Abstract—Conventional finite-element methods (FEMs) rely on an underlying tessellation to describe the geometry and the basis functions that are used to represent the unknown quantity. Alternatively, however, it is possible to represent both the geometry and basis as a set of points. This alternative scheme has been used extensively in solid mechanics to compute stress and strain distributions. This paper presents an adaptation of the scheme to the analysis of electromagnetic problems in both the static and quasi-static regimes. It validates the proposed model against both analytical solutions and benchmarked FEMs. The paper demonstrates the efficacy of the proposed method by applying it to a range of problems.

Index Terms—Electromagnetic analysis, finite-element method, meshless method, nondestructive evaluation.

I. INTRODUCTION

The finite-element method (FEM) is an established analysis technique that has been successfully applied in solving various problems in science and engineering. This method is based on the fundamental idea that a continuous function over the entire solution domain can be replaced by a piecewise continuous approximation, usually based on polynomials, over a set of subdomains called finite elements. The interconnecting structure of the elements via nodes is called a finite-element (FE) mesh. The need for an FE mesh leads to some distinct disadvantages of the FEM. Generation of a mesh for a complex geometry is in itself a difficult and time-consuming task [1]. For example, in electromagnetic computation, problems that involve geometrical deformation such as inverse shape optimization, or large dynamic geometrical changes such as propagating cracks, the use of an underlying mesh creates difficulties in the treatment of discontinuities that might not necessarily coincide with the element boundaries. Usually remeshing is required to handle the discontinuities at every step of the shape reconstruction [2]–[4]. For such problems, developing a numerical method that does not rely on an underlying mesh is desirable.

Recently, a new technique known as the “meshless method” has been developed where the unknown function is approximated entirely in terms of “local” functions defined at a set of nodes. Elements, or usual relationships between nodes and elements, are not necessary to construct a discrete set of equations. With the implementation of moving least squares (MLS) approximation in Galerkin formulation, meshless methods are now widely applied to problems in fracture mechanics and static electromagnetics [5]–[9]. These methods are promising, although more expensive than the FEM in terms of construction of the mass matrices.

This disadvantage notwithstanding, the principal attractive feature of meshless methods is the possibility of: 1) working with a cloud of points that describes the underlying structure, instead of relying on a tessellation and 2) using this method for inverse problems. The principal contributions of this paper are twofold: a) the development and validation of the element-free Galerkin (EFG) method as applied to static and quasi-static electromagnetics problems via comparison against analytical models and the standard FEM and b) formulation and application of this technique to static and quasi-static electromagnetic problems in both two and three dimensions.

The organization of this paper is as follows. Section II defines the problems that will be analyzed. Section III outlines the EFG method in general, while Section IV presents details of numerical implementation. Section V presents a series of results that both validate this model as well as demonstrate its applicability. Finally, Section VI summarizes the contribution of this paper.

II. MATHEMATICAL PRELIMINARIES

The interaction of static and low-frequency electromagnetic fields with materials is the basis of nondestructive evaluation (NDE) of conducting samples. In order to model this interaction, the fields are typically expressed in terms of potentials and the governing equations for either the Poisson or the diffusion equation are solved together with appropriate boundary and initial conditions. In what follows, we analyze the application of the EFG method to the numerical solution of both these equations.

Consider a domain of interest denoted by $\Omega$ that is bounded by $S_1 \cup S_2 = \partial \Omega$, where $S_1$ is the Dirichlet boundary and $S_2$...
is the Neumann boundary. The outward pointing normal to the boundary is denoted by \( \hat{n} \).

### A. Poisson Equation

The Poisson equation with inhomogeneous boundary conditions in Cartesian coordinates can be expressed as

\[
\nabla^2 V(x) = \frac{\rho(x)}{\varepsilon}, \quad \text{for} \quad x \in \Omega
\]

\[
V(x) = V_0(x), \quad \text{for} \quad x \in S_1
\]

\[
\frac{\partial V(x)}{\partial n} = g(x), \quad \text{for} \quad x \in S_2
\]

where \( V(x) \) is the electric scalar potential, \( \rho(x) \) is electric charge density, and \( \varepsilon \) is the material permittivity. Note that \( x \) represents the position vector in either two or three dimensions.

