Random uncertainty modeling and vibration analysis of a straight pipe conveying fluid

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Abstract A new procedure on random uncertainty modeling is presented for vibration analysis of a straight pipe conveying fluid when the pipe is fixed at both ends. Taking real conveying condition into account, several randomly uncertain loads and a motion constraint are imposed on the pipe and its corresponding equations of motion, which are established from the Euler–Bernoulli beam theory and the nonlinear Lagrange strain theory previously. Based on the stochastically nonlinear dynamic theory and the Galerkin method, the equations of motion are reduced to the finite discretized ones with randomly uncertain excitations, from which the vibration characteristics of the pipe are investigated in more detail by some previously developed numerical methods and a specific Poincaré map. It is shown that, the vibration modes change not only with the frequency of the harmonic excitation but also with the strength and spectrum width of the randomly uncertain excitations, quasi-periodic-dominant responses can be observed clearly from the point sets in the Poincaré’s cross-section. Moreover, the nonlinear elastic coefficient and location of the motion constraint can be adjusted properly to reduce the transverse vibration amplitude of the pipe.

Keywords Pipe conveying fluid · Random uncertainty · Motion constraint · Specific Poincaré map · Vibration analysis

1 Introduction

In modern industrial practice, there are a large variety of solid particles used as raw materials, catalysts, energy resources, and so on. In comparison with gas or liquid, these solid particles are usually inconveniently processed, stored, or conveyed, so the fluidization technology has been developed from chemical industry and energy conversion, where the solid particles are fluidized by mixing with gas or liquid, which is one of the multi-phase fluidization processes [1–3]. In the early stages of development of the fluidization technology, the understanding of the fluidization process was mostly intuitive or empirical. Gradually, because of the increasing demand for more reliable information, and along with an improved availability of analytical tools, progress has been made not only in the field of basic predictions [4–6] but also in the detailed characterization of the multi-phase flow in pipes [7,8].

In the literature [3], the authors reviewed the procedures for investigating the fluid-dynamic behavior of gas–solid fluidized beds using pressure signals as obtained from modeling or experiments. In 1989,
Stringer suggested that the gas–solid fluidized bed’s dynamic behavior might be interpreted as being chaotic [9]. Since then, several studies had been carried out to characterize this dynamic behavior by means of chaos invariants, to name a few, see [10, 11]. Recently, some approaches for investigation of the modeled and measured signals were discussed in [3] from the time domain, the frequency domain, the time-frequency domain, and the state space, and the time series analysis technique was also recommended to study the pressure signals of pipes conveying multi-phase flow.

From nonlinear dynamical points of view, modeling of vibration of a pipe conveying multi-phase flow is a challenging task. The vibration of a pipe conveying multi-phase flow is inevitably affected by some randomly uncertain factors, and the result predicted by the deterministic dynamic theory will deviate from the real ones. The fluidization condition, such as the distribution of liquid and solid particles in the fluidized bed, the mass flow rate, bubbles and coalescence behavior of solid particles, and so on, can result in the strong time-variation and random fluctuations of the pipe pressure; the adhesion of solid particles in the pipe wall will also induce the random variation of pipe’s material property. Narayanan [12] investigated the stochastic stability of fluid-conveying tubes with random velocity fluctuations and derived the mean square stability conditions for the cases of simply supported, cantilevered, and fixed–fixed tubes. For the problem of stochastic stability of a pipe conveying pulsating fluid, Vedula and Sri Namchchivaya obtained an asymptotic approximation of the moment Lyapunov exponent [13].

An important question of interest, which to our knowledge has not been addressed in most previous studies, is the effect of irregular random pressure on the vibration of a pipe conveying multi-phase flow, and few contributions are concerned with random vibration or random nonlinear dynamics of such pipe based on a model with random parameters and/or random excitations. Here, we develop a kind of random uncertainty modeling technique for a straight pipe conveying fluid and also present a new procedure on discretizing the partial equations of motion by combining the customary Galerkin method with the triangular series approximation of each almost ergodic realization of given random process. The vibration analysis of such pipe is then performed based on the discretized equations and a specific Poincaré map.

In Sect. 2, we firstly develop the effective forms of the randomly uncertain excitations by taking the real multi-phase fluid conveying conditions into account, and these excitations and some motion constraint are then imposed on the equations of motion of the pipe, which are previously established from the general Lagrange strain theory and the Euler–Bernoulli beam theory. In Sect. 3, the almost ergodic realizations for the randomly uncertain excitations are approximately generated by the theory of stochastic process, and the partial differential equations (PDEs) developed in Sect. 2 are discretized to some finite ordinary differential equations (ODEs) based on the customary Galerkin method for the deterministic realizations, and then we introduce a specific Poincaré map from the time-limited invariance condition for further stability analysis of the pipe’s vibration. In Sect. 4, several numerical examples are studied in more detail to illustrate the validity of our algorithm and investigate the pipe’s dynamical evolution trend and the influence of the randomly uncertain excitations and the motion constraint. In the final section, the results of this work are summarized and discussed.

2 Randomly uncertain modeling of a straight pipe conveying fluid

The pipe shown in Fig. 1 has been extensively investigated by many scholars, say, Paidoussis [14] and Lee et al. [15]. Here, we further assume that the pipe is subjected to a motion constraint at some location and used to convey multi-phase fluid. In Fig. 1, $L$ is the length of pipe, $d$ and $h$ are the outer diameter and the thickness, respectively, the position coordinates along the pipe axis is $0 \leq x, x_b \leq L$. $p(x, t)$ is the external excitation on the pipe wall at the position $x$ and the instant $t$, $U(t)$ and $\dot{U}(t)$ are the velocity and the acceleration of the inner multi-phase fluid, respectively, while $u(x, t)$ and $v(x, t)$ are the longitudinal and transverse displacements of the pipe at the position $x$ and the instant $t$, respectively.

