An integrated model for slack-based measure of super-efficiency in additive DEA☆

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A B S T R A C T

As a non-radial approach, a super-efficiency model, Super SBM, was proposed by Tone [15] to rank efficient DMUs. Du et al. [7] extends the Super SBM model to the additive (slacks-based) DEA model. To obtain the super-efficiencies of the DMUs, one needs to identify the efficient DMUs first and then apply the additive super-efficiency model to those efficient DMUs. In this paper, we propose an integrated model so that the efficiencies of the inefficient DMUs and the super-efficiencies of the efficient DMUs can be obtained by a single model. The efficiency scores obtained by our integrated model are the same as those obtained by Du et al. [7] and the additive DEA model.

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1. Introduction

Ever since Charnes et al. [3] introduced data envelopment analysis (DEA), many researchers have endeavored to propose innovative methodologies and explore new applications in DEA. The measure proposed by Charnes et al. [3] is referred to as the radial efficiency measure. With radial efficiency measure, a DMU with the efficiency score equal to one might be inefficient because it could not account for all efficiency components of a DMU [12]. Under radial efficiency measure, a DMU with the efficiency equal to one is usually referred to as weakly efficient and a weakly efficient DMU with zero slacks is referred to as strongly efficient. To account for all inefficiencies, Charnes et al. [2] proposed an additive model to evaluate the performance of a DMU in terms of the input and output slacks (excesses and shortfalls) directly. A DMU is efficient if it has zero slacks. However, the additive model does not provide an efficiency measure. As a representative of non-radial measures, an efficiency measure based on slacks directly was proposed by Tone [14], which is referred to as the SBM (slack-based measure). With SBM, a DMU having efficiency score equal to one is strongly efficient.

To differentiate efficient DMUs under the radial efficiency measure, Andersen and Petersen [1] proposed a radial super-efficiency model so that efficient DMUs can have efficiency scores larger than one and the efficiency scores of the inefficient DMUs remain unchanged. Usually, the super-efficiency model proposed in [1] assumes constant returns to scale (CRS). If variable returns to scale (VRS) is assumed, the super-efficiency model may suffers from infeasibilities. These detail discussions can be found in [4–6,9–11,13]. To differentiate efficient DMUs under slacks-based measure, Tone [15] proposed the Super SBM model based on SBM model to rank efficient DMUs. An alternative approach to compute the super-efficiency based on SBM is proposed by Fang et al. [8].

Du et al. [7] extended the Super SBM model to the additive (slacks-based) DEA model. One needs to identify the efficient DMUs first then applies the additive super-efficiency model to those efficient DMUs. These steps could be cumbersome in the implementation of real applications because it is time-consuming, especially in the application involving large scale of data. In current paper, we propose a unified model so that the efficiencies of the inefficient DMUs and the super-efficiencies of the efficient DMUs can be obtained by a single model. However, the projections identified by the super-efficiency model proposed by Du et al. [7] might be weakly efficient. Our one-stage model guarantees that the projections identified are strongly efficient. By contrast, our approach is more efficient than the method proposed by Du et al. [7]. Our model can be used to compute both efficiency scores and super-efficiency scores in one step. This saves the need to switch between two models and lots of efforts in large scale of real applications. The contribution of our method is twofold. First, the strongly efficient projection of the target DMU is identified. Second, it is time-saving compared to the method proposed by Du et al. [7].
The rest of the paper is organized as follows. Section 2 briefly reviews the SBM and Super SBM models proposed by Tone. In Section 3, the additive super-efficiency model proposed by Du et al. [7] is reviewed. Section 4 presents our integrated additive super-efficiency model. Illustrative examples are demonstrated in Section 5. Finally, some remarks are made in Section 6.

2. Slack-based measure super-efficiency

Assume we deal with a set of \( n \) DMUs under evaluation. Each DMU has \( m \) inputs and \( s \) outputs. The \( i \)-th input and the \( r \)-th output of the DMU \( j \) are denoted as \( x_{ij} (i = 1, ..., m; j = 1, ..., n) \) and \( y_{jr} (r = 1, ..., s; j = 1, ..., n) \) respectively. Tone [14] proposed the following SBM model to evaluate the efficiency of the DMU \( k \):

\[
1 - \frac{1}{m} \sum_{i=1}^{m} z_{i}^{-} / x_{ik} \\
1 + \frac{1}{r} \sum_{r=1}^{s} z_{r}^{+} / y_{rk} \\
\text{s.t.} \quad x_{ik} = \sum_{j=1}^{n} x_{ij} \lambda_{j} + z_{i}^{-}, \text{ } i = 1, ..., m \\
y_{rk} = \sum_{j=1}^{n} y_{jr} \lambda_{j} - z_{r}^{+}, \text{ } r = 1, ..., s \\
\lambda_{j} \geq 0, j = 1, ..., n \\
z_{i}^{-} \geq 0, i = 1, ..., m \\
z_{r}^{+} \geq 0, r = 1, ..., s \\
(1)
\]