### B. Diffusion Equation

The diffusion equation for eddy current problem in Cartesian coordinates can be written as [10]

\[
\nabla \times \left( \frac{1}{\mu} \right) \nabla \times A(x) + \nabla \left( \frac{1}{\mu} \nabla \cdot A(x) \right) + j\omega \sigma A(x) + \sigma \nabla V(x) = J_S(x)
\]

\[
\nabla \cdot ( j \omega \sigma A(x) + \sigma \nabla V(x) ) = 0
\]

where \( \mu \) is the permeability, \( \sigma \) is the conductivity, \( A(x) \) is the vector potential, and \( J_S(x) \) is the current density.

### III. EFG METHOD

In the EFG method, a set of nodes is used to construct the discrete equations. However, to implement the Galerkin procedure, it is necessary to compute the integrals over the solution domain; this is done by defining the support of the basis functions using either a set of quadrature points or a background mesh. The manner in which this is numerically implemented is elucidated in Section IV.

Consider a function \( u(x) \) that is to be approximated. The EFG method utilizes a MLS approach, which relies on three components: 1) a weight function; 2) a polynomial basis; and 3) a set of position-dependent coefficients. The weight function is nonzero only over a small subdomain around a particular node, which is termed the domain of influence of that node.

### A. MLS Approximation

In MLS approximation, the interpolant \( u^h(x, \overline{x}) \) is given by [5]

\[
u^h(x, \overline{x}) = \sum_{j=0}^{m} p_j(\overline{x}) a_j(x) = p^T(\overline{x})a(x)
\]

where \( x \) is the approximation point, \( \overline{x} \) is a particular node, \( p_j(\overline{x}) \) are monomial basis functions, \( m \) is the number of terms in the basis function, and \( a_j(x) \) are coefficients that depend on the position \( x \). The coefficients \( a_j(x) \) are determined by minimizing the difference between the local approximation and the nodal parameters \( u_j \), i.e., by minimizing the following quadratic form:

\[
J = \sum_{j=1}^{n} w(x - x_j) \left( u^h(x_j) - u_j \right)^2
\]

\[
= \sum_{j=1}^{n} w(x - x_j) \left[ \left( \sum_{i=0}^{m} p_i(x_j) a_i(x) - u_j \right) \right]^2.
\]

Here, \( w(x - x_j) \) is a weight function with compact support, and \( n \) is the number of nodes in the neighborhood of \( x \) where the weight function does not vanish. In matrix notation, (7) can be rewritten as

\[
J = (P a - u)^T W(x) (P a - u)
\]

where \( u^T = (u_1, u_2, \cdots u_n) \) are the unknowns

\[
P = [p_i(x_j)]_{m \times n},
\]

\[
W(x) = \alpha \text{diag}[w(x-x_1), w(x-x_2), \cdots, w(x-x_n)].
\]

The minimization of \( J \) with respect to \( a(x) \) leads to

\[
B(x) a(x) - C(x) u = 0
\]

where

\[
B(x) = P^T W(x) P
\]

\[
C(x) = P^T W(x).
\]

It follows that

\[
a(x) = B^{-1}(x) C(x) u.
\]

Substituting (14) into (6), and letting \( \overline{x} = x \), the MLS approximation can be written as

\[
u^h(x) = \sum_{j=1}^{n} \phi_j(x) u_j
\]

where the shape functions \( \phi_j \) are given by

\[
\phi_j(x) = \sum_{i=0}^{m} p_i(x) \left( B^{-1}(x) C(x) \right)_{i,j} = p^T B^{-1} C_j.
\]

Note that the shape function does not satisfy the Kronecker delta criterion: \( \phi_j(x_k) \neq \delta_{jk} \); therefore, \( u^h(x_j) \neq u_j \), which makes it difficult to impose essential boundary conditions. Techniques that can be applied to address this issue include using either a Lagrange multiplier or coupling with standard finite elements at the boundary. Note, the construction of the shape function is identical in both two and three dimensions.

