) ABAQUS FEM grid.

Fig. 3. Displacement history for the left end of the bar (point A).
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Fig. 4. Velocity history for the left end of the bar (point A).

The 1D bar is next given an initial velocity of \( V_0 = 5 \text{ m s}^{-1} \), thus putting the bar in compression initially. This is done to determine the effect of the UC formulation on the stress oscillation or “saw-tooth” behavior encountered in the standard or conventional SPH approach \( (R = 0.5) \). Putting the bar initially in compression allows a comparison between the UC and the conventional SPH, since the latter is not stable in tension. Figure 7 shows a snapshot of the stress \( \sigma_{11} \) along the entire bar at \( t = 1.0 \times 10^{-5} \text{ s} \). The UC results with \( R = 0.25 \) are indicated by the solid line, and the conventional SPH results in which \( R = 0.5 \) are depicted by the dashed line. In general, we see that the UC results are much more stable than the conventional SPH results.
8. CONCLUSIONS

In this work, tension instability has been addressed for SPH methods. A new unconventional (UC) approach has been presented in which the stresses are computed at points away from the SPH nodes. The location of these two stress points within a 1D SPH element (particle) is controlled by the parameter $R$. The UC approach removed the tension instability in the bar considered for $R$ in the approximate range of $0 < R < 0.45$. No optimum value of $R$ was found in general. UC displacement and velocity results were in excellent agreement with a comparable ABAQUS finite element model. Axial stress comparison to the ABAQUS model was not quite as good, but that probably is to be expected given the coarseness of the meshes and the non-local nature of the SPH method, which interpolates outside the SPH element (unlike FEM). In a comparison to standard SPH ($R = 0.5$) for the bar in compression, initially the UC seemed to alleviate the stress oscillation or "sawtooth" effect. The UC approach was successful in removing tension instability, but only for the option $JSP = 2$, in which eqns (6) and (9) were used for the stress and momentum calculations. Equations (6) and (9) are not widely used in the SPH literature, and eqn (9) does not conserve momentum. These are important points to note with the UC approach.

Unfortunately, the $JSP = 4$ option (like $JSP = 1$ and 3) failed to remain stable for the UC method. If this option had been successful, modifications to existing SPH codes would have been easier and less expensive, since only one stress point (the centroid) within each particle would have to be tracked.

Overall, based on the results from the simple 1D bar, the use of the unconventional approach for SPH calculations is very encouraging. This approach can be extended to 2D and 3D problems, but this will be computationally expensive, since additional stress points within each SPH element have to be tracked in the analysis, besides just the centroid. Also, extension to 2D and 3D may require a rethinking of the overall particle concept. Finally, perhaps a method analogous to hourglass control for reduced inte-
migration in finite element techniques [11] may be possible with the UC method.

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