Note that due to the objective function, the model requires that all data are positive, i.e., \( x_{ij}, y_{jr} > 0 \) for all possible \( i = 1, ..., m; r = 1, ..., s; j = 1, ..., n \).

For a SBM-efficient DMU \( k \), Tone [15] proposed the following model to identify its super-efficiency:

\[
\frac{1}{m} \sum_{i=1}^{m} x_{i}/x_{ik} \\
\text{max} \quad \frac{1}{r} \sum_{r=1}^{s} y_{r}/y_{rk} \\
\text{s.t.} \quad x_{i} \geq \sum_{j=1}^{n} x_{ij} \lambda_{j}, \text{ } i = 1, ..., m \\
y_{r} \leq \sum_{j=1}^{n} y_{jr} \lambda_{j}, \text{ } r = 1, ..., s \\
\lambda_{j} \geq 0, j = 1, ..., n, j \neq k \\
x_{ik} \geq 0, i = 1, ..., m \\
y_{rk} \geq 0, r = 1, ..., s \\
(2)
\]

Note that model (2) always provides efficiency scores greater than or equal to one. For an inefficient DMU, the efficiency score obtained by model (2) will be one. Therefore, efficient DMUs and inefficient DMUs can be not be differentiated by model (2). To evaluate the efficiency and super-efficiency scores for all DMU with models (1) and (2), usually one applies model (1) to all DMUs and then applies model (2) to the efficient DMUs filtered out by model (1) for their super-efficiency scores. However, as noted by Fang et al. [8], one can reverse the order. That is, model (2) is first applied to all DMUs. If the super-efficiency score is larger than 1, the DMU is efficient. Those DMUs with super-efficiency scores equal to 1 are singled out. They may be inefficient or efficient. Model (1) is then applied to these DMUs to identify the inefficient DMUs. If the efficiency score is less than 1, the DMU is inefficient.

3. Additive super-efficiency

Du et al. [7] extended model (2) and developed super-efficiency DEA models based upon the additive model [2]. Charnes et al. [2] proposed the following additive DEA model to evaluate the DMU \( k \):

\[
\text{max} \quad \mu_{k} = \sum_{i=1}^{m} s_{ik} + \sum_{r=1}^{s} s_{rk}^{+} \\
\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} \lambda_{j} = x_{ik} - s_{ik}^{-}, i = 1, ..., m \\
\sum_{j=1}^{n} y_{jr} \lambda_{j} = y_{rk} + s_{rk}^{+}, r = 1, ..., s \\
\lambda_{j} \geq 0, j = 1, ..., n \\
s_{ik}^{-} \geq 0, i = 1, ..., m \\
s_{rk}^{+} \geq 0, r = 1, ..., s \\
(3)
\]

where \( s_{ik}^{-} \) and \( s_{rk}^{+} \) are input and output slacks representing input excess and output shortage from the frontier respectively. DMU \( k \) is efficient if and only if all slacks are zero. It is easy to show that DMU \( k \) is efficient under the additive model (3) if and only if DMU \( k \) is efficient under the Tone’s slacks-based measure model (1).

For an efficient DMU \( k \) under model (3), Du et al. [7] proposed the following additive super-efficiency model:

\[
\text{min} \quad \alpha_{k} = \sum_{i=1}^{m} t_{ik}^{+} + \sum_{r=1}^{s} t_{rk}^{+} \\
\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} \lambda_{j} \leq x_{ik} + t_{ik}^{+}, i = 1, ..., m \\
\sum_{j=1}^{n} y_{jr} \lambda_{j} \geq y_{rk} - t_{rk}^{+}, r = 1, ..., s \\
\lambda_{j} \geq 0, j = 1, ..., n, j \neq k \\
t_{ik}^{+} \geq 0, i = 1, ..., m \\
t_{rk}^{+} \geq 0, r = 1, ..., s \\
(4)
\]

where \( t_{ik}^{+} \), \( t_{rk}^{+} \) are slacks representing the input savings and output surplus from the frontier respectively.