### B. Weight Function

In two dimensions, the solution domain is covered by the domains of influence of all nodes; while the choice of shape of this influence domain is arbitrary, a circular or rectangular domain is typically used. In our implementation, a rectangular domain is
used and its corresponding weight function is derived as a tensor product weight, which at any given point can be expressed as [5]

\[ w(\mathbf{x} - \mathbf{x}_j) = u_x \cdot u_y = w(r_x) \cdot w(r_y) \]  \hspace{1cm} (17)

where \( w(r_i) \), for \( i = x, y \), are either Gaussians, exponentials, or cubic splines. These functions take the form

\[ w(r) = \begin{cases} 1 - e^{-(r-1)^2}, & \text{for } r \leq 1 \\ 0, & \text{for } r > 1 \end{cases} \quad \text{(Gaussian)} \]  \hspace{1cm} (18)

\[ w(r) = \begin{cases} e^{-\left(\frac{r}{\alpha}\right)^2}, & \text{for } r \leq 1 \\ 0, & \text{for } r > 1 \end{cases} \quad \text{(Exponential)} \]  \hspace{1cm} (19)

\[ w(r) = \begin{cases} \frac{3}{2} - 4r^2 + 4r^3, & \text{for } r \leq \frac{1}{2} \\ \frac{3}{2} - 4r + 4r^2 - \frac{4}{3}r^3, & \text{for } \frac{1}{2} < r \leq 1 \\ 0, & \text{for } r > 1 \end{cases} \quad \text{(Cubic spline)} \]  \hspace{1cm} (20)

where \( \alpha = 0.5 \) yields best convergence for the exponential weight, and

\[ r_x = \frac{|x - x_j|}{d_{mx}}, \quad r_y = \frac{|y - y_j|}{d_{my}} \]  \hspace{1cm} (21)

\[ d_{mx} = d_{max} \cdot c_x, \quad d_{my} = d_{max} \cdot c_y. \]  \hspace{1cm} (22)

Here, \( d_{max} \) is a scaling factor, and \((c_x, c_y)\) is the difference between node \( \mathbf{x}_j \) and its nearest neighbor. Note, \( d_{max} \) is chosen such that matrix \( B \) is nonsingular. In three dimensions, the weight is a natural extension of those presented earlier for two dimensions.

C. Discontinuities Approximation

NDE problems typically involve multiply connected regions with each region characterized by different conductivities and permeabilities. This results in discontinuities of the normal component of current density when passing from one material to another, which in turn implies that the derivatives of the shape function or the shape function itself should be discontinuous at the interface. Since continuity of shape functions is inherited from continuity of the weight function, it is necessary to introduce discontinuity into the weight function. This is realized by using the visibility criterion [1].

As shown in Fig. 1, if there is no material discontinuity, the domain of influence is the total area of the square. However, in the presence of a discontinuity, the domain of influence of node \( \mathbf{x}_j \) shrinks to the area covered by the dashed horizontal line, and the weight function vanishes outside of that area. This procedure directly results in the discontinuity of weight function, which in turn introduces the discontinuity of shape function and its derivatives.

![Fig. 1. Domain of influence of node adjacent to material discontinuity.](image)

IV. NUMERICAL IMPLEMENTATION

A. Static Problem

The weak form solution to the Poisson equations (1)–(3), \( V \), is computed by minimizing the functional [12]

\[ F(V) = \int_{\tilde{S}_2} \left( \nabla V \right)^2 d\Omega + 2 \int_{\tilde{S}_1} gV dS + \int \lambda(V - V_0) dS \]  \hspace{1cm} (23)

where \( V \in \text{vector space } W \) that is spanned by the shape functions \( \{ \Phi_k \} \) and satisfies the Dirichlet boundary conditions, and \( \lambda \) is the Lagrange multiplier used to impose the boundary conditions.

