Robust approach to repetitive controller design for uncertain feedback control systems

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Abstract: In many applications, add-on type repetitive controllers have been reported to have prominent capability of attenuating periodic disturbances and/or tracking periodic reference commands. However, the effective information such as performance weighting functions for the design of feedback controllers has not been considered sufficiently on the design of repetitive controllers. In this study, we deal with a problem of a robust repetitive controller design for an uncertain feedback control system using its explicit performance information. We first show that a robust stability condition of repetitive control systems has a similar form with the well-known robust performance condition of general feedback control systems. The repetitive controller is designed using the performance weighting function for the design of the robust feedback controller. It is also shown that a steady-state tracking error of the repetitive control system is described in a simple form without time-delay term. This result yields that the repetitive control system has a much larger loop gain in the steady state than the feedback control system. Moreover, this paper provides sufficient conditions ensuring that the power of the steady-state tracking error in the repetitive control system is less than or equal to that of the feedback control system. Based on the obtained results, we present repetitive controller design method using the design information of the feedback control system. Finally, application studies on the track-following control system of optical disk drives are performed to show the validity of the proposed method.

1 Introduction

Repetitive control is a specialised control scheme to reduce a tracking error of the control system with periodic reference signals and/or disturbances. Generally, a periodic signal generator in the repetitive controller produces highly accurate tracking property. However, the positive feedback loop and the time-delay term to generate the periodic signal decrease the stability margin. For this reason, tradeoff between stability and tracking performance has been considered as an important factor in the repetitive control system design. Hara et al. [1] derived sufficient conditions for the stability of a repetitive control system and a modified repetitive control system, which sacrifices tracking performance at high frequencies for system stability. Güvenç [2] applied the structured singular value to repetitive control systems in order to determine their stability and performance robustness in the presence of structured parametric modelling error in the plant. Li and Tsao [3] addressed analysis and synthesis of repetitive control systems with robust stability and robust performance. Doh and Chung [4] proposed a method to design a repetitive control system ensuring robust stability for linear systems with time-varying uncertainties. Tsai and Yao derived upper and lower bounds of the repetitive controller parameters ensuring the stability and the desired performance [5] and an upper bound for square integral of the tracking error over one-time period of periodic input signals based on Fourier analysis [6], respectively. Doh and Ryoo presented a robust stability condition for the repetitive control system from the robust performance condition by selecting the performance weighting function as a $q$ filter [7].

However, there are two explicit problems to be solved during the repetitive controller design. First, although a repetitive controller is added on to an existing feedback control system, we have considered the repetitive controller design as a totally different issue with the feedback controller design [3, 4, 8, 9]. In other words, the valuable information such as performance weighting functions used in design of feedback controllers has not been utilised sufficiently on the design of repetitive controllers. To solve the problem, it is highly recommended to use the information to obtain the cutoff frequency and the order of the $q$ filter in the repetitive controller with more effectiveness. Second, not only the delay term and the positive feedback loop in the repetitive controller decrease the phase margin, but also plant uncertainty threatens overall system stability. Hence, the added repetitive controller should guarantee robust stability.

In this paper, a systematic design method of the add-on type repetitive control system is presented for the feedback control system with plant modelling perturbation. A real plant is represented as a multiplicative uncertainty model. It is assumed that the feedback controller is already designed.
based on the performance weighting function describing the reference servo and ensuring robust performance. We first propose a robust stability condition of repetitive control systems which is similar with the well-known robust performance condition of general feedback control systems. Using the proposed stability condition, this paper introduces to design a q filter using the performance weighting function. It is also shown that if the robust stability condition is satisfied, the steady-state tracking error can be described in a simple form without the time-delay term. Then we can achieve a repetitive control system that has a larger loop gain in the steady state than the feedback control system. Moreover, through an analysis on the steady-state tracking errors in the sense of power, we propose conditions which can be ensured that the steady-state tracking error of the repetitive control system is less than or equal to that of the feedback control system. From the obtained results, several design criteria of repetitive controllers are proposed to not only improve the tracking performance but also satisfy the robust stability condition. The properties of q filter are obtained from those of the performance weighting function. As a result, a q filter can be easily designed without solving an additional design problem. Application studies on the track-following control system of optical disk drives are performed to show the feasibility of the proposed method.