From Fig. 1, the pipe also suffers a motion constraint at the location $x = x_b$. Previous studies have shown that the effect of the constraint can be replaced by a spring, e.g., Chen [16] studied a fluid-conveying pipe clamped at the upper end and supported by a displacement spring and found that the pipe could lose its stability within certain ranges of the parameters even with the spring
support. Jiyavan [17] considered a cantilevered pipe, which was stimulated by transverse fluid flow with motion limiting stopper at the free end, and proposed a general theory for forecasting the pipe motion and the pipe baffle impact force. Here, the elastic restraint takes the same form as the one given by Wang [18], i.e.,

$$F(y) = ay^3$$  \tag{1}$$

As mentioned in the previous section, the excitations exerted on the pipe wall by the pulsating fluid, inside and outside temperature fluctuation, ground motion, repetitive on-off operation, and others will cause the pipe’s longitudinal and transverse vibration, and these excitations usually fluctuate uncertainly, which will bring together with the pipe’s stability and reliability problems in practical applications. In the following, these uncertain factors are taken into account to establish the equations of motion of the pipe conveying fluid.

We firstly assume that random uncertainty is included in the external excitation from the pulsating fluid, which is a sum of a periodic excitation and a relatively weak random excitation. Moreover, considering that the pipe wall may suffer strong external forces somewhere, strong random excitations should also be included at some positions along the pipe axis. Thus, the excitation $p(x, t)$ can be expressed in the following form:

$$p(x, t) = p_0 \sin(\Omega t + \alpha) + \varepsilon_0 f_0 \xi_0(t) + \sum_{i=1}^{N_r} f_i \delta(x - x_i) \xi_i(t)$$  \tag{2}$$

where $p_0$ and $\Omega$ are the amplitude and the circular frequency of the periodic excitation, $\alpha$ is the initial phase angle, $0 \leq x_i \leq L$ is the position along the pipe axis, $N_r$ is the number of the stronger random excitations along the pipe axis, $f_0$ and $f_i (i = 1, 2, \ldots, N_r)$ are the constants, $0 \leq \varepsilon_0 \ll 1$ is a small quantity, and $\delta(\cdot)$ is the Dirac function. Moreover, $\xi_i(t)$ and $\xi_0(t)$ are the random excitation processes, which are assumed to be the independent bounded noises [19]:

$$\xi_i(t) = A_i \sin[\Omega_i t + \sigma_i B_i(t) + \gamma_i]$$  \tag{3}$$

where $A_i$ is the noise amplitude, $\Omega_i$ is the central frequency, $\sigma_i$ is a constant, $B_i(t)$ is a unit Wiener process, and $\gamma_i$ is a random variable uniformly distributed in $[0, 2\pi)$. From [19], the mean of the bounded noise is zero, and the noise’s correlation function is given by

$$E[\xi_i(t_1)\xi_i(t_2)] = \frac{1}{2} \cos \Omega_i |t_1 - t_2| \exp\left(-\frac{\sigma_i^2}{2} |t_1 - t_2| \right)$$  \tag{4}$$

and its two-sided power spectral density function is expressed as

$$S_{\xi_i\xi_i}(\omega_i) = \frac{\sigma_i^2}{2\pi} \left[ \frac{1}{4(\Omega_i - \omega_i)^2 + \sigma_i^2} + \frac{1}{4(\Omega_i + \omega_i)^2 + \sigma_i^2} \right]$$  \tag{5}$$

The bounded noise $\xi_i(t)$ is a generalized stationary random process, and the shape of its spectral density is determined by $\Omega_i$ and $\sigma_i$. In general, it has two symmetrical peaks, symmetrically located in the positive and negative frequency domains, their bandwidth depends on $\sigma_i$, and their locations depend on $\Omega_i$, to a less degree, and also on $\sigma_i$. When $\Omega_i/\sigma_i \gg 1$, $\xi_i(t)$ becomes a narrow-band random process, while in the limit as $\sigma_i$ approaches infinity, the random process becomes a “white noise” of constant spectral density [19]. From previous studies, the measured pressure fluctuations are usually shown as irregular narrow band signals (see [3] and other references therein), so we employ the...
bounded noise $\xi_i(t)$ ($i = 0, 1, 2, \ldots, N_r$) to simulate these signals due to its aforementioned features, and it is included in the excitation $p(x, t)$ to establish a reasonable and effective dynamic model for the pipe conveying multi-phase fluid. One can, of course, find an alternative colored noise to simulate these pressure signals, which also depends on actual situation. Nevertheless, from the discretization process presented in Sect. 3, similar analysis can be performed.

Next, we briefly introduce the modeling process presented in [15] based on the general Lagrange strain theory (the nonlinear strains and the linear stresses) and Euler–Bernoulli beam theory. For the pipe conveying fluid shown in Fig. 1, applying the generalized Hamilton principle yields

$$
\int_{t_1}^{t_2} (\delta K - \delta P + \delta W_{nc} - \delta M) dt = 0, \quad (6)
$$

where $\delta$ is the variational operator, $K$ is the kinetic energy, $P$ is the potential energy, $\delta W_{nc}$ is the variation of the virtual work done by the nonconservative forces, while $\delta M$ is the variation of the virtual momentum change. From [15], they can be expressed as

$$
K = \frac{1}{2} m_p \int_0^L \left[ \dot{u}^2(x, t) + v^2(x, t) \right] dx 
+ \frac{1}{2} m_f \int_0^L \left\{ \left[ \dot{u}(x, t) + U(t)(1 + u'(x, t)) \right]^2 \right. \quad (7a)
+ \left. \left[ \dot{v}(x, t) + U(t)v'(x, t) \right]^2 \right\} dx, \quad (7b)
$$