Let the EFG shape functions \( \{ \Phi_k \} \) be the basis in \( W \), then \( V = \sum_{k=1}^{N} \Phi_k V_k \), where \( N \) is the total number of nodes in domain \( \Omega \). The functional \( F(V) \) is minimized with respect to values of function \( V \) at all nodes in the domain \( \Omega \), that is, with respect to vector \( V^T = [V_1, V_2, \ldots, V_N] \). The minimization of functional \( F(V) \) results in the following equations:

\[ \frac{\partial F}{\partial \mathbf{V}} = \left[ \frac{\partial F}{\partial V_1}, \frac{\partial F}{\partial V_2}, \ldots, \frac{\partial F}{\partial V_N} \right]^T = 0 \]  \hspace{1cm} (24)

which can be expressed as

\[ \sum_{i=1}^{N} \int_{\tilde{S}_1} \nabla \Phi_i \cdot \nabla \Phi_j d\Omega \quad V_i + \sum_{k=1}^{NUK} \int_{\tilde{S}_1} N_k \Phi_j dS \lambda_k = -\sum_{j=1}^{N} \int_{\tilde{S}_2} \frac{\partial}{\partial x} \Phi_j d\Omega \quad + \int_{\tilde{S}_1} g \Phi_j dS, \quad \text{for } j = 1, 2, \ldots, N. \]  \hspace{1cm} (25)

Combining the above set of equations with Dirichlet boundary conditions

\[ \sum_{i=1}^{N} \int_{\tilde{S}_1} N_k \Phi_i dS \quad V_i - \int_{\tilde{S}_1} N_k V_0 dS = 0 \hspace{1cm} \text{for } k = 1, 2, \ldots, NUK \]  \hspace{1cm} (26)

yields a set of equations that can be written in matrix form. In the above equations, \( \lambda_k \) is a Lagrange multiplier, \( N_k \) is the shape function associated with the Lagrange multiplier, and \( NUK \) is the number of nodes on \( \tilde{S}_1 \) where a Dirichlet boundary condition is imposed. As mentioned before, a background mesh is required for integration calculations that appear in (25) and (26).
since evaluation of $d\Omega$ or $dS$ corresponds to computation of the Jacobian of a background integration cell.

B. Quasi-Static Problem

In this section, Galerkin formulation and Lagrange multiplier techniques are applied to obtain the numerical solution of the eddy-current equations (4) and (5). The first task is to express potentials as a linear combination of shape functions, such as $A = \sum_{j=1}^{3N} \Phi_j A_j$, and $V = \sum_{j=1}^{N} V_j \Phi_j$, where $\Phi_j = \Phi_j \hat{x}$, $\Phi_j \hat{y}$ or $\Phi_j \hat{z}$. Next, substituting the above expressions into (4) and (5), performing the inner product on each of the resulting equations with a test function (shape function will be used), and applying the Lagrange multiplier method

$$
\begin{align*}
\sum_{j=1}^{3N} \left[ \int_{\Omega} \left( \frac{1}{\mu} \left( \nabla \times \Phi_j \right) \cdot \left( \nabla \times \Phi_j \right) + \frac{1}{\mu} \left( \nabla \cdot \Phi_j \right) \left( \nabla \cdot \Phi_j \right) \right) d\Omega \right. &+ j\omega \sigma \Phi_j \cdot \Phi_j \right] A_j + \sum_{j=1}^{N} \left[ \int_{\Omega} \sigma \Phi_i \cdot \nabla \Phi_j d\Omega \right] V_j \\
&+ \sum_{k=1}^{3NUK} \left[ \int_{\partial \Omega} \Phi_j \cdot N_k dS \right] \alpha_k = 0 
\end{align*}
$$

(27)

$$
\begin{align*}
\sum_{j=1}^{3N} \left[ \int_{\Omega} j\omega \sigma \nabla \Phi_i \cdot \nabla \Phi_j d\Omega \right] A_j + \sum_{j=1}^{N} \left[ \int_{\Omega} \sigma \nabla \Phi_i \cdot \nabla \Phi_j d\Omega \right] V_j \\
&+ \sum_{j=1}^{NUK} \left[ \int_{\partial \Omega} \Phi_i N_k dS \right] \beta_k = 0
\end{align*}
$$

(28)

$$
\begin{align*}
\sum_{j=1}^{3N} \left[ \int_{\partial \Omega} N_k \cdot \Phi_j dS \right] A_j - \int_{\partial S} N_k \cdot A_0 dS = 0 \\
\sum_{j=1}^{N} \left[ \int_{\partial \Omega} N_k \Phi_j dS \right] V_j - \int_{\partial S} N_k V_0 dS = 0
\end{align*}
$$