2 Robust stability condition of repetitive control systems

Consider the feedback control system in Fig. 1. In this figure, y(t) is the reference trajectory and is assumed to be periodic and bounded within a known period T, y(t) is the plant output, and u(t) is the feedback control input. C(s) is the given feedback controller that stabilises the feedback control system and ensures robust performance. The plant G(s) is described in the following multiplicative uncertainty form

\[ G(s) = (1 + \Delta(s)W_p(s))G_u(s) \]  

where \( G_u(s) \) is a nominal plant model, \( W_p(s) \) is a known stable uncertainty weighting function, and \( \Delta(s) \) is an unknown stable function satisfying \( \| \Delta \|_\infty \leq 1 \).

The following lemma, which is widely known as the robust performance condition in the robust control theory, will be used to derive our results.

Lemma 1 [10]: Consider the feedback control system in Fig. 1 with the plant \( G(s) \) described as (1). Then a necessary and sufficient condition for robust performance is

\[ \| W_uT_n \|_\infty < 1 \text{ and } \| W_pS_n \|_{1+\Delta W_uT_n} < 1 \]

which is equivalent to

\[ \| W_pS_n \| + \| W_uT_n \|_\infty < 1 \]  

where \( W_p(s) \) is assumed to be a known stable performance weighting function, \( S_n(s) = 1/(1 + G_u(s)C(s)) \) is the nominal sensitivity function, and \( T_n(s) = 1 - S_n(s) \) is the nominal complementary sensitivity function.

In order to track the periodic reference signal effectively, the repetitive controller \( C_{\text{rc}}(s) \) is added on to the existing feedback control system shown in Fig. 2 where \( q(s) \) is a low-pass filter to ensure system stability. Note that \( C_{\text{rc}}(s) \) is equivalent to the modified repetitive controller with \( a(s) = 1 \) as proposed by Hara et al. [1].

Theorem 1: Consider the repetitive control system in Fig. 2. Then the repetitive control system is robustly stable if there exists a \( q(s) \) satisfying

\[ \| qS_n \| + \| W_uT_n \|_\infty < 1 \]  

Proof: To prove robust stability, we consider the repetitive control system shown in Fig. 2, but ignoring inputs. The transfer function from the output of \( e^{-Ts} \) around to the input of \( e^{-Ts} \) is equal to \( M(s) = qS_n/(1 + \Delta W_uT_n) \), so the repetitive control system shown in Fig. 2 can be converted as an equivalent system composed of \( e^{-Ts} \) with \( M(s) \) shown in Fig. 3. Since the time delay \( e^{-Ts} \) is stable and its norm is 1, the maximum loop gain equals to \( \| M(s)e^{-Ts} \|_\infty \), which is less than 1 for \( e^{-Ts} \) if and only if the small-gain condition [10, 11]

\[ \frac{\| qS_n \|}{1 + \Delta W_uT_n} < 1 \]  

holds. Hence, it is clear that (4) is a sufficient condition for robust stability of the repetitive control system by the small-gain theorem. Finally, similar to Lemma 1, (4) is guaranteed under the condition (3).

Theorem 1 shows that a design problem of repetitive controller is equivalent to find a \( q(s) \), that is, \( q \) filter satisfying (3).

![Fig. 2 Repetitive control system](image2)

![Fig. 3 Equivalent system](image3)
Remark 1: The considered feedback control system ensures \(\|W_pT_n\|_\infty \leq 1\) and \(S_n(s)\) has properties of a high-pass filter. Thus, to satisfy (3), \(q\) filter should be a kind of low-pass filter with the magnitude of 1 or less than 1 for all frequencies, which is a well-known design scheme in the repetitive control area.

In case that \(W_p(s)\) is selected as a \(q(s)\), the repetitive control system in Fig. 2 is robustly stable if the robust performance condition (2) of general feedback control systems is guaranteed. This result is equivalent to that of [7]. The feedback controller satisfying the robust performance condition can directly guarantee the robust stability of the repetitive control system. Therefore there is no need to design a \(q(s)\) in the repetitive controller ensuring robust stability in comparison with other methods [2–6]. In other words, \(W_p(s)\) with \(\|W_p(s)\|_\infty \leq 1\), which is used to design the feedback control system satisfying the robust performance condition, plays a role of \(q\) filter to stabilise the uncertain repetitive control system. However, since \(\|W_p\|_\infty\) is generally much larger than 1 to reduce the tracking error in the design of the feedback control system, the robust stability condition should be modified to solve practical problems as following.