The posterior additive efficiency score of (3) is defined to be

\[
\bar{\mu}_{k} = \alpha_{k} / \mu_{k} \\
(5)
\]

where \( s_{ik}^{-}^{*} \) and \( s_{rk}^{+}^{*} \) are the optimal solutions of (3).

The posterior additive super-efficiency score of (4) is defined to be

\[
\bar{\sigma}_{k}^{*} = \alpha_{k} / \mu_{k} \\
(6)
\]

where \( t_{ik}^{+}^{*} \) and \( t_{rk}^{+}^{*} \) are the optimal solutions of (4).

Note that the posterior efficiency scores in (5) and (6) are defined following the convention of the SBM.

To compute the additive super-efficiency scores for a set of DMUs, two models must be used. Similar to the usage of the models (1) and (2), model (3) is first used to identify those efficient DMUs; model (4) is then applied to the efficient DMUs subsequently.
Du et al. [7] noted that to ensure model (3) is unit invariant, we can rewrite model (3) as the following model:

\[
\text{max } \eta_k = \frac{1}{m} \left( \sum_{i=1}^{m} \tilde{s}_{ik} / x_{ik} + \sum_{r=1}^{s} \tilde{t}_{ik} / y_{rk} \right)
\]

s.t. \( \sum_{j=1}^{n} x_{ij} \delta_j = x_{ik} - \tilde{s}_{ik}, \ i = 1, \ldots, m \)

\( \sum_{j=1}^{n} y_{ij} \delta_j = y_{rk} + \tilde{t}_{ik}, \ r = 1, \ldots, s \)

(7)

Also, model (4) can be rewritten so that the resulting model is unit invariant:

\[
\text{min } \rho_k = \frac{1}{m} \left( \sum_{i=1}^{m} t_{ik} - \tilde{s}_{ik}, \ i = 1, \ldots, m \right)
\]

s.t. \( \sum_{j=1}^{n} x_{ij} \delta_j \leq x_{ik} + \tilde{t}_{ik}, \ i = 1, \ldots, m \)

\( \sum_{j=1}^{n} y_{ij} \delta_j \geq y_{rk} - \tilde{t}_{ik}, \ r = 1, \ldots, s \)

(8)

The efficiency scores of (7) and (8) can be defined in a way similar to (5) and (6).

Two models must be used to compute the efficiencies and super-efficiencies of all DMUs. Model (3) is first used for all DMUs. The efficiencies scores of the inefficient DMUs are then obtained by model (4). To simplify the computation, in the following section, we are going to propose an integrated model (one stage model) which integrates model (3) and model (4) so that either the additive efficiencies (for the inefficient DMUs) or the additive super-efficiencies (for the efficient DMUs) can be identified by the integrated model in one stage.

4. The integrated model

To distinguish slacks resulted from inefficient and super-efficient circumstances, we call \( \tilde{s}_{ik} \) and \( \tilde{t}_{ik} \) as the inefficiency slacks, and \( t_{ik}^{+} \) and \( t_{ik}^{-} \) as the super-efficient slacks. In the application of model (3) and model (4), model (3) is solved first and model (4) is then employed. We identify the inefficiency slacks first and then the super-efficiency slacks. That is, the objective function of the model (3) is maximized first and the objective function of the model (4) is minimized secondly. Intuitively, the objective functions of the models (3) and (4) can be integrated as

\[
\text{max} \left( \sum_{i=1}^{m} \tilde{s}_{ik} + \sum_{r=1}^{s} \tilde{t}_{ik} - \epsilon \left( \sum_{i=1}^{m} t_{ik}^{+} + \sum_{r=1}^{s} t_{ik}^{-} \right) \right),
\]

where \( \epsilon \) is a relatively small positive number so that \( \left( \sum_{i=1}^{m} \tilde{s}_{ik} + \sum_{r=1}^{s} \tilde{t}_{ik} \right) \) is maximized first and then \( \left( \sum_{i=1}^{m} t_{ik}^{+} + \sum_{r=1}^{s} t_{ik}^{-} \right) \) is minimized. The constraints of the models (3) and (4) can be integrated as

s.t. \( \sum_{j=1}^{n} x_{ij} \delta_j = x_{ik} + t_{ik}^{+} - \tilde{s}_{ik}, \ i = 1, \ldots, m \)