(29)

(30)

where $A_j$ is one of the three components of the magnetic vector potential and $V_j$ is the electric scalar potential, $\alpha_k$, $\beta_k$ are Lagrange multipliers, $N_k$ is the shape function associated with the Lagrange multipliers, and $NUK$ is the number of nodes where the Dirichlet boundary condition is imposed. In matrix notation, (27)–(30) may succinctly be written as

$$
GA = Q.
$$

(31)

As in the standard FEMs, the matrix $G$ is sparse, and (31) is solved using a standard nonstationary iterative solver such as transpose-free quasi-minimal residual (TFQMR) [11]. The choice of TFQMR is in some sense arbitrary. Although any iterative solver can be used, TFQMR as opposed to biconjugate gradient (BCG) and QMR, has excellent convergence properties.

V. APPLICATIONS AND RESULTS

In this section, we first present the validation of the EFG method and compare solutions using this method to those ob-

![Fig. 2. Surface plot of numerical solution to (32).](image)

![Fig. 3. Error estimation in $L^2$ (dashed) and $H^1$ (solid) norm for the linear EFG, as a function of mesh density.](image)

ained using traditional FEM for two-dimensional (2-D) problems. This technique is then applied to analyze 2-D quasi-static (eddy-current) NDE problems. Finally, both static and quasi-static problems are studied in three dimensions.

A. Convergence Study and Comparison With Traditional FEMs

1) Effect of Mesh Density: Consider the Poisson equation

$$
\nabla^2 u(x, y) = 8\pi^2 \sin(2\pi x) \sin(2\pi y)
$$

for $(x, y) \in [-0.5, 0.5] \times [-0.5, 0.5]$ (32)

with boundary condition

$$
u(x, y) = \sin(2\pi x) \sin(2\pi y).$$

(33)

Uniform meshes of $9 \times 9$, $17 \times 17$, $33 \times 33$, and $65 \times 65$ nodes are used with $4 \times 4$ Gaussian quadrature in each cell. The EFG method with linear bases is used in the study, and the results are compared with those obtained by linear FEM. A typical surface plot of a numerical solution is shown in Fig. 2. The error estimations in $L^2$ and $H^1$ norm are shown in Figs. 3 and 4. The error functions and convergence rate $\tau$ are computed as

$$
L^2_{\text{err}}(h) = \left\{ \frac{1}{\Omega} \left( u^{\text{num}} - u^{\text{exact}} \right)^2 d\Omega \right\}^{1/2}
$$

(34)
Fig. 4. Error estimation in $L^2$ (dashed) and $H^1$ (solid) norm for different weight functions, as a function of mesh density.

\[
H^{1\text{err}}(h) = \left\{ \int_{\Omega} \left[ (u_{\text{num}} - u_{\text{exact}})^2 \right] + \sum_{\alpha=x,y} (u_{\text{num}} - u_{\text{exact}})^2 \right\}^{1/2} dh \right\}^{1/2} \quad (35)
\]

\[
r = \log_2 \left( \frac{\text{err}(h)}{\text{err}(2h)} \right) \quad (36)
\]

where $u_x$ and $u_y$ are the derivatives of $u$ with respect to $x$ and $y$, respectively, and $h$ is the space between nodes.

It is observed that: 1) the convergence rate of EFG depends on the scaling factor defined in (22) and 2) the convergence rate of the EFG method is higher than that of linear FEM in both $L^2$ and $H^1$ norm, which varies from 2.1 to 3.1 in $L^2$ norm for different and varies from 1.0 to 2.0 in $H^1$ norm. We should mention that the EFG method is inherently a high-order scheme that requires more computational overhead than the linear FEM that is used for comparison.

2) Effect of Weight Function: As mentioned before, choice of weight function is crucial to the convergence of the numerical solution. Cubic splines, exponentials, and Gaussians are some of the commonly used weight functions. It is found that cubic spline has the best convergence property, while the exponential and Gaussian are better when the scaling factor is small as shown in Fig. 4. This is largely due to the fact that the latter two functions reduce the domain of influence, which in turn reduces the number of nonzero entries in the stiffness matrix. This reduction saves both computational time and memory.