Corollary 1: Consider the repetitive control system in Fig. 2. Then the repetitive control system is robustly stable if there exists a \(q(s)\) ensuring

\[
\|q(s)/W_p(s)\|_\infty \leq 1
\]  
(5)

Proof: By Lemma 1 and (5), the following inequality is obtained

\[
|qS_n| + |W_pT_n| \leq |W_pS_n| + |W_pT_n| < 1, \quad \forall \omega
\]  
(6)

Therefore since \(C(s)\) satisfies (2), (5) satisfies (3) automatically.

Corollary 1 provides a design guideline of \(q(s)\) to robustly stabilise the repetitive control system when the repetitive controller is added on to the feedback control system satisfying robust performance. In general, since \(W_p(s)\) has the information about the controlled system such as the control bandwidth, the magnitude of the tracking error, and so on, this approach is efficient to determine the bandwidth and the degree of \(q(s)\) using \(W_p(s)\).

Remark 2: According to (5), the relative degree of \(q(s)\) is more than or equal to that of \(W_p(s)\). The cutoff frequency of \(q(s)\) is selected to ensure that the magnitude envelope of \(q(s)\) is lower than that of \(W_p(s)\). By this result, we can draw a profile of \(q(s)\) only using \(W_p(s)\).

Using the condition (3), the relationship between the robustness and the bandwidth of \(q\) filter can be explained. For this purpose, we rewrite (3) as

\[
\|qS_n| + |W_pG_nCS_n|\|_\infty = \|S_n(|q| + |W_pG_nC|)\|_\infty = \rho
\]  
(7)

where \(\rho\) is defined as a robustness measure of the repetitive control system. In other words, if a small \(\rho\) can be achieved by designing an adequate \(q\) filter, then the repetitive control system can be stabilised in spite of large plant uncertainties. Since \(|W_pG_nC|\) and \(S_n\) are already fixed in (7), \(\rho\) can be determined according to the properties of \(q\) filter. \(\rho\) increases as the magnitude of \(q\) filter approaches 1 irrespective of the other terms of (7). Since a low-pass filter is selected as \(q\) filter, the bandwidth and the relative degree of \(q\) filter have significant effects on the robustness. The narrower bandwidth of \(q\) filter is, the smaller \(\rho\) can be obtained. Moreover, a large relative degree of \(q\) filter makes \(\rho\) small since the magnitude of \(q\) filter decrease abruptly as the frequency increases.

Remark 3: As explained qualitatively, the robustness of the repetitive control system can be better as the bandwidth of \(q\) filter is narrower. Also, in case of the same bandwidth, the increase of relative degree of \(q\) filter makes the robustness improved.

3 Analysis on the steady-state tracking error

The following theorem shows that the steady-state tracking error of the repetitive control system in Fig. 2 can be obtained irrespective of the time-delay term if the robust stability condition of the repetitive control system is satisfied.

Theorem 2: Consider the repetitive control system in Fig. 2. The tracking error \(e(t)\) approaches to

\[
e_s(t) = \lim_{t \to \infty} \mathcal{L}^{-1} \left\{ \frac{S_n(1-q)}{1+\Delta W_pT_n - qS_n} Y_r(s) \right\}
\]  
(8)

as \(t \to \infty\) if the repetitive control system satisfies the condition (3).