\( \sum_{j=1}^{n} y_{ij} \delta_j = y_{rk} - t_{ik}^{-} + \tilde{t}_{ik}, \ r = 1, \ldots, s \)

Therefore, we can combine models (3) and (4) into the following model:

\[
\text{max } \alpha_k = \frac{1}{m} \left( \sum_{i=1}^{m} \tilde{s}_{ik} + \sum_{r=1}^{s} \tilde{t}_{ik} - \epsilon \left( \sum_{i=1}^{m} t_{ik}^{+} + \sum_{r=1}^{s} t_{ik}^{-} \right) \right)
\]

s.t. \( \sum_{j=1}^{n} x_{ij} \delta_j = x_{ik} + t_{ik}^{+} - \tilde{s}_{ik}, \ i = 1, \ldots, m \)

\( \sum_{j=1}^{n} y_{ij} \delta_j = y_{rk} - t_{ik}^{-} + \tilde{t}_{ik}, \ r = 1, \ldots, s \)

(9)

However, the objective function of (9) is unbounded, which is proven in the following theorem:

**Theorem 1.** Model (9) is unbounded.

**Proof.**

Assume that \( (\bar{s}_{ik}^{s+}, \bar{s}_{ik}^{-}, \bar{t}_{ik}^{+}, \bar{t}_{ik}^{-}) \) is the optimal solution of (9) and let \( c \) be a positive constant. The solution \( (\bar{s}_{ik}^{s+} + c, \bar{s}_{ik}^{-} + c, \bar{t}_{ik}^{+} + c, \bar{t}_{ik}^{-} + c) \) is also feasible. However, the objective function for \( (\bar{s}_{ik}^{s+} + c, \bar{s}_{ik}^{-} + c, \bar{t}_{ik}^{+} + c, \bar{t}_{ik}^{-} + c) \) is larger than the objective function for \( (\bar{s}_{ik}^{s+}, \bar{s}_{ik}^{-}, \bar{t}_{ik}^{+}, \bar{t}_{ik}^{-}) \), which contradicts that \( (\bar{s}_{ik}^{s+}, \bar{s}_{ik}^{-}, \bar{t}_{ik}^{+}, \bar{t}_{ik}^{-}) \) is the optimal solution.

As noted by Fang et al. [8], the sequence of applying SBM and Super-SBM can be reversed. Hence we optimize the objects of the models (3) and (4) in an opposite order. We identify the super-efficiency slacks first and then the inefficiency slacks. That is, \( (\sum_{i=1}^{m} t_{ik}^{+} + \sum_{r=1}^{s} t_{ik}^{-}) \) is first minimized and then

---

**Table 1**

A simple data set.

<table>
<thead>
<tr>
<th>DMU</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( y_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 2**

The results of the additive super-efficiency model for Table 1.

<table>
<thead>
<tr>
<th>DMU</th>
<th>Model (3)</th>
<th>Model (4)</th>
<th>Projection</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( s_{ik} )</td>
<td>( s_{ik}^{+} )</td>
<td>( s_{ik}^{-} )</td>
</tr>
<tr>
<td>A</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
The frontier formed by all DMUs excluding DMU $k$ is maximized:

$$\min \hat{\alpha}_k = \left( \sum_{i=1}^{n} \lambda_{ik} + \sum_{r=1}^{s} t_{rk} \right)-\varepsilon \left( \sum_{i=1}^{n} s_{ik} + \sum_{r=1}^{s} s_{rk} \right)$$

s.t. $\sum_{j=1, j \neq k}^{n} x_{ij} \alpha_j = x_{ik} + t_{ik} - s_{ik}, i = 1, \ldots, m$

$\sum_{j=1, j \neq k}^{n} y_{ij} \alpha_j = y_{ik} - t_{ik} + s_{rk}, r = 1, \ldots, s$

$\lambda_j \geq 0, j = 1, \ldots, n, j \neq k$

$s_{ik} \geq 0, i = 1, \ldots, m$

$s_{rk} \geq 0, r = 1, \ldots, s$

$t_{ik} \geq 0, i = 1, \ldots, m$

$t_{rk} \geq 0, r = 1, \ldots, s$

(10)

Let $s_{ik}^+, s_{ik}^-, t_{ik}^+$ and $t_{ik}^-$ denote the optimal solutions of the model (10). If the super-efficiency slacks identified by (10) are zeros, that is, $t_{ik}^+ = t_{ik}^- = 0$, there will be two possibilities for the target DMU $k$. First, DMU $k$ is on the frontier formed by all DMUs excluding DMU $k$. The frontier formed by all DMUs excluding DMU $k$ is the same as the frontier formed by all DMUs. DMU $k$ is efficient and has the efficiency score of DMU $k$.