$$
P = \frac{1}{2} EA \int_0^L \left[ u'^2(x, t) + \frac{1}{2} u'(x, t) \right] \times \left[ u^2(x, t) + v'^2(x, t) \right] dx, \quad (7c)
$$

$$
\delta W_{nc} = \int_0^L \left[ p(x, t) + F(v)\delta(x - x_b) \right] dv dx, \quad (7d)
$$

$$
\delta M = \left\{ m_f U(t) \left[ \dot{u}(x, t) + U(t)(1 + u'(x, t)) \right] \delta u 
+ \frac{1}{2} m_f U(t)v'(x, t) \delta v \right\} \int_0^L, \quad (7e)
$$

where $m_p$ and $m_f$ are the mass densities of the pipe and the fluid per unit pipe length, respectively, $E$ is the Young’s elastic modulus, $A$ is the cross-sectional area of the pipe, and $I$ is the area moment of inertia. Substituting Eqs. (7a)–(7d) into Eq. (6), the coupled nonlinear equations of motion for the pipe shown in Fig. 1 can be derived as follows:

$$
(m_p + m_f)\ddot{u} + m_f\dot{U}(1 + u') + 2m_f U\dot{u}' + m_f U^2\dddot{u}' = E\left( u'' + \frac{3}{2} u''u'' + \frac{1}{2} v''v'' \right) - \frac{3}{2} E I v''v''(3)^2 = 0, \quad (8a)
$$

$$
(m_p + m_f)\ddot{v} + m_f\dot{U}v' + 2m_f U\dot{v}' + m_f U^2\dddot{v}' = -\frac{1}{2} E\left( u''v' + u'v'' \right) + E I (v^{(4)} + \frac{3}{2} u'^{(4)}) + 3u''v''(3)^4 + \frac{3}{2} u''(3)^4) = p(x, t) + F(v)\delta(x - x_b), \quad (8b)
$$

where the functions $F(v)$ and $p(x, t)$ in Eq. (8b) refer to Eqs. (1)–(3), and the boundary conditions are given by

$$
u = v' = 0 \quad x = 0, \quad L. \quad (8c)
$$

3 Discretized ODEs subjected to the time-limited invariance condition

Prior to the dynamical analysis on stochastically nonlinear dynamical systems, some ergodic hypotheses need to be proposed, which are usually based on the assumption of fully developed chaos. Since the equations of motion of the pipe shown in Fig. 1, i.e., Eqs. (8a)–(8c), involve the randomly uncertain excitations, the customary Galerkin method cannot be directly applied, and some preliminary treatments should be done by the random process theory and the stochastically nonlinear dynamics. In the following sections, we deal with the randomly uncertain dynamic model developed in Section 2 from a purely deterministic point of view by generating the deterministic almost ergodic realization of the bounded noise $\xi_i(t)$ ($i = 0, 1, 2, \ldots, N_r$).

From [20], each almost ergodic realization of the noise $\xi_i(t)(i = 0, 1, 2, \ldots, N_r)$ can be approximated by

$$
\bar{\xi}_i(t) \approx \sum_{k=1}^{N_i} A_i \cos(\omega_{i,k}t + \phi_{i,k}) \quad (i = 0, 1, \ldots, N_r; N_i \to \infty), \quad (9)
$$

where $A_i = \sqrt{2S_i / \Delta \omega_i}$, $\{\omega_{i,k}|i = 0, 1, \ldots, N_r; k = 1, 2, \ldots, N_i\}$ is an independent and nonnegative random variable over the interval $[\omega_{i,1}, \omega_{i,N_i}]$, $\{\phi_{i,k}|i = 0, 1, \ldots, N_r; k = 1, 2, \ldots, N_i\}$ is an independent and nonnegative random variable over the interval $[\phi_{i,1}, \phi_{i,N_i}]$. The parameter $\Delta \omega_i$ is the thickness of the adjacent frequency band, and $S_i$ is the corresponding power density.
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0, 1, ..., \(N_p\); \(k = 1, 2, \ldots, N_l\) is the random phase uniformly distributed in the interval \([0, 2\pi]\), \(N_l(i = 0, 1, \ldots, N_p)\) is a sufficiently large positive integer, and \(\Delta \omega_i = (\omega_{i,r} - \omega_{i,l}) / N_l\) is a frequency increment. Since each realization of the noise \(\xi_i(t) (i = 0, 1, 2, \ldots, N_p)\) is approximated by the sum of \(N_l(i = 0, 1, \ldots, N_p)\) harmonic functions, it can be treated as a deterministic function once it is generated, and then the customary Galerkin method can be applied to derive the finite discretized ODEs from the pipe’s equations of motion, i.e., Eqs. (8a)–(8c).

From Eq. (9), the equations of motion of the pipe are transformed into the deterministic ones, and the discretization process by the Galerkin method can be performed. Since the discretization process is nearly the same as that in the work by Lee and Chung [15] except for several additional randomly uncertain excitations and a motion constraint in the equations, we only present some necessary steps and equations for further numerical analysis. We also introduce the trial functions as follows:

\[
\begin{align*}
  u(x, t) &= \sum_{n=0}^{N} U_n(x) \tilde{T}_n^{u}(t), \quad (10a) \\
  v(x, t) &= \sum_{n=0}^{N} V_n(x) \tilde{T}_n^{v}(t), \quad (10b)
\end{align*}
\]

and the weighting functions corresponding of the above trial functions are given by

\[
\begin{align*}
  \tilde{u}(x, t) &= \sum_{n=0}^{N} \bar{U}_n(x) \tilde{T}_n^{u}(t), \quad (11a) \\
  \tilde{v}(x, t) &= \sum_{n=0}^{N} \bar{V}_n(x) \tilde{T}_n^{v}(t), \quad (11b)
\end{align*}
\]