Proof: From Fig. 2, the tracking error is given by

\[
E(s) = Y_r(s) - Y(s) = Y_r(s) - \frac{GC}{1-ge^{-Ts}} E(s)
\]

From the above equation, we obtain

\[
E(s) - ge^{-Ts} E(s) = Y_r(s) - ge^{-Ts} Y_r(s) - GCE(s)
\]

Using \(q/(1 + G_n(1 + \Delta W_pC)) = qS_n/(1 + \Delta W_pT_n)\), the tracking error becomes

\[
E(s) = \frac{S_n(1-ge^{-Ts})}{1+\Delta W_pT_n - qS_ne^{-Ts}} Y_r(s)
\]  
(9)

Since \(\|qS_n/(1 + \Delta W_pT_n)\|_\infty < 1\), the tracking error can be obtained by expanding the denominator in terms of a power
Consider the feedback control system in Fig. 2, the sum of \( \|k_1E_1\|_{\infty} \) is bounded, \( A_{r}(s) = \frac{S_n(1+4W_n)}{1+\Delta W_n T_n} \) and \( S_n(1+\Delta W_n T_n) - q S_n Y_s(s) \) is bounded, there exists inverse Laplace transforms for each of the terms in (10). The tracking error in the time domain is

\[
e(t) = e_0(t) + \sum_{k=1}^{\infty} e_k(t-kT)
\]

where \( e_0(t) = \mathcal{L}^{-1}\{E_0(s)\} \) is the tracking error generated only by the feedback control system in Fig. 1 and \( e_k(t-kT) = \mathcal{L}^{-1}\{E_k(s)e^{-kT}\} \) for all \( k \in \mathbb{N} \). Thus, the steady-state tracking error becomes

\[
e_\infty(t) = \lim_{t \to \infty} e(t)
\]

Since \( \|q S_n(1+\Delta W_n T_n)\|_{\infty} < 1 \), the sum of \( E_0(s) \) and the infinite series \( \sum_{k=1}^{\infty} E_k(s) \) is \( S_n(1+\Delta W_n T_n) - q S_n Y_s(s) \). Finally, the steady-state tracking error can be described as (8).

From Theorem 2, we analyse the loop gain closely related with the steady-state tracking error. Using (8), let the transfer function \( S_{rc}(s) \) from \( y_1(t) \) to \( e_\infty(t) \) of the repetitive control system be defined as

\[
S_{rc}(s) = \frac{1}{1-q + CG_n(1+\Delta W_n)}
\]

which means the sensitivity function of the repetitive control system in the steady state. Similarly, the sensitivity function of the feedback control system is written as

\[
S_{fb}(s) = \frac{1}{1+CG_n(1+\Delta W_n)} = \frac{1}{1+L_{fb}}
\]

where \( L_{fb}(s) = C(s)G_n(s)(1+\Delta(s)W_n(s)) \) is the loop transfer function of the feedback control system. Equation (13) can be rewritten as the similar form with (14)

\[
S_{rc}(s) = \frac{1}{1+(CG_n(1+\Delta W_n))(1-q)} = \frac{1}{1+L_{rc}}
\]

where \( L_{rc}(s) = L_{fb}(s)/(1-q(s)) \).

Corollary 2: Consider the feedback control system in Fig. 1 and the repetitive control system in Fig. 2. \( |L_{rc}(s)| \) is greater than or equal to \( |L_{fb}(s)| \) if

\[
||1-q(s)||_{\infty} \leq 1
\]

Proof: (16) means that \( 1/|1-q(j\omega)| \geq 1 \), \( \forall \omega \). Multiplying \( |L_{fb}(j\omega)| \) on the left and right sides, we can obtain

\[
\frac{|L_{fb}(j\omega)|}{|1-q(j\omega)|} \geq |L_{rc}(j\omega)|, \quad \forall \omega
\]

which means that \( |L_{rc}(j\omega)| \geq |L_{fb}(j\omega)|, \quad \forall \omega \).
control system because \(|S_p(j\omega_0)|\) may be less than or equal to \(|S_c(j\omega_0)|\) in some frequency regions. In other words, even if the condition (16) is satisfied, there are some frequency regions where \(|S_p(j\omega_0)|\) may be less than or equal to \(|S_c(j\omega_0)|\). Now, using \(S_p(s)\) and \(S_c(s)\) directly related with the steady-state tracking error, we determine conditions under which it can be ensured that the steady-state tracking error of the repetitive control system is less than or equal to that of the feedback control system in the sense of power. Before analysing the steady-state tracking error from the viewpoint of power, we first define the powers of steady-state tracking errors of the repetitive control system and the feedback control system as follows

\[
P_{rc} = \frac{1}{T} \int_T |y_r(t)\|^2 \, dt
\]

