Second-Order Consensus for Multiagent Systems via Intermittent Sampled Data Control

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Abstract—In this paper, a periodic intermittent sampled data control strategy, which both reduces the load of updating rate of the controller and cuts down the working time of the controller in each sampling interval, is proposed to investigate the second-order consensus of multiagent systems. In order to reach consensus, a necessary and sufficient condition depending on the coupling gains, the communication width, the sampling period, and the spectrum of the Laplacian matrix, is established. Furthermore, the feasible region of communication width is derived for given coupling gains and structure of network. Besides, the coupling gains are also carefully designed. On the other hand, when the time delay exists in the sampling process, a time delayed protocol is proposed. A necessary and sufficient condition based on the communication width, the sampling period, the coupling gains, the time delay, and the network structure is also derived for achieving consensus. It is amazing found that the sampling period should have both lower and upper bounds for given time delay and communication width for the sake of reaching consensus. Finally, several numerical simulations are presented to demonstrate the effectiveness of the theoretical results.

Index Terms—Communication width, consensus, coupling gains, intermittent sampled data control, multiagent systems.

I. INTRODUCTION

CONSSENSUS of multiagent systems is one of the important and fundamental problems in the area of cooperative control. Consensus means that a group of agents converge to a common state or reach an agreement under appropriate distributed control. The main challenging problem is to design an appropriate protocol-based only on the local relative information [1]–[4]. In the past decades, consensus of multiagent systems has received significant research due to its widespread applications in sensor networks [5], [6], information control [7], robotic teams, spacecraft formation flying [8], and so on.

The general framework of consensus problem for first-order multiagent systems with fixed or switching topology was proposed by Olfati-Saber and Murray [9]. Furthermore, the consensus condition was relaxed by Ren and Beard [10]. They obtained that the consensus of first-order multiagent systems can be reached if and only if the switching topology contains a spanning tree frequently enough. Besides, many significant results about the first-order consensus of multiagent systems have been obtained [11]–[15]. However, in reality, many class of multiagent systems are modeled by second-order dynamics [3]. Therefore, the second-order consensus has been studied in [16]–[20]. In addition, many complicated multiagent systems including nonlinear dynamics, high-order dynamics, and stochastic dynamics have been considered as well in [21]–[23].

It is worth noting that most of the aforementioned works on consensus problems were focused on the continuous time control, which required the continuous communication among agents and information updated continuously by controllers. These control laws may be infeasible or impractical in many real applications since excessive consumption of limited resources. To deal with this issue, intermittent control strategy was proposed. Wen et al. [24] investigated the first-order consensus of multiagent systems with nonlinear dynamics and external disturbance via intermittent communication. In [25]–[28], the second-order consensus problems of multiagent systems with and without time delays were studied by using intermittent control. In [29], the consensus of multiagent system via adaptive intermittent control was investigated. The advantage of intermittent control is that it can shorten the working time of the controllers, while the deficiency is that the information updating rates of controllers cannot be reduced. For the sake of reducing the load of controllers updating, Yu et al. [30] studied the consensus of second-order multiagent systems via periodic sampled data control, and some necessary and sufficient conditions for reaching consensus were obtained. Yu et al. [31] investigated the consensus of second-order multiagent systems via sampled data control, where the protocol was designed based on both current and delayed position states. Since the relative velocities of agents are more difficult to measure as compared to the relative positions, Huang et al. [32] studied the consensus of multiagent systems using only sampled position data protocol. Guan et al. [34] studied the consensus of multiagent system via impulsive sample control. The former works were based on periodic sampled data control. Thus, an aperiodic sampled data control strategy was used for investigation the
consensus of multiagent systems in [35]. In addition, the consensus of multiagent systems via event-triggered strategy was considered in [36]–[40]. All these works about the consensus of multiagent systems in [30]–[40] only reduced the update rates of controllers actually, which did not shorten the working time of controllers due to the information transmission among agents remain ongoing in every sampling interval. Therefore, it is desirable to design novel control schemes, such that both working time of controllers can be shortened and the load of update rates of controllers can be reduced significantly.

Motivated by the aforementioned discussions, in this paper, a novel control strategy, the intermittent sampled data control protocol, which both shorten the working time of controllers and reduce the update rates of controllers compared with continuous control, are designed for second-order multiagent systems.

The rest of this paper is organized as follows. Some preliminaries about algebraic graph theory, definitions, lemmas, and model formulation are provided in Section II. The main results are derived and presented in Sections III and IV. Several simulation examples are given to verify the effectiveness of the theoretical results in Section V. A short conclusion is given in Section VI.