where \(N\) is the total number of the basis functions, \(\tilde{T}_n^{u}(t)\) and \(\tilde{T}_n^{v}(t) (n = 0, 1, \ldots, N)\) are unknown functions of time to be determined, \(\tilde{T}_n^{u}(t)\) and \(\tilde{T}_n^{v}(t) (n = 0, 1, \ldots, N)\) are arbitrary functions of time, while \(U_n(x)\) and \(V_n(x) (n = 0, 1, \ldots, N)\) are the comparison functions for the longitudinal and transverse displacements and written as

\[
\begin{align*}
  U_n(x) &= a_n x^{n+1} (L - x), \quad (12a) \\
  V_n(x) &= b_n x^{n+2} (L - x)^2, \quad (12b)
\end{align*}
\]

where \(a_n\) and \(b_n\) \((n = 0, 1, \ldots, N)\) are the constants satisfying the following normalization conditions:

\[
\int_0^L U_n dx = \int_0^L V_n dx = 1. \quad (13)
\]

Introducing \(u(x, t)\) and \(v(x, t)\) by Eqs. (10a)–(10b) into Eqs. (8a)–(8b), multiplying \(\tilde{u}(x, t)\) and \(\tilde{v}(x, t)\) by Eqs. (11a)–(11b), summing all the equations, integrating them over the length \(L\), and then collecting all the terms about \(\tilde{T}_n^{u}(t)\) and \(\tilde{T}_n^{v}(t) (n = 0, 1, \ldots, N)\), we can get the finite discrete equations as follows:

\[
\begin{align*}
  \sum_{n=0}^{N} & \left[ m_{nl}^{u} \ddot{T}_n^{u} + 2 U_n s_{nl} T_n^{u} + (k_{nl}^{u} + U_n^2 h_{nl}^{u} + \dot{U}_n s_{nl}^{u}) T_n^{u} \right] = f_t^{u} \quad (14a) \\
  \sum_{n=0}^{N} & \left[ m_{nl}^{v} \ddot{T}_n^{v} + 2 U_n s_{nl} T_n^{v} + (k_{nl}^{v} + U_n^2 h_{nl}^{v} + \dot{U}_n s_{nl}^{v}) T_n^{v} \right] = f_t^{v} \quad (14b)
\end{align*}
\]

where

\[
\begin{align*}
  m_{nl}^{u} &= (m_p + m_f) \int_0^L U_n U_j dx, m_{nl}^{v} \\
  & = (m_p + m_f) \int_0^L V_n V_j dx, s_{nl}^{u} = m_f \int_0^L U_n \frac{dU_j}{dx} dx, \\
  s_{nl}^{v} &= m_f \int_0^L V_n \frac{dV_j}{dx} dx, h_{nl}^{u} \\
  & = m_f \int_0^L U_n \frac{d^2 U_j}{dx^2} dx, h_{nl}^{v} \\
  k_{nl}^{u} &= -EA \int_0^L U_n \frac{d^2 U_j}{dx^2} dx, k_{nl}^{v} = EI \int_0^L V_n \frac{d^4 V_j}{dx^4} dx, \\
  \alpha_{jnl}^{u} &= -\frac{3}{2} EA \int_0^L U_n \frac{dU_j}{dx} \frac{d^2 U_j}{dx^2} dx, \\
  \alpha_{jnl}^{v} &= -\frac{3}{2} EI \int_0^L V_n \frac{dV_j}{dx} \frac{d^2 V_j}{dx^2} dx
\end{align*}
\]
\[ \alpha_{jnl} = -\frac{1}{2} EA \int_0^L V_n \frac{d}{dx} \left( \frac{dU_j}{dx} \cdot \frac{dV_i}{dx} \right) dx + \frac{3}{2} EI \]

\[ \times \int_0^L V_n \left( \frac{dU_j}{dx} \frac{d^4V_i}{dx^4} + 2 \frac{d^2U_j}{dx^2} \frac{d^3V_i}{dx^3} + \frac{d^3U_j}{dx^3} \frac{d^2V_i}{dx^2} \right) dx, \]

\[ f_i^n = -m_j \ddot{U} \int_0^L U_j dx, \]

\[ f_i^n = m_j \int_0^L V_i p(x, t) dx, \]

\[ = A \sin(\Omega t + \alpha) + \epsilon_0 f_0 \xi_0(t) + \sum_{i=1}^{N_r} f_i V_i(x_i) \xi_i(t) \]

\[ -K \left[ \sum_{j=0}^{N_r} V_j(x_0) T_n^j(t) \right]^3 V_0(x_0). \]

The above Eqs. (14a)–(14b) can be rewritten in the same vector-matrix form as that in [15]:

\[ \mathbf{M} \ddot{\mathbf{T}}(t) + 2U \mathbf{G} \dot{\mathbf{T}}(t) + (\mathbf{K} + U^2 \mathbf{H} + \dot{U} \mathbf{G}) \mathbf{T}(t) + \mathbf{N}(\mathbf{T}(t)) = \mathbf{F}(t), \quad (15) \]

where the \( \mathbf{T}(t) \) is displacement vector, \( \mathbf{M} \) is the mass matrix, \( \mathbf{G} \) is the matrix associated with the gyroscopic force, \( \mathbf{K} \) is the structural stiffness matrix, \( \mathbf{H} \) is the matrix associated with the fluid centrifugal force, \( \mathbf{N}(\mathbf{T}(t)) \) is the nonlinear internal force vector, and \( \mathbf{F}(t) \) is the external force vector affected by the realizations of the randomly uncertain excitations and the motion constraint.

Traditionally, the terminology “stability” means a weak sensitivity of a solution to the small changes in initial conditions, and, therefore, starting points for the stable trajectories belong to some compact sets in phase space. In Eqs. (14a)–(14b) or Eq. (15), although each individual realization of a randomly uncertain excitation can be treated as a deterministic function rather than a stochastic one, even a weak noise may accelerate the phase space transport between some different stable regions as \( t \to \infty \). It implies that all trajectories are unstable in the limit \( t \to \infty \), which makes a deterministic description of long-term dynamics senseless.