\[
P_{fb} = \frac{1}{T} \int_T |y_s(t)\|^2 \, dt
\]

where \(y_r(t) = L^{-1}\{S_c(s)\}\) and \(y_s(t) = L^{-1}\{S_p(s)\}\), respectively. The complex Fourier series representation of \(y_s(t)\) is given by

\[
y_s(t) = \sum_{k=-N_0}^{N_0} c_k e^{j\omega_0 t}
\]

where \(c_k\) is the \(k\)th Fourier coefficient and \(\omega_0 = 2\pi/T\). Let \(q(s)\) be selected as a low-pass filter with unity gain. In the low-frequency region, \(|S_c(j\omega_0)|\) approaches 0 and is much less than \(|S_p(j\omega_0)|\) as \(q(j\omega) \approx 1\) and in the high-frequency region, \(|S_c(j\omega_0)| \approx |S_p(j\omega_0)|\) as \(q(j\omega) \approx 0\). However, there are some frequency regions between the low frequency and in the high frequency where \(|S_c(j\omega_0)| - |S_p(j\omega_0)|\) is larger than zero. By solving the equation

\[
|S_c(j\omega_0)| = |S_p(j\omega_0)|
\]

we can find the frequencies where the sign of \(|S_c(j\omega_0)| - |S_p(j\omega_0)|\) is changed. Let \(q(j\omega)\) and \(S_p(j\omega_0)\) be defined as \(|q(j\omega)| e^{i\phi_c(j\omega)}\) and \(|S_p(j\omega_0)| e^{i\phi_p(j\omega)}\), respectively, where \(\phi_c(j\omega) = \angle q(j\omega)\) and \(\phi_p(j\omega) = \angle S_p(j\omega)\). After eliminating a common term \(|S_p(j\omega_0)|\) in the both sides and some mathematical manipulation, we can get

\[
1 - |S_p(j\omega_0)|^2 \geq 2 |S_c(j\omega_0)| - 2 |S_p(j\omega_0)| |\phi_c + \phi_p|
\]

which is a simple form of (21). The following theorem gives analysis results on the steady-state tracking error in the sense of power.

**Theorem 3:** Consider the repetitive control system shown in Fig. 2. Assume that \(q(s)\) has the frequency characteristics:

(a) \(|q(j\omega)| \sim 1, |\omega| \leq \omega_{\phi_0}\)

(b) \(|q(j\omega)| < 1, |\omega| > \omega_{\phi_0}\)

and \(\omega_{\phi_1} > \omega_{\phi_0}\) is the least frequency satisfying (22).

(i) Let \(y_s(t)\) be a band-limited signal represented as (20) and \(N_0\omega_{\phi_0} \leq \omega_{\phi_0}\). Then \(P_{rc}\) is zero.

(ii) Let \(y_s(t)\) be a band-limited signal represented as (20) and \(N_0\omega_{\phi_0} > \omega_{\phi_0}\). Then \(P_{rc} \leq P_{fb}\) if

\[
\sum_{k=-N_0}^{N_0} |c_k|^2 (|S_c(j\omega_0)|^2 - |S_p(j\omega_0)|^2)
\]

\[
\leq -2 \sum_{k=N_0+1}^{\infty} |c_k|^2 (|S_c(j\omega_0)|^2 - |S_p(j\omega_0)|^2)
\]

Proof: Since \(y_s(t)\) is given by the Fourier series representation

\[
S_c(t) * y_s(t) = \sum_{k=-N_0}^{N_0} S_c(j\omega_0) c_k e^{j\omega_0 t}
\]

\[
S_p(t) * y_s(t) = \sum_{k=-N_0}^{N_0} S_p(j\omega_0) c_k e^{j\omega_0 t}
\]

By Parseval’s theorem, \(P_{rc}\) and \(P_{fb}\) are represented as

\[
P_{rc} = \sum_{k=-N_0}^{N_0} |c_k|^2 |S_c(j\omega_0)|^2
\]

\[
P_{fb} = \sum_{k=-N_0}^{N_0} |c_k|^2 |S_p(j\omega_0)|^2
\]

respectively.