Second, they can be further improved. Assume the optimal solutions for (4) are the optimal solutions for (10) which is equivalent to model (3). Hence $s_{ik}^+ = s_{ik}^-$ and $s_{ik}^+ = s_{ik}^-$. □

**Theorem 2.** Let $s_{ik}^+$ and $s_{ik}^-$ be the optimal solutions of the model (3). If DMU $k$ is inefficient, then $s_{ik}^+ = s_{ik}^-$ and $s_{ik}^+ = s_{ik}^-$. □

**Proof.**

If DMU $k$ is inefficient, DMU $k$ lies in the super-efficiency slacks that model (4) does, which is stated in the following Theorem 3.

**Theorem 3.** Let $t_{ik}^+$ and $t_{ik}^-$ be the optimal solutions of the model (4). If DMU $k$ is efficient, $t_{ik}^+ = t_{ik}^-$ and $t_{ik}^- = t_{ik}^-$. □

**Proof.**

If DMU $k$ is efficient, $t_{ik}^+ = t_{ik}^-$ and $t_{ik}^- = t_{ik}^-$. It would be also feasible for model (10). If $t_{ik}^+ = t_{ik}^-$ and $t_{ik}^- = t_{ik}^-$ are not optimal, they can be further improved. Assume the optimal solutions are $t_{ik}^+$ and $t_{ik}^-$ where $t_{ik}^+ < t_{ik}^-$ and $t_{ik}^- < t_{ik}^-$. The solutions $t_{ik}^+$ and $t_{ik}^-$ will also be feasible for model (4) which contradicts that $t_{ik}^+ = t_{ik}^-$ and $t_{ik}^- = t_{ik}^-$. □

To ensure that model (10) provides the same ranking that model (3) does when DMU $k$ is inefficient and provides the same ranking that model (4) does when DMU $k$ is efficient, we define the posterior efficiency score of the integrated model (10) as

\[ \hat{\alpha} = \min \left( \sum_{i=1}^{n} \lambda_{ik} + \sum_{r=1}^{s} t_{rk} \right)-\varepsilon \left( \sum_{i=1}^{n} s_{ik} + \sum_{r=1}^{s} s_{rk} \right) \]

s.t. $\sum_{j=1, j \neq k}^{n} x_{ij} \alpha_j = x_{ik} + t_{ik} - s_{ik}, i = 1, \ldots, m$

$\sum_{j=1, j \neq k}^{n} y_{ij} \alpha_j = y_{ik} - t_{ik} + s_{rk}, r = 1, \ldots, s$

$\lambda_j \geq 0, j = 1, \ldots, n, j \neq k$

$s_{ik} \geq 0, i = 1, \ldots, m$

$s_{rk} \geq 0, r = 1, \ldots, s$

(10)
The following theorem states that model (10) gives then same efficiency that model (4) does when DMU k is efficient. That is, the ranking of those efficient DMUs obtained by model (4) is the same as that obtained by the proposed model (10).

**Theorem 5.** If DMU k is efficient, then \( \hat{\delta}_k^* = \delta_k^* \).

**Proof.**

Let \( s_k^+ \), \( s_k^+ \), \( t_k^+ \) and \( t_k^+ \) be the optimal solutions of the model (10). Let \( s_k^+ \) and \( s_k^+ \) denote the optimal solutions of the model (3). If DMU k is inefficient, \( t_k^+ = t_k^+ = 0 \) and \( s_k^+ = s_k^+ = s_k^+ = s_k^+ \). Hence \( \hat{\delta}_k^* = \delta_k^* \). □