To investigate the vibration response and dynamic behavior of the present pipe, further numerical analysis is limited to a finite time interval, say, \([0, T_0]\), where \( T_0 \) can be adjusted as required for various engineering applications. Under the restriction on time duration, we can introduce the previously developed time-limited invariance condition to Eqs. (14a)–(14b) or Eq. (15), i.e., if any set in the phase space at \( t = 0 \) transforms to itself at \( t = T_0 \) without mixing, then it corresponds to an ensemble of trajectories that are stable by Lyapunov within the interval \([0, T_0]\), see [21–23], and some customary deterministic approaches can be employed to explore the sets of stable trajectories.

From this point of view, we can establish a specific Poincaré map as follows:

\[ \Phi : \Sigma_0^{\phi_0} \rightarrow \Sigma_0^{\phi_0}, \]

\[ (u_0(0), v_0(0)) \rightarrow (u_0(T_0), v_0(T_0)), \quad (16) \]

where \( \Sigma_0^{\phi_0} \) is a global cross-section in the phase space of Eqs. (14a)–(14b) or Eq. (15). For each given realization of the bounded noise excitation \( \xi(t) (i = 0, 1, \ldots, N_r) \), we integrate Eqs. (14a)–(14b) or Eq. (15) within the interval \([0, T_0]\) from the initial conditions \( u(0) = u_0 \) and \( v(0) = v_0 \) at \( t = 0 \) by proper numerical integration approaches, say, the well-known Runge Kutta method, the results of \( u(t) \) and \( v(t) \) at \( t = T_0 \) can be obtained. Then, \( u(T_0) \) and \( v(T_0) \) are used as the new initial conditions to calculate \( u(2T_0) \) and \( v(2T_0) \) for the same realization within the interval \([0, T_0]\), and so on. Thus, the \( n_1 \)-th iteration of the above specific Poincaré map (16) is given by

\[ \Phi^{n_1} : \Sigma_0^{\phi_0} \rightarrow \Sigma_0^{\phi_0}, \]

\[ (u_0(0), v_0(0)) \rightarrow (u(n_1 T_0), v(n_1 T_0)). \quad (17) \]

In this way, we can repeatedly observe the dynamic transitions of the pipe excited by the same physical realizations of the bounded noise \( \xi(t) (i = 0, 1, 2, \ldots, N_r) \) within the time interval \([0, T_0]\), i.e.,

\[ \xi(t + n_1 T_0) = \xi(t), \quad t \in [0, T_0]; \quad i=0, 1, \ldots, N_r. \quad (18) \]

From Eq. (18), the original equations with randomly uncertain excitations are replaced by the periodically driven ones. For further analysis, a comment should be made on the \( n_1 \)-th iteration of the Poincaré map. When the excitation imposed on a system is a stationary random process, there may still exist some stable ensembles of trajectories by Lyapunov in phase space after long-time evolution from the work by Makarov [21] and our previous studies [22,23], where the phenomenological structures of the stable ensembles of trajectories are found to be similar for various almost ergodic physical realizations. This can be explained as follows: since \( N_r(i = 0, 1, \ldots, N_r) \) is unconcerned with \( t \) during the generating process for each physical realization, see Eq. (9), \( N_r(i = 0, 1, \ldots, N_r) \) can be large enough.
to make each physical realization sufficiently ergodic within the time interval \([0, T_0]\), and the phenomenological structures of any hyperbolic trajectory in different ergodic realization excitations appear to be similar from the stochastically nonlinear dynamics. Moreover, from the literature [23], if the randomly uncertain perturbation is weaker than the harmonic excitation, \(T_0\) can be chosen as the period of the harmonic excitation, from which we can discuss the effects of the band-limited randomly uncertain pressure signals on the pipe’s vibration.

4 Illustrating examples

As mentioned previously, there have been many contributions on the irregular pressure signals from the fluidized beds, which are of significance for the studies on the vibration of a straight pipe conveying fluid. Depending on the properties of the solid particles and the fluidization conditions (e.g., temperature of fluidizing gas or liquid), the particles may agglomerate, which will subsequently lead to defluidization parts of the pipeline and even blockage of the pipe, resulting in maldistribution of the gas or liquid, local hot spots, local increase of the pressure drop to the inner pipe wall, and so forth.

In Sect. 2, the excitations exerted on the pipe wall with a motion constraint are assumed to come from the pulsating fluid, a randomly uncertain perturbation of the fluid, and several strong randomly uncertain pressures at some positions along the pipe axis due to severe fluidization condition, see Fig. 1, from which the coupled nonlinear equations of motion of the pipe and their discrete ODEs are derived from the Euler-Bernoulli beam theory, the nonlinear Lagrange strain theory, the stochastically nonlinear dynamic theory, and the Galerkin method.

In the following, several numerical examples are used to investigate the vibration of the fluid-conveying pipe with randomly uncertain excitations by the above procedure. First, the natural frequencies of a pipe are calculated by the linearized one of Eq. (5), and the vibration responses are investigated when the randomly uncertain excitations are absent in the model, i.e., \(f_i = 0(i = 0, 1, 2, \ldots, N_r)\). Second, the influence of the randomly uncertain excitations on the vibration response and stability of the pipe without motion constraint is studied in more detail, where the almost ergodic physical realizations of the bounded noises are approximately generated by Eq. (9), and the specific Poincaré map by Eq. (16) or Eq. (17) is employed to perform the stability analysis. Finally, in the presence of randomly uncertain excitations, the motion constraint is imposed on the pipe to observe the influences of the elastic coefficient and the location of the constraint on the pipe’s vibration.