(i) Since \(S_c(j\omega_0) \geq 0\) for \(|\omega| \leq \omega_{\phi_0}\), \(P_{rc}\) of (27) is zero.

(ii) In the case of \(\omega_{\phi_0} < N_0\omega_{\phi_1} \leq \omega_{\phi_1}\), \(|S_c(j\omega_0)| \leq 0\) or \(|S_c(j\omega_0)| \leq |S_p(j\omega_0)|, \forall k \in [-N_0, N_0].\) Therefore from (27) and (28), we can easily show that \(P_{rc} \leq P_{fb}\).

(iii) \(s_c(t) * y_s(t)\) in (18) is written as

\[
s_c(t) * y_s(t) = \sum_{k=-N_0}^{N_0} c_k S_c(j\omega_0) e^{j\omega_0 t} + \sum_{k=0}^{N_0} c_k S_p(j\omega_0) e^{j\omega_0 t}
\]
Then $P_{rc}$ in (18) is given by

$$
P_{rc} = \frac{1}{T} \int \left\{ \left( \sum_{k=-N_1}^{N_1} c_k S_{rc}(j k \omega_0) e^{j k \omega_d t} \right)^2 \right. \\
+ \left. \left( \sum_{k=N_1+1}^{N_0} c_k S_{rc}(j k \omega_0) e^{j k \omega_d t} \right)^2 \right. \\
+ \left. 2 \sum_{k=N_1+1}^{N_0} c_k S_{rc}(j k \omega_0) e^{j k \omega_d t} \times \sum_{k=N_1+1}^{N_0} c_{-k} S_{rc}(-j k \omega_0) e^{-j k \omega_d t} \right. \\
+ \left. \left( \sum_{k=N_1+1}^{N_0} c_{-k} S_{rc}(-j k \omega_0) e^{-j k \omega_d t} \right)^2 \right\} \, dt \tag{31}
$$

Since

$$
\int \left( \sum_{k=N_1+1}^{N_0} e^{j k \omega_d t} \right) \, dt = \int \left( \sum_{k=N_1+1}^{N_0} e^{-j k \omega_d t} \right) \, dt = 0
$$

c_{-k} = c_k^* \text{, and } S_{rc}(-j k \omega_0) = S_{rc}^*(j k \omega_0), \ P_{rc} \text{ can be simplified as}

$$
P_{rc} = \sum_{k=-N_1}^{N_1} |c_k|^2 |S_{rc}(j k \omega_0)|^2 + 2 \sum_{k=N_1+1}^{N_0} |c_k|^2 |S_{rc}(j k \omega_0)|^2
$$

Similarly

$$
P_{fb} = \sum_{k=-N_1}^{N_1} |c_k|^2 |S_{fb}(j k \omega_0)|^2 + 2 \sum_{k=N_1+1}^{N_0} |c_k|^2 |S_{fb}(j k \omega_0)|^2
$$

Therefore to ensure that $P_{rc} \leq P_{fb}$, (23) should be satisfied.

(iv) By extending $N_0$ to $\infty$ in the proof of iii), we obtain the condition (24) to ensure that $P_{rc} \leq P_{fb}$. $\square$

Through Fourier analysis, Theorem 3 shows that a repetitive controller that is added on to the feedback control system effectively reduces the steady-state tracking error.

Remark 5: It was assumed that $T$ is known. However, there may be some uncertainties to measure $T$. The uncertainty on $T$ has effects on the robustness and the performance of the repetitive control system. For this case, refer to Na et al.’s results [12, 13].

4 Application studies

To verify the feasibility of the proposed method, we perform application studies on the track-following control system in DVD drives. A repetitive track-following control system is shown in Fig. 4 where $d(t)$ with unknown magnitude has the same form with $y(t)$ in iii) of Theorem 3, $e_{pd}(t)$ is an amplified signal of the tracking error $e(t)$ by the photo detector gain $K_{PD}$.