**Table 6**
The results of the integrated model for Table 4.

<table>
<thead>
<tr>
<th>DMU</th>
<th>Model (10) Projection</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( s_{ik} ) ( t_{ik} ) ( t_{ik} ) ( t_{ik} ) ( s_{ik} ) ( s_{ik} ) ( s_{ik} ) ( s_{ik} )</td>
</tr>
<tr>
<td>A</td>
<td>1 0 0 0 0 0 0 0.875</td>
</tr>
<tr>
<td>B</td>
<td>4 0 0 0 0 0 0 0.714286</td>
</tr>
<tr>
<td>C</td>
<td>0 0 0 0 0 0 0 1.25</td>
</tr>
<tr>
<td>D</td>
<td>0 0 0 0 0 0 0.2 1.25</td>
</tr>
<tr>
<td>E</td>
<td>0 3 0 0 0 0.5 2</td>
</tr>
<tr>
<td>F</td>
<td>2 0 0 0 0 0 0.9</td>
</tr>
<tr>
<td>G</td>
<td>4 0 0 0 0 0 0 0.83333</td>
</tr>
</tbody>
</table>

**Table 7**
The data for 6 DMUs from Tone [15].

| DMU | \( x_1 \) \( x_2 \) \( x_3 \) \( x_4 \) \( y_1 \) \( y_2 \) |
|-----|------|------|------|------|
| D1  | 80   | 60   | 54   | 8    | 90   | 5    |
| D2  | 65   | 200  | 97   | 1    | 58   | 1    |
| D3  | 83   | 40   | 72   | 4    | 60   | 7    |
| D4  | 40   | 1000 | 75   | 7    | 80   | 10   |
| D5  | 52   | 600  | 20   | 3    | 72   | 8    |
| D6  | 94   | 700  | 36   | 5    | 96   | 6    |

**Table 8**
The results of the additive super-efficiency model for Table 7.

<table>
<thead>
<tr>
<th>DMU</th>
<th>Model (4) Projection</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( t_{ik} ) ( t_{ik} ) ( t_{ik} ) ( t_{ik} ) ( t_{ik} ) ( t_{ik} ) ( s_{ik} ) ( s_{ik} ) ( s_{ik} ) ( s_{ik} ) ( s_{ik} ) ( s_{ik} )</td>
</tr>
<tr>
<td>D1</td>
<td>0.000</td>
</tr>
<tr>
<td>D2</td>
<td>0.000</td>
</tr>
<tr>
<td>D3</td>
<td>0.000</td>
</tr>
<tr>
<td>D4</td>
<td>0.000</td>
</tr>
<tr>
<td>D5</td>
<td>0.000</td>
</tr>
<tr>
<td>D6</td>
<td>0.000</td>
</tr>
</tbody>
</table>
Model (10) is unit variant. It can be rewritten as follows so that the resulting model is unit invariant:

$$\min \lambda_i = \frac{1}{1+\tau} \left( \sum_{j=1}^{n} x_{ij} \lambda_j + \sum_{r=1}^{s} t_{r+}^i s_{r+}^i - \sum_{r=1}^{s} t_{r-}^i s_{r-}^i, r = 1, \ldots, m \right)$$

s.t. \n
$$\sum_{j \neq k}^{n} y_{jk} \lambda_j = y_{rk} - t_{r-}^i + s_{r+}^i, r = 1, \ldots, s$$

$$\lambda_j \geq 0, j = 1, \ldots, n, j \neq k$$

$$s_{r+}^i \geq 0, i = 1, \ldots, m$$

$$s_{r-}^i \geq 0, r = 1, \ldots, s$$

$$t_{r-}^i \geq 0, i = 1, \ldots, m$$

$$t_{r+}^i \geq 0, r = 1, \ldots, s$$

(12)

5. Illustration

In this section, numerical examples will be examined to demonstrate our contributions. Namely, our method identifies the strongly efficient projection and obtains the efficiency scores and super-efficiency scores in a single step. Consider the simple data set in Table 1. Table 2 shows the results of the additive super-efficiency model proposed by Du et al. [7]. First, model (3) is applied. DMU A and DMU D are identified as the efficient DMUs. Second, the super-efficiency model is applied to DMU A and DMU D. The super-efficiencies of DMU A and DMU D are both 2. The projection of DMU D is (1, 1.5, 0.5).