For the present numerical examples, we take some proper parameters according to GB/T 12771-2008 of China for a welded stainless steel pipe conveying fluid, i.e., \(m_p = 4.0137 \, \text{kg/m}, \ E = 2.06 \times 10^{11} \, \text{Pa}, \ L = 8\, \text{m}, \ d = 0.11\, \text{m}, \ h = 0.0015\, \text{m}; \) while the fluid in the pipe is assumed to be crude oil with mass density \(m_f = 7.3735 \, \text{kg/m}^3\) at a standard temperature according to GB/T 1985–1998 of China. The total number of the basis functions is set to be \(N = 5\), which is found enough from the subsequent numerical results. For the deterministic pressure term in Eq. (2), we choose \(p_0 = 1 \, \text{N/m}, \ \Omega = 10\pi\), and \(\alpha = 0\) from previous studies (cf. [18]). For the bounded noise terms in Eqs. (2) and (3), we set \(N_0 = 2, \ omega_0 = 2\pi, \ \sigma_0 = 10, \ x_1 = L/4, \ x_2 = 3L/4, \ \omega_1 = 1, \ \omega_2 = 2, \ \sigma_1 = 2, \ \sigma_2 = 1, \ omega_{0,l} = \omega_{1,l} = \omega_{2,l} = 120\pi, \) and \(\omega_{0,r} = \omega_{1,r} = \omega_{2,r} = 2000\), to simulate the irregular narrow-band pressure signals. It should be noted that these choices for the banded noises seem a little subjective, which depend on real engineering application. However, for other choices and even for other random processes, similar simulations can be complemented. To make the physical realizations almost ergodic, we set \(N_0 = N_1 = N_2 = 20000\). Without loss of generality, the motion constraint is placed at the location \(x_{b} = L/2\), in the subsequent simulations, unless otherwise stated.

4.1 Natural frequencies and vibration responses of the pipe in the deterministic case

The linearized equations of motion near the equilibrium point are the same as those by Lee and Chung [15], i.e.,

\[
\mathbf{M} \ddot{\mathbf{T}}(t) + 2U \mathbf{G} \dot{\mathbf{T}}(t) + (\mathbf{K} + U^2\mathbf{H})\mathbf{T}(t) = \mathbf{0}
\]

and can be rewritten by

\[
\mathbf{A} \ddot{\mathbf{Y}}(t) + \mathbf{B} \dot{\mathbf{Y}}(t) = \mathbf{0},
\]

where

\[
\mathbf{A} = \begin{bmatrix} 0 & 1 \\ \mathbf{M} & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 2UG & \mathbf{K} + U^2\mathbf{H} \end{bmatrix},
\]

\[
\mathbf{Y}(t) = \begin{bmatrix} \dot{\mathbf{T}}(t) \\ \mathbf{T}(t) \end{bmatrix}.
\]
Assume that each solution of Eq. (20) is expressed as
\[ Y(t) = Y_0 e^{\lambda_n t}, \]
the complex eigenvalue \( \lambda_n \) can be calculated by
\[ \det(B + \lambda_n A) = 0. \]  
(22)

In the absence of randomly uncertain excitations and motion constraint, nearly the same natural frequencies and vibration responses can be obtained from our numerical codes if we take the same parameters in Eqs. (19) or (20) as those in [15], and the results are not repeated here. For the present pipe with the parameters described above, Table 1 shows the convergence characteristics of the natural frequencies with the increase in the number \( N \) of basis functions when the pipe is stationary, i.e., \( U = \dot{U} = 0 \). Obviously, the calculated natural frequencies are close to the accurate values given by Blevins [24] when \( N = 5 \).

To further study the variation of the complex eigenvalue with the fluid velocity, we introduce the dimensionless eigenvalue \( \bar{\lambda}_n \) and the dimensionless fluid velocity \( \bar{U} \) as in [25]:
\[ \bar{\lambda}_n = \lambda_n L^2 \sqrt{\frac{m_p + m_f}{EI}}, \quad \bar{U} = \frac{UL}{\sqrt{EI}}. \]  
(23)

The results are shown in Fig. 2, where \( \dot{U} = 0 \), and the first critical fluid velocity is about \( \bar{U}_c = 6.28 \), see also Païdousis’s work [25]. Within the interval \( 6.28 < \bar{U} < 8.99 \), the eigenvalue of the first mode is a real number, which associates with the divergence instability of the present pipe. The critical velocity for flutter instability is about \( \bar{U}_c = 9.46 \), where the first two modes have the same eigenvalues [25].

In the absence of randomly uncertain excitations and motion constraint, we present several transverse vibration responses of the present pipe based on the nonlinear models developed in [15] by Lee et al and [25] by Païdousis, respectively, for a comparison. All the equations given in these references are discretized by the Galerkin method with the same basis functions, and the dynamic responses are calculated by the generalized-\( \alpha \) method [26]. The dynamic responses of the transverse displacement at \( x = L/2 \) are shown in Figs. 3 and 4 when the fluid velocities are set to be \( U = 0 \) and \( U = 3 \text{ m/s} \), respectively. In both cases, there are almost no differences between the responses calculated by the models with different nonlinear terms in [15] and [25], respectively, since the fluid velocity in real application is far less than the critical velocity of the pipe. Nevertheless, the equations in [15] are derived consistently by only assuming linearized stress which has been shown more reasonable than those by Païdousis [25] based on the order-of-magnitude approximation and the infinitesimal strain theory, so they are further developed to investigate the dynamic response of the pipe subjected to randomly uncertain excitations and motion constraint in the present study.