**Notation:** In this paper, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. Let $\|u\|$, $\mathcal{R}(u)$, and $\mathcal{I}(u)$ be the modulus, real, and imaginary parts of a complex number $u$, respectively. For a vector $x$, $\|x\|$ denotes its Euclidean norm. For a square matrix $A$, $A^T$ represents its transpose and det$(A)$ denotes its determinant. For two sets $B$ and $C$, $B \setminus C$ represents the difference set. $\otimes$ represents the Kronecker product. diag$(\cdot)$ represents the diagonal matrix. $i$ is the imaginary unit.

## II. Preliminaries

In this section, some important preliminaries about algebraic graph theory, definitions, lemmas, and model formulation are briefly introduced.

Consider a network composed by $N$ agents. The communication topology among agents is described by a directed graph $G = (\mathcal{V}, \mathcal{E}, A)$, where each agent is regarded as a node of $G$. $\mathcal{V} = \{v_1, \ldots, v_N\}$ denotes the node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the edge set. $e_{ij} = (v_i, v_j)$ is the edge of $G$. The set of neighbors of node $v_i$ is denoted by $N_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. $A$ is weighted adjacency matrix and its element is defined as $a_{ij} = 0$ if $e_{ij} \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. The Laplacian matrix $\mathcal{L} = \{\ell_{ij}\}_{N \times N}$ with respect to the graph $G$ is defined as $\ell_{ij} = \sum_{j \neq i}^N a_{ij}$. $\ell_{ij} = -a_{ji}$ for $i \neq j$. A directed path from node $v_i$ to $v_j$ is a sequence of edges of form $(v_i, v_1), (v_1, v_2), \ldots, (v_k, v_j)$ in the directed graph with distinct node $v_i \in \mathcal{V}$ for $l = 1, 2, \ldots, k$. The graph $G$ is connected if between any pair of distinct nodes $v_1$ and $v_j$ in $G$, there exists a path between $v_i$ and $v_j$. A node is called root if it has a directed path to every other node. A tree is a connected subgraph without closed paths. A directed spanning tree is a directed tree containing all nodes of the graph with only one root.

**Lemma 1 [30]:** Given a complex coefficient polynomial of order two as follows:

$$g(s) = s^2 + (\xi_1 + i\xi_2)s + \xi_0 + i\xi_0$$

where $\xi_1, \gamma_1, \xi_0$, and $\xi_0$ are real constants. Then, $g(s)$ is stable if and only if $\xi_1 > 0$ and $\xi_1(\gamma_1) > 0$. $\xi_0^2 - \xi_0^2 > 0$.

**Lemma 2 [41]:** Let equation $ax^3 + bx^2 + cx + d = 0$, where $a$, $b$, $c$, $d \in \mathbb{R}$ ($a \neq 0$). Let further $A = b^2 - 3ac$, $B = bc - 9ad$, $C = c^2 - 3bd$, and $D = B^2 - 4AC$. Then, the following results hold.

1) The equation has three real roots if and only if $D \leq 0$.
   a) If $A = 0$, the equation has a triple root. The root is
      $$x_1 = x_2 = x_3 = \frac{-b}{3a} = \frac{-c}{b} = \frac{-3d}{c}.$$
   b) If $D = B^2 - 4AC = 0$, the equation has a simple root and a double root. The roots are
      $$x_1 = \frac{-b}{a} + K, \quad x_2 = x_3 = -\frac{K}{2}$$
      where $K = (B/A)$.
   c) If $D = B^2 - 4AC < 0$, the equation has three different real roots. They are
      $$x_1 = \frac{-b - 2\sqrt{A} \cos \frac{\varphi}{3}}{3a}$$
      $$x_{2,3} = \frac{-b + \sqrt{A} \left(\cos \frac{\varphi}{3} \pm \sqrt{3} \sin \frac{\varphi}{3}\right)}{3a}$$
      where $\varphi = \arccos \Theta$, $\Theta = [(2ba - 3aB)/\sqrt{A^3}]$ ($A > 0$, $-1 < \Theta < 1$).

2) The equation has one real root and a pair of conjugate complex roots if and only if $D > 0$. Further more, all roots are

$$x_1 = \frac{-b - (\sqrt{Y_1} + \sqrt{Y_2})}{3a}$$

$$x_{2,3} = \frac{-b + \frac{1}{2}(\sqrt{Y_1} + \sqrt{Y_2}) \pm \frac{1}{2}\sqrt{(\sqrt{Y_1} - \sqrt{Y_2})^2}}{3a}$$

where $Y_{1,2} = bA + 3a((-B + \sqrt{B^2 - 4AC})/2)$, $i^2 = -1$. 