Table 3 presents the results of the integrated model. As shown in Table 3, for inefficient DMUs, model (10) identifies the same efficient scores that model (3) does, and for efficient DMUs, model (10) identifies the same super-efficiency scores that model (4) does. In other words, the ranking of those inefficient DMUs obtained by model (3) is the same as the proposed model (10), and the ranking of those efficient DMUs obtained by model (4) is the same as our model (10). However, the projection of DMU D identified by model (10) is strongly efficient and different from the projection identified by model (4) as shown in Table 2.

Two examples in Tone [15] are used for demonstration. The data sets are shown in Tables 4 and 7 respectively. Table 5 presents the results of Table 4 yielded by the additive model proposed by Du et al. [7] and Table 6 presents the results yielded by our approach. It demonstrates that our approach yields the same results as model (3) and model (4) except for the projections. Four DMUs (DMU A, B, F and G) are inefficient and three DMUs (DMU C, D and E) are efficient. The efficiency scores yielded by our approach are presented in the eighth column of Table 6. From Tables 5 and 6, it can be seen that the integrated model (10) provides the same efficiency scores as those obtained by model (3) when the DMUs are inefficient. Tables 5 and 6 also indicate that the integrated model (10) yields the same super-efficiency scores as those obtained by model (4) when the DMUs are efficient. From Tables 5 and 6, we find that the projection of the DMU E identified by model (4) is weakly efficient while the projection of the DMU E identified by model (10) is strongly efficient. The DMUs in Table 7 are all efficient. From Tables 8 and 9, it can be seen that the super-efficiency scores yielded by the integrated model (10) are the same as those obtained by model (4). Note that projections identified by model (10) are the same as the projections identified by model (4).

Model (10) assumes constant returns to scale (CRS). To demonstrate the effects of the integrated model under variable...
returns to scale (VRS), we provide another data set in Table 10.

**Table 11**
The results of the additive super-efficiency model for Table 10 under VRS.

<table>
<thead>
<tr>
<th>DMU</th>
<th>Model (3)</th>
<th>Model (4)</th>
<th>Projection</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$s^*_E$</td>
<td>$s^*_C$</td>
<td>$r^*_F$</td>
</tr>
<tr>
<td>A</td>
<td>2</td>
<td>1</td>
<td>0.166667</td>
</tr>
<tr>
<td>B</td>
<td>0.5</td>
<td>0</td>
<td>0.833333</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>E</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 12**
The results of the integrated model for Table 10 under VRS.

<table>
<thead>
<tr>
<th>DMU</th>
<th>Model (10)</th>
<th>Projection</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$s^*_E$</td>
<td>$s^*_C$</td>
</tr>
<tr>
<td>A</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>E</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Fig. 1.** A simple example under VRS.

Table 11 shows the results yielded by model (3) and model (4) under VRS. Table 12 shows the results yielded by the integrated model (10) under VRS. From Tables 11 and 12, we can see that the efficiency scores yielded by the integrated model (10) are the same but the projections identified by the integrated are strongly efficient. For example, the projection of the DMU E identified by model (4) under VRS is (3, 2), which is weakly efficient. As shown in Fig. 1, the point (3, 2) is between DMU A and DMU B and DMU B outperforms it. If we employ the integrated model (10) under VRS to evaluate DMU E, the projection would be B(3, 3). Although the super-efficiency score is the same as that identified by model (4), DMU B is strongly efficient.

6. Concluding remarks

This paper extends the work of Du et al. [7] and develops an integrated model based on the additive DEA. Our new model differs from the additive super-efficiency model proposed by Du et al. [7] in two aspects. The first is that our model calculates the super-efficiencies in one stage instead of two stages. The second is that the projections identified by our model are strongly efficient.

In addition to the formal proofs of the related theorems, we also provide numerical examples to demonstrate that our integrated model gives the same super-efficiency scores in one stage instead of two stages, which are required by the method proposed by Du et al. [7]. Note that the discussion in this paper is based on the constant returns to scale assumption. The integrated model can be extended to the variable returns to scale assumption and similar results can be obtained.

Our method has managerial implications in the implementation of the real practices, especially when computing the SLM based Malmquist productivity index over a long panel. The method proposed by Du et al. [7] involves three steps. First, we apply the additive model to all DMUs. Then we single out the efficient DMUs, and finally apply the super-efficiency model to the efficient DMUs. This is time-consuming. It will be cumbersome and unmanageable in large scale applications involving huge volume of data. Our method overcomes the problem of switching between different models and provides an efficient approach toward the problem of evaluating efficiency and super-efficiency scores when the problem size is large.

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**References**