### 4.2 Influence of randomly uncertain excitations on transverse vibration responses of the pipe without motion constraint

As mentioned in the foregoing, a pipe conveying fluid unavoidable suffers from randomly uncertain excitations, especially when the pipe is employed to convey fluidized particles. The excitation \( p(x, t) \) shown
Table 1  Convergence characteristics of the natural frequencies (rad/s) when $U = \dot{U} = 0$

<table>
<thead>
<tr>
<th>$N$</th>
<th>Transverse vibration</th>
<th>Longitudinal vibration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>First</td>
<td>Second</td>
</tr>
<tr>
<td>1</td>
<td>40.93</td>
<td>114.72</td>
</tr>
<tr>
<td>2</td>
<td>40.79</td>
<td>114.72</td>
</tr>
<tr>
<td>3</td>
<td>40.79</td>
<td>112.47</td>
</tr>
<tr>
<td>4</td>
<td>40.79</td>
<td>112.43</td>
</tr>
<tr>
<td>5</td>
<td>40.79</td>
<td>112.43</td>
</tr>
<tr>
<td>6</td>
<td>40.79</td>
<td>112.43</td>
</tr>
<tr>
<td>7</td>
<td>40.79</td>
<td>112.43</td>
</tr>
</tbody>
</table>

Fig. 4  The transverse displacement responses at $x = L/2$ based on the model by a Lee et al. [15], and b Paidousis [25], respectively, where $U = 3 \text{ m/s}$

in Fig. 1 is written in the form of Eq. (2) to study the influence of randomly uncertain excitations on the pipe’s vibration by varying their strength and spectrum widths. Here, we only focus on the time history of the pipe’s transverse vibration and the point sets in the Poincaré’s cross-section, where the sampling period is set to be the harmonic one, and further identification on the noise-contaminated responses of the pipe can refer to our previous work [22, 23] and other references therein.

First, we investigate the effect of the circular frequency of the harmonic component on the pipe’s transverse vibration in the presence of the randomly uncertain excitations. From Fig. 5, where $\varepsilon_0 f_0 = f_1 = f_2 = 0.05$, $U = 3 \text{ m/s}$, the transverse vibration of the pipe will vary from not only the amplitude but also the phase. Interestingly, among the four cases, the transverse vibration of the pipe is very sensitive to the weak bounded noise excitations only when $\Omega = 10\pi \text{ rad/s}$, and the noise-contaminated characteristic is more obvious, see Fig. 5c, d, where the phase portrait becomes more complex and the point sets in the Poincaré’s cross-section seems to be in a mess in comparison with the other three nearly closed curves, so this special case is picked out for a further study in the following. Moreover, when $\Omega = 12\pi \text{ rad/s}$, the amplitude of the transverse displacement of the pipe is larger than those in the other three cases, since this frequency is more close to the first order natural frequency.

Figure 6 shows the vibration responses corresponding to $\varepsilon_0 f_0 = 0$, 0.03, 0.1, 0.5 and $f_1 = f_2 = 0$. With the increase of $\varepsilon_0 f_0$, the phase portraits of the transverse vibration of the pipe obviously differ from each other from Fig. 6a, c, e, g, and the regular closed curve in the deterministic case will be broken up to be dispersed, and irregular point sets till a mess in the Poincaré’s cross-section, see Fig. 6b, d, f, h. If we neglect the randomly uncertain fluctuation but increase the values of $f_1$ and $f_2$, similar results can be obtained, but the variation ranges in the horizontal and vertical directions, which represent the transverse vibration amplitude and the velocity, respectively, change more slowly in comparison with the former, see Fig. 7.

The spectrum width of each bounded noise excitation $\xi_i(t)$ ($i = 0, 1, 2, \ldots, N_r$) can be widened by increasing the value of parameter $\sigma_i$ ($i = 0, 1, 2, \ldots, N_r$). From Fig. 8, the amplitude of the displacement and the velocity of the pipe’s transverse vibration change little with the increase of $\sigma_0$, while the point
Fig. 5 Phase portraits of the transverse vibration at \( x = L/2 \) and the point sets in the Poincaré’s cross-sections, where (a) and (b) \( \Omega = 8\pi \text{rad/s} \), (c) and (d) \( \Omega = 10\pi \text{rad/s} \), (e) and (f) \( \Omega = 12\pi \text{rad/s} \), (g) and (h) \( \Omega = 16\pi \text{rad/s} \).

Sets in the Poincaré’s cross-section will change from a dispersed loop to a nearly closed curve, see Fig. 8b, d, f, h, which implies that the transverse vibration will become more and more regular-dominant and almost no prick appears when \( \sigma_0 \geq 200.0 \). When we increase the value of other parameter \( \sigma_i (i = 1, 2, \ldots, N_r) \), similar results can be obtained, which are not given here.
4.3 Effect of additional motion constraint on the pipe’s transverse vibration

From the previous subsections, the pipe will vibrate when it is subjected to pulsating pressure from the inner flow and randomly uncertain excitations, and the amplitude of the transverse displacement can increase following the increase of the flow velocity or the strength and frequency of the pulsating pressure. In many engineering applications, a motion constraint is added at some location of the pipeline to avoid potential hazards from large amplitude of the transverse displace-
Fig. 7  Phase portraits of the transverse vibration at $x = L/2$ and the point sets in the Poincaré’s cross-sections, where $\varepsilon_0 f_0 = 0$, a and b $f_1 = f_2 = 0$, c and d $f_1 = f_2 = 0.05$, e and f $f_1 = f_2 = 0.3$, g and h $f_1 = f_2 = 1.0$.

ment, see Fig. 1. Here, we investigate the effect of the additional motion constraint on the pipe’s transverse vibration by varying the nonlinear elastic coefficients $a$ in Eq. (1) and the location of the additional constraint. In both the cases, the pipe is subjected to the pulsating pressure and the randomly uncertain excitations, and we set $\varepsilon_0 f_0 = f_1 = f_2 = 0.05$ and $\Omega = 14\pi \text{ rad/s}$.