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Lemma 3 [32]: Given a coefficient polynomial of order three in the form of \( f(s) = s^3 + (p_2 + iq_2)s^2 + (p_1 + iq_1)s + p_0 + iq_0. \) Then, \( f(s) \) is stable if and only if \( p_2 > 0, p_2q_1 + p_2^2p_1 - q_1^2 - p_2p_0 > 0, \) and
\[
p_2 \text{ det } \begin{pmatrix} q_2 & -p_1 & -q_0 \\ p_2 & q_1 & -p_0 \\ 1 & q_2 & -p_1 \\ 0 & p_2 & q_1 \end{pmatrix} - \text{ det } \begin{pmatrix} q_1 & -p_0 & 0 \\ p_2 & q_1 & -p_0 \\ 1 & q_2 & -p_1 \\ 0 & p_2 & q_1 \end{pmatrix} > 0.
\]

Lemma 4 [30]: Given a real coefficient polynomial of order three as follows:
\[
f(s) = a_3s^3 + a_2s^2 + a_1s + a_0.
\]
Then, \( f(s) \) is stable if and only if \( a_3, a_2, a_1, \) and \( a_0 \) are positive and \( a_1a_2 > a_0a_3. \)

Consider a group of \( N \) identical agents with second-order dynamics. The dynamics of each agent \( i \) is described as
\[
\begin{align*}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= u_i(t), \quad i = 1, 2, \ldots, N
\end{align*}
\]
where \( x_i(t) \in \mathbb{R}^n \) and \( v_i(t) \in \mathbb{R}^n \) represent the position and velocity states of agent \( i \), respectively; \( u_i(t) \) is the control input. For notational simplicity, \( n = 1 \) is considered throughout this paper. For case \( n > 1 \), all results can be easily generalized from the case \( n = 1 \) by using Kronecker product operations.

Definition 1: Second-order consensus of multiagent system (1) is said to be reached if, for any initial conditions
\[
\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \\
\lim_{t \to \infty} \|v_i(t) - v_j(t)\| = 0, \quad i, j = 1, 2, \ldots, N.
\]

For second-order multiagent system (1), some researchers have considered the following control protocols [30], [33]:
\[
u_i(t) = \alpha \sum_{j=1,j \neq i}^{N} a_{ij}(x_j(t_k) - x_i(t_k)) + \beta \sum_{j=1,j \neq i}^{N} a_{ij} \times (v_j(t_k) - v_i(t_k))
\]
and
\[
u_i(t) = \alpha \sum_{j=1,j \neq i}^{N} a_{ij}(x_j(t_k - \tau) - x_i(t_k - \tau)) + \beta \sum_{j=1,j \neq i}^{N} a_{ij} \times (v_j(t_k - \tau) - v_i(t_k - \tau))
\]
(2)

Note that these protocols are designed by using sampled data control strategies. However, the controller is remain working in each sampling interval. It should be noticed that the controller equipped in each agent may have limited energy resources to gather information or communicate with neighbor agents. Hence, it is desirable to design a novel control scheme such that both the working time and the update rate of controller for each agent can be reduced. Besides, in the process of information transmission, the time delay is always inevitable. Therefore, in consideration of these practical situations, two kinds of intermittent sampled data control protocols are proposed as follows:
\[
u_i(t) = \left\{ \begin{array}{ll}
\alpha \sum_{j=1,j \neq i}^{N} a_{ij}(x_j(t_k) - x_i(t_k)) + \beta \sum_{j=1,j \neq i}^{N} a_{ij} \\
\times (v_j(t_k) - v_i(t_k)), & t \in [t_k, t_k + \theta]
\end{array} \right.,
\]
(3)
\[
u_i(t) = \left\{ \begin{array}{ll}
\alpha \sum_{j=1,j \neq i}^{N} a_{ij}(x_j(t_k - \tau) - x_i(t_k - \tau)) + \beta \sum_{j=1,j \neq i}^{N} a_{ij} \\
\times (v_j(t_k - \tau) - v_i(t_k - \tau)), & t \in [t_k, t_k + \theta]
\end{array} \right.,
\]
(4)
where \( \{t_k : k \in \mathbb{N}\} \) is the sampling sequence, which satisfies \( 0 = t_0 < t_1 < \ldots < t_k < \ldots \), and \( t_{k+1} - t_k = T \) with \( T > 0 \) being the sampling period, \( \theta \) is the communication width and \( 0 < \theta \leq T \), \( \alpha \) and \( \beta \) are the coupling gains, \( 0 < \tau < T \) is the time delay.