A nominal tracking actuator is modelled as

$$
G_n^0(s) = \frac{K \cdot \omega_n^2}{s^2 + 2 \zeta \omega_n s + \omega_n^2} \tag{29}
$$

Although (29) is nearly exact in the low-frequency region, $G_n^0(s)$ is different from the actual tracking actuator $G(s)$ in the high-frequency region. In addition to the unmodelled component, resonances at high frequencies are barely considered in the plant model. To take the effect of the unmodelled dynamics into account, $G_n^0(s)$ should be modelled with consideration of the uncertainty. For the purpose, a multiplicative uncertainty model is adequate [10, 11] and $G_n^0(s)$ is given as

$$
G_n^0(s) = (1 + \Delta(s))W_n(s)G_n^0(s) \tag{30}
$$

where $W_n(s)$ is a known uncertainty weighting function given by $2s^2/(s + 4000 \pi)^2$ and $\Delta(s)$ is an unknown stable function satisfying $\|\Delta\|_{\infty} \leq 1$. $K_{PD}$ is the conversion ratio of the position sensor from the distance between the track centre and the laser spot to the electrical error signal. The track pitch of DVD disk is 0.74 $\mu$m and the measurable range is $\pm 0.37 \mu$m, which corresponds to the tracking error signal amplitude of $\pm 2.0$ V. Therefore, $K_{PD}$ is $5.4 \times 10^6$ V/m. The plant $G(s)$ is $K_{PD}G_n^0(s)$ and has the characteristics given by Table 1.

A feedback compensator satisfying robust performance is designed based on the reference servo for track-following recommended by the DVD standard [14]. Let the allowable disturbance, the maximum tracking error, and the expected maximum radial acceleration be $\pm 50 \mu$m, $\pm 0.022$ mm/s, and 1.1 m/s$^2$, respectively, as explained in the DVD standard. Then, a minimal open-loop gain $|H_{min}(j\omega)|$ (dash-dot) of the reference servo for track-following is depicted in Fig. 5. For an open-loop transfer function $H(s)$ of the reference servo, $|1 + H(j\omega)|$ is limited as schematically shown by the shaded surface of Fig. 5. A performance weighting function is selected from the relationship between the robust performance condition and the reference servo.

Since (2) is equivalent to

$$
\|W_p(s)S(s)\|_{\infty} < 1 \tag{31}
$$

Table 1 Parameters of the plant

| first resonance frequency ($\omega_0$) | 62 Hz | $\zeta$ | 0.08 |
| second resonance frequency | 20 kHz | $K$ | $5.031 \times 10^{-4}$ |
| second resonance magnitude | 10 dB | $K_{PD}$ | $5.4 \times 10^6$ (V/m) |

where $S(s)$ is described as

$$
S(s) = \frac{1}{1 + G(s)C(s)} = \frac{1}{1 + H(s)}
$$

(32)

(31) and (32) lead to

$$
|W_p(j\omega)| < |1 + H(j\omega)|, \quad \forall \omega
$$

(33)

A less strict condition for (33) is written as

$$
|W_p(j\omega)| < |H(j\omega)|, \quad \forall \omega
$$

(34)

Thus, the performance weighting function should be determined to satisfy (34) and is given by

$$
W_p(s) = \frac{6.0755 \times 10^7}{s^2 + 145.14s + 21066}
$$

(35)

A feedback controller is designed as

$$
C(s) = \frac{19.5(s + 17592)(s + 35186)}{(s + 87965)(s + 105560)}
$$

(36)

to consider the minimum stability margin for stable track-following pull-in [15] and cope with a certain level of disturbance [16].

Based on the results obtained in Section III, a few design criteria are derived, which are summarised in Remark 1–4.

**(Design Criteria)**

1. A basic form of $q$ filter: A $q$ filter is a kind of low-pass filter with its magnitude of 1 or less than 1 for all frequencies.
2. Robust stability: The relative degree of $q(s)$ is more than or equal to that of $W_p(s)$ and the cutoff frequency of $q(s)$ is less than or equal to that of $W_p(s)$.
3. Robustness of repetitive control systems: The robustness is improved as the bandwidth of $q$ filter is narrower and the relative degree of $q$ filter increases.
4. Steady-state tracking error: The steady-state tracking error is reduced as the relative degree of $q$ filter decreases and the bandwidth of $q$ filter is wider.