B. Model Validation—2-D

1) Circular Disk: We consider a 2-D eddy-current problem. The governing equation in $A = V$ formulation is derived from Maxwell equations [10]

\[
\nabla \times \frac{1}{\mu} \nabla \times A(x,y) + j \sigma \omega A(x,y) + \nabla V(x,y) = 0 \quad (37)
\]

in a unit disk $r \leq 1$, with Dirichlet boundary conditions

\[
A(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}, \quad V(x,y) = 0 \quad (38)
\]

on a unit circle $r = 1$, where $r = (x^2 + y^2)^{1/2}$. The analytical solution for this problem is

\[
A(x,y) = \frac{I_1 \left( j \frac{1}{2} (\omega \sigma \mu)^{1/2} r \right)}{r I_1 \left( j \frac{1}{2} (\omega \sigma \mu)^{1/2} \right)} \begin{pmatrix} -y \\ x \end{pmatrix}, \quad V(x,y) = 0 \quad (39)
\]

where $I_1(x)$ is the modified Bessel function of the first kind. The numerical solution is shown in Fig. 5, where $\omega = 1$, $\sigma = 1$, and $\mu = 1$. The domain was uniformly discretized using a $6 \times 6$ grid of nodes, $4 \times 4$ cells for integration, and $4 \times 4$ Gaussian quadrature in each quadrature cell. As is expected, the eddy currents flow in circular paths (see Fig. 5). The convergence of the EFG method is analyzed by studying increasingly denser discretization; more specifically, $9 \times 9, 17 \times 17, 33 \times 33$, and $65 \times 65$ nodal points are used in this numerical experiment. The EFG method with linear bases was used in this study, and the results obtained were compared against those using linear FEM for the
same discretization. The error estimations in $L^2$ and $H^1$ norm are shown in Fig. 6, as a function of $h$.

The convergence rate defined in (36) of the EFG method is 2.39 in the $L^2$ norm, which is higher than that of the linear FEM, whose convergence rate is 2.0. The convergence rate of the EFG method is 1.35 in $H^1$ norm, which is also higher than that of linear FEM.

2) Plate With Tight Crack: Assume an infinite-size conducting plate with a tiny crack along the $y$ direction (width = 1 cm) placed under a time-varying harmonic current source in the $x$ direction. The induced current will be along the $-x$ direction in the absence of crack, and will be perturbed in the presence of a crack. This geometry was modeled using the EFG method, and results showing the induced current distribution are presented in Fig. 7. The induced current flows around the crack due to the electrical discontinuity confirming the validity of EFG methods to model tight cracks. As shown in Figs. 8 and 9, a tight crack is modeled without introducing any additional nodes (total 144 nodes) in the region in contrast to conventional FEM, which requires a larger number of nodes (total 180 nodes) and elements since at least two layers of elements are needed to model the crack. Fig. 10 shows the difference in the eddy-current distributions, indicating that the two solutions are very close to each other.

3) Conducting Plate in Time-Varying Field: The third geometry considered for the purpose of model validation is a conducting plate, which is immersed in a time-varying harmonic magnetic field as shown in Fig. 11. Using $\omega = 1$, $\sigma = 1$, and $\mu = 1$, the induced eddy currents are calculated with both EFG and FE methods. The induced current calculated by EFG and the difference between EFG and FE methods are shown also in Fig. 11. Both methods correctly predict the current continuity conditions and the eddy current flow around the sharp corners in the geometry. Both models use a discretization of 23 x 23 nodes uniformly distributed in region $[-1.15, 1.15] \times [-1.15, 1.15]$ cm. The relative total energy difference ($R$) between them is 6.82%, which mainly occurs at the corners

\[
R = \left\{ \int_{\Omega} \left( J_x^\text{EFG} - J_x^\text{FEM} \right)^2 + \left( J_y^\text{EFG} - J_y^\text{FEM} \right)^2 d\Omega \right\}^{\frac{1}{2}}.
\]

\[
\left\{ \int_{\Omega} \left( J_x^\text{FEM} \right)^2 + \left( J_y^\text{FEM} \right)^2 d\Omega \right\}^{\frac{1}{2}}. \quad (40)
\]