The calculated transverse vibration responses of the pipe with additional constraint at $x = L/2$ are shown in Fig. 9 corresponding to the nonlinear elastic coeffi-
of the transverse vibration is nearly twice that in Fig. 9a or c when $a = 10^{10} N/m^3$. By contrast with the results shown in Fig. 9e, f, a weaker transverse vibration is expectedly achieved when $a = 5 \times 10^{11} N/m^3$, see Fig. 9g, h, from which the amplitude of the transverse vibration is less than half of that in Fig. 9a–d. This complex phenomenon is explained in [18] as follows:

Fig. 8 Phase portraits of the transverse vibration at $x = L/2$ and the point sets in the Poincaré’s cross-sections, where $\varepsilon_0 f_0 = 0.03$, $f_1 = f_2 = 0$, $a$ and $b$ $\sigma_0 = 1.0$, $c$ and $d$ $\sigma_0 = 10.0$, $e$ and $f$ $\sigma_0 = 100.0$, $g$ and $h$ $\sigma_0 = 200.0$

On the whole, the transverse vibration appears to be a quasi-periodic-dominant motion even though the pipe is suffered from the randomly uncertain excitations. When $a = 10^8 N/m^3$, the pipe’s transverse vibration response is nearly the same as that without the motion constraint, see Fig. 9a–d. Unexpectedly, the amplitude of $a = 0, 10^8, 10^{10}, 5 \times 10^{11} N/m^3$, respectively.

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$\varepsilon_0 f_0 = 0.03$, $f_1 = f_2 = 0$, $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$.
Fig. 9  Phase portraits of the transverse vibration at $x = \frac{L}{2}$ and the point sets in the Poincaré’s cross-sections, where (a) and (b) $a = 0$, (c) and (d) $a = 10^8 N/m^3$, (e) and (f) $a = 10^{10} N/m^3$, (g) and (h) $a = 5 \times 10^{11} N/m^3$

the mean deformation-induced tension dominates the nonlinearity at a small-value range of $a$; however, for a large-value range of $a$, the nonlinearity is dominated by both the mean deformation-induced tension and the motion constraint. This means that if we want to add the constraint to reduce the vibration, the material with nonlinear elastic coefficient $a$ must be properly chosen. When $a$ is too small or improper, the constraint is difficult to realize the damping effect and may even aggravate the transverse vibration of the pipe.

In Fig. 10, we present the transverse vibration response of the pipe without the motion restraint (or $a = 0$) and with the constraint located at $x = \frac{L}{4}$, $\frac{L}{2}$, $\frac{3L}{4}$, and the nonlinear elastic coefficient is set to be $a = 5 \times 10^{11} N/m^3$. As expected, the effect of the motion restraint located at $x = L/2$ is much better than that located at $x = L/4$ or $3L/4$. It can be reasonably concluded that the midpoint of the straight pipeline is the best location for the motion constraint to suppress the pipe’s transverse vibration. Moreover,
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the noise-contaminated characteristics of the responses in Fig. 10c, d, g, h are more obvious than those in Fig. 10a, b, e, f, and the responses in Fig. 10a, b, e, f can be easily identified as the quasi-periodic-dominant ones.

5 Conclusions

Vibration analysis of a pipe conveying fluid has been an important subject of numerous studies due to their wide engineering applications; the deterministic vibra-
tion and stability of pipes conveying fluid have been extensively studied for different types of boundary conditions. To investigate the vibration of a straight pipe conveying fluid under irregular pressure from fluidized particles in many chemical industrial fields, we propose a randomly uncertain model for such pipe based on the nonlinear model by Lee et al, in which the bounded noise excitation is employed to simulate the narrow-band pressure signals due to its statistical features, and the influence of the motion constraint on the pipe is also taken into account.

For the partial equations of motion of the pipe, we develop a discretization procedure. From the stochastic dynamical theory, each almost ergodic realization of the bounded noise excitation can be approximately generated by a sufficiently large sum of harmonic functions, so the customary Galerkin method can be applied to derive the discretized ODEs, from which the generalized-$\alpha$ method is used to calculate the vibration responses, and a specific Poincaré map is established to investigate the noise-contaminated characteristics of the transverse vibration responses. Our numerical codes are validated by comparing the results with those from Lee et al and the theoretically predicted values. When the randomly uncertain excitations and the motion constraint are present, we come to the conclusions as follows:

1. The bounded noise excitation does small contribution on the amplitude of transverse displacement at the midspan of the pipe, but with the increase of the strength of the excitation, the velocity of the transverse vibration will change more dramatically within a widened region, and the regular closed curve in the deterministic case will be broken up to be dispersed and irregular point sets till a mess in the Poincaré’s cross-section, i.e., the noise-contaminated characteristic will be more apparent. As the spectrum width of the bounded noise increases, the point sets in the Poincaré’s cross-section will change from a dispersed loop to a nearly closed curve, i.e., the quasi-periodic-dominant characteristic can be observed more clearly.

2. The transverse vibration of the pipe will vary from not only the shape but also the regular-dominant period and the amplitude when the circular frequency of the harmonic excitation changes. Moreover, the amplitude of the transverse displacement of the pipe is larger than those in other cases as the frequency of the harmonic excitation is close to the first order natural frequency. When a motion constraint is tried to reduce the vibration of the pipe, improper material, and location of the constraint can even aggravate the pipe’s vibration. From the present study, it is better to place the constraint with large nonlinear elastic coefficient at the midspan of the pipe.

Here, the irregular pressure is modeled by a bounded noisy excitation, despite other colored noisy excitations, the randomness of the pressure appears not only in the form of time but also of space, i.e., a spatiotemporal random process may be more feasible than the current ones in practice. Moreover, in view of fault diagnosis, some fault features can be extracted from irregular vibration signals by the identification on various noise-contaminated signals and experimental analysis. All these deserve further study.

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