Fig. 5 Reference servo for track-following: $|1 + H(j\omega)|$ (the shaded surface), $|H_{\text{min}}(j\omega)|$ (dash-dot)

**Fig. 6** Frequency properties of the designed $q(s)$ based on the reference servo for track-following.

Fig. 6 depicts frequency properties of the designed $q(s)$ based on the reference servo for track-following. $q(s)$ should have unity DC gain between the maximum rotational frequency, $f_{\text{max}}$ (23.1 Hz) and the 0 dB cross-over frequency of $W_p(s)$, $f_c$ (1.24 kHz) by the proposed criteria. By Criterion 1, $q(s)$ is a low-pass filter with unity DC gain. To satisfy Criterion 2, the relative degree of $q(s)$ is higher than or equal to 2 and the cutoff frequency of $q(s)$, $f_c$ is less than or equal to $f_c$. Moreover, to satisfy both Criteria 3 and 4, we select a value between $f_{\text{max}}$ and $f_c$ as $f_c$. Therefore $|q(j\omega)|$ should be rolled off in the region represented by diagonal lines. Considering the design specifications, we select the following second order low-pass filter with $f_c$ of 1 kHz and unity DC gain

$$
q(s) = \frac{3.948 \times 10^7}{s^2 + 8885.8s + 3.948 \times 10^7}
$$

(37)

as $q(s)$. Fig. 7 shows the magnitude plots of the open loop with the real plant, $W_p(s)$, $H_{\text{max}}(s)$, and $q(s)$ obtained from (35), (36) and (37). This result leads to

$$
\|qS_n\| + \|W_pT_n\|_\infty = 0.671
$$

$$
\|W_pT_n\|_\infty = 0.897
$$

respectively, as shown in Fig. 8. Therefore the robust stability of the repetitive control system is ensured with preserving the robust performance of the feedback control system.

The track-following control system including the repetitive controller was digitally implemented on a 32-bit floating point DSP, TMS320C6727. The program for the track-following control was executed at a sampling rate of 200 kHz, which is an extremely high sampling rate for control applications but common in commercial DVD drives. The controller designed in continuous-time domain was transformed to a discrete-time controller by the pole-zero-mapping method.

In the experiment, the repetitive controller was turned on at 0.43 s. Fig. 9 shows the tracking error before and after the application of the repetitive controller. Although the results are affected by the measurement noise, the effect of a repetitive controller is evident in the results. The external disturbance of the disk rotational frequency 12 Hz is almost completely attenuated by the repetitive controller. As explained in Theorem 3, the repetitive controller enabled
the track-following control system to reduce the tracking error to a value (+0.01 μm) below the maximum allowable boundary (+0.022 μm), resulting in more reliable reading/writing of data from/to the optical disk. The improved performance is explicitly illustrated by the fast Fourier transform (FFT) results shown in Fig. 10. The repetitive controller reduced the tracking error remarkably at 12 Hz, corresponding to the frequency of disk rotation, which leads to the improved tracking accuracy. However, because the bandwidth of $q(s)$ is restricted to 1 kHz, high-order harmonics of tracking error is hardly decreased.

**Remark 6**: The first objective of this paper is to design a robust repetitive controller effectively using the information of the feedback control system such as the performance weighting function rather than to obtain an optimal or suboptimal solution to improve the performance of the repetitive control system by solving an additional design problem for a repetitive controller. Therefore, the proposed method does not always make a better performance than other methods. For example, the repetitive controller designed by Doh et al.'s method [17] has a $q$ filter of 1st order low-pass filter with the cutoff frequency of 1.24 kHz as following

$$q(s) = \frac{7791}{s + 7791}$$

which makes a better tracking performance than the proposed method because (38) has a wider bandwidth and a lower relative degree than (37).

5 **Conclusions**

This paper considered the problem of repetitive controller design for an uncertain feedback system. The robust stability condition of the repetitive control system was obtained using the small-gain theorem, which is closely
related with the robust performance condition of the feedback control system. Through the analysis on the steady-state tracking error, the loop and the power of the steady-state tracking error in the repetitive control system were compared with those of the feedback control system. Based on the obtained results, several design criteria were proposed to design a repetitive controller. Finally, the application study on the track-following control in optical disk drives were performed and the experimental results were presented to validate the effectiveness of the proposed method. We verified that repetitive control is an effective auxiliary controller of the feedback controller to improve the tracking performance.

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7 References