C. Model Validation—Three-Dimensional (3-D) Static Field

Consider the Poisson equation

\[
\nabla^2 u(x, y, z) = 12\pi^2 \sin(2\pi x) \sin(2\pi y) \sin(2\pi z)
\]

for $(x, y, z) \in [-0.5, 0.5] \times [-0.5, 0.5] \times [-0.5, 0.5] \quad (41)$
with homogeneous boundary condition

$$u(x, y, z) = 0.$$  \hspace{1cm} (42)$$

The 3-D numerical model consists of $n \times n \times n$ integration cells with $4 \times 4$ Gaussian quadrature in each cell. Linear basis is applied to construct the shape function. The error estimations in $L^2$ and $H^1$ norm are shown in Fig. 12.

Again, the convergence rate of EFG depends on the scaling factor. The convergence rate varies from 2.16 to 3.10 in $L^2$ norm for EFG, in contrast to 1.89 for linear FEM. The convergence rate in $H^1$ norm varies from 1.10 to 2.16 for EFG, also higher than that of linear FEM. The results for computation cost are summarized in Fig. 13. For a fixed size of the mesh, FEM is much less expensive than EFG; however, for a given precision EFG uses less computation time than FEM. In this paper, the EFG has a high setup cost to assemble the matrix, which is a bottleneck of meshless methods.

D. Experimental Geometry—3-D Quasi-Static Field

To illustrate the application of the EFG method to 3-D eddy current problems, we used a test geometry consisting of a single-layer aluminum plate 3 mm thick and of infinite width, with defects, placed under a time-varying harmonic current sheet.

1) An infinite sinusoidal ac current foil 1 mm thick and 12 mm wide is placed above the conducting plate. Both the foil and plate are of infinite extent along the current direction, as shown in Fig. 14. Current density is $1 \text{ A/m}^2$. The solution region is $[-12, 12] \text{ mm} \times [-12, 12] \text{ mm} \times [-5, 5] \text{ mm}$, and $7 \times 7 \times 21$ uniformly distributed nodes were used for discretization. Figs. 15 and 16 show the $X$ component of the magnetic vector potential at various excitation frequencies. It is observed that numerical results agree very well with the analytical solutions.

2) Next, we introduce a defect, a square notch of 5 mm width and 1.5 mm depth in the conducting plate, and set the excitation frequency at 1 kHz. The solution region is chosen as $[-10, 10] \text{ mm} \times [-10, 10] \text{ mm} \times [-5, 2] \text{ mm}$ with 11
nodes in the $X$ and $Y$ directions, and eight nodes in the $Z$ direction. This produces a discretization with a total of 700 integration cells and 968 nodes (3872 unknowns).

Fig. 17 shows the induced eddy currents in the conducting plate. It is observed that the induced currents flow around the defect without passing across, which agrees with the underlying physics and experiments. Fig. 18 shows the induced currents nearby a tight crack of 5 mm length and 3 mm depth (cutting through the sample plate). The EFG method does not require finer meshing around the crack, and predicts the characteristics of the induction phenomenon accurately.

VI. CONCLUSION

This paper presented a detailed description of the formulation, imposition of essential and interface boundary conditions, and implementation of the EFG method. A number of examples in 2-D and 3-D were studied. Through these examples, we see that the EFG method can be successfully applied to the electromagnetic field problems in either static or quasi-static fields. Comparison of conventional FE and EFG methods with respect to accuracy, computation time, and data storage is summarized in Tables I–IV for different mesh discretization. The major advantage of EFG methods is that they avoid the difficulties of
large mesh changes in problems involving tight cracks, commonly encountered in NDE applications.


dr. Udpa is an Associate Technical Editor of the American Society of Nondestructive Testing Journals on Materials Evaluation and Research Techniques in NDE.

Dr. Shanker is a full member of the USNC-URSI Commission B.

Liang Xuan (M’04) was born in China in 1972. He received the B.S. and M.S. degrees in mathematics from the University of Science and Technology of China, Hefei, in 1994 and 1997, respectively, and the Ph.D. degree in electrical engineering from Iowa State University, Ames, in 2002.

Since January 2003, he has been a Postdoctoral Scholar in the Electrical and Computer Engineering Department, University of Kentucky, Lexington. His research interests include electromagnetic computation and microwave circuit modeling.

Zhiwei Zeng (M’04) was born in China in 1974. He received the B.S. degree in electrical engineering from Nanjing University of Aeronautics and Astronautics, Nanjing, China, in 1996 and the Ph.D. degree in electrical engineering from Iowa State University, Ames, in 2003.

Currently, he is a Postdoctoral Research Associate in the Department of Electrical and Computer Engineering, Michigan State University, East Lansing. His research interests include aspects of computational electromagnetics, nondestructive evaluation, and applied statistics.

Balasubramaniam Shanker (M’96–SM’01) received the B.Tech degree from the Indian Institute of Technology, Madras, India, in 1989 and the M.S. and Ph.D. degrees from The Pennsylvania State University, State College, in 1992 and 1993, respectively.

From 1993 to 1996, he was a Research Associate in the Department of Biochemistry and Biophysics, Iowa State University, Ames, where he worked on the Molecular Theory of Optical Activity. From 1996 to 1999, he was with the Center for Computational Electromagnetics at the University of Illinois at Urbana-Champaign, Urbana, as a Visiting Assistant Professor, and from 1999 to 2002, he was with Iowa State University as an Assistant Professor in the Department of Electrical and Computer Engineering. Currently, he is an Associate Professor in the Electrical and Computer Engineering Department, Michigan State University, East Lansing. He has authored or coauthored over 130 articles in archival journals and conference proceedings. His research interests include all aspects of computational electromagnetics, and electromagnetic wave propagation in complex media.

REFERENCES


TABLE III
Comparison of Performances of FE and EFG Methods for the 2-D Poisson Problem (64 × 64 Nodes)

<table>
<thead>
<tr>
<th>Mesh size (64 × 64)</th>
<th>L_2 error</th>
<th>H^1 error</th>
<th>Assembling time (seconds)</th>
<th>Solver time (seconds)</th>
<th>Data storage (MB/yr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear term</td>
<td>1.203e-3</td>
<td>0.126e-2</td>
<td>4.421</td>
<td>1.547</td>
<td>1.608</td>
</tr>
<tr>
<td>EFG (dE = 2.0)</td>
<td>1.930e-6</td>
<td>3.464e-4</td>
<td>5.235</td>
<td>2.859</td>
<td>3.110</td>
</tr>
<tr>
<td>EFG (dE = 2.5)</td>
<td>4.948e-6</td>
<td>2.989e-3</td>
<td>5.641</td>
<td>3.609</td>
<td>5.070</td>
</tr>
</tbody>
</table>

TABLE IV
Comparison of Performances of FE and EFG Methods for the 2-D Poisson Problem (90 × 90 Nodes)

<table>
<thead>
<tr>
<th>Mesh size (90 × 90)</th>
<th>L_2 error</th>
<th>H^1 error</th>
<th>Assembling time (seconds)</th>
<th>Solver time (seconds)</th>
<th>Data storage (MB/yr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear term</td>
<td>6.089e-4</td>
<td>8.941e-2</td>
<td>1.078</td>
<td>2.578</td>
<td>1.157</td>
</tr>
<tr>
<td>EFG (dE = 1.5)</td>
<td>7.995e-5</td>
<td>3.978e-2</td>
<td>16.81</td>
<td>5.325</td>
<td>3.257</td>
</tr>
<tr>
<td>EFG (dE = 2.0)</td>
<td>9.064e-7</td>
<td>2.072e-4</td>
<td>17.50</td>
<td>9.375</td>
<td>6.141</td>
</tr>
<tr>
<td>EFG (dE = 2.5)</td>
<td>2.313e-6</td>
<td>2.108e-3</td>
<td>19.58</td>
<td>13.14</td>
<td>9.887</td>
</tr>
</tbody>
</table>