Remark 1: Note that the sampled data control and event-triggered control strategies [30]–[40] reduce the update rate of the control signal, while they need the controller to work on the whole sampling interval. Although the intermittent control [24]–[29] shortens the controller’s working time, the controller needs to update the information continuously at each communication interval. However, the intermittent sampled data control proposed in this paper overcomes these shortcomings. It both reduces the update rate of the control signal and cuts down the working time of the controller in each sampling interval. In particular, when \( \theta = T \), the protocols (4) and (5) reduce to protocols (2) and (3), respectively.

III. INTERMITTENT SAMPLED CONTROL WITHOUT DELAY

In this section, distributed intermittent sampled protocol (4) is concerned for system (1). Under the protocol (4) and combination with the definition of Laplacian matrix, system (1) can be rewritten as follows:
\[
\dot{x}_i(t) = v_i(t)
\]
\[
\dot{v}_i(t) = \left\{ \begin{array}{ll}
-\alpha \sum_{j=1}^{N} \epsilon_{ij}x_j(t_k) - \beta \sum_{j=1}^{N} \epsilon_{ij}v_j(t_k), & t \in [t_k, t_k + \theta]
\end{array} \right.,
\]
and
\[
u_i(t) = \left\{ \begin{array}{ll}
\alpha \sum_{j=1,j \neq i}^{N} a_{ij}(x_j(t_k) - x_i(t_k)) + \beta \sum_{j=1,j \neq i}^{N} a_{ij} \\
\times (v_j(t_k) - v_i(t_k)), & t \in [t_k, t_k + \theta]
\end{array} \right.,
\]
(2)

Let \( \eta_i(t) = (x_i(t), v_i(t))^T \), \( A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \).

System (6) can be cast to
\[
\dot{\eta}_i(t) = \begin{pmatrix} A\eta_i(t) - \sum_{j=1}^{N} \epsilon_{ij}B\eta_j(t_k), \quad t \in [t_k, t_k + \theta] \\ A\eta_i(t), \quad t \in [t_k + \theta, t_{k+1}], \quad k \in \mathbb{N} \end{pmatrix}
\]
(7)
Let \( \eta(t) = (\eta_1^T(t), \ldots, \eta_N^T(t))^T \) and system (7) can be rewritten as the following form:
\[
\dot{\eta}(t) = \begin{pmatrix} (I_N \otimes A)\eta(t) - (C \otimes B)\eta(t_k), \quad t \in [t_k, t_k + \theta] \\ (I_N \otimes A)\eta(t), \quad t \in [t_k + \theta, t_{k+1}], \quad k \in \mathbb{N} \end{pmatrix}
\]
(8)
For Laplacian matrix \( L \), there exists a nonsingular matrix \( P \) such that \( L = PJP^{-1} \), where \( J \) is the Jordan form associated with the Laplacian matrix \( L \). Introduce the state transformation 
\[ y(t) = (P^{-1} \otimes I_2)z(t), \]
with (9), one has 
\[ \dot{y}(t) = \begin{cases} \left( (I_N \otimes A) - (J \otimes B) \right) y(t_k), & t \in [t_k, t_k + \theta) \\ \left( (I_N \otimes A) \right) y(t_k), & t \in [t_k + \theta, t_{k+1}), \end{cases} \]
where \( k \in \mathbb{N} \).

If the graph \( G \) is directed, some eigenvalues of \( J \) may be complex, and \( J = \text{diag}(J_1, \ldots, J_r) \), where \( J_i \in \mathbb{R}^{N_i \times N_i} \), \( i = 1, \ldots, r \), are Jordan blocks with the diagonal elements being the eigenvalues of \( L \) and \( N_1 + \cdots + N_r = N \). Particularly, when the graph \( G \) is undirected, then \( J \) is a diagonal matrix with non-negative real eigenvalues.

With the above development, the following assumptions and lemmas are presented.

**Assumption 1**: The graph \( G \) contains a directed spanning tree.

**Assumption 2**: The graph \( G \) is undirected and connected.

**Lemma 5**: Under Assumption 1, the second-order consensus in system (1) with protocol (4) can be achieved if and only if, in (9) \( \lim_{t \to \infty} \|y(t)\| = 0 \), \( i = 2, \ldots, N \).

**Proof**: The proof is similar to [30, Th. 1].

**Corollary 1**: Under Assumption 1, the second-order consensus in system (1) with protocol (4) can be achieved if and only if the following \( N-1 \) systems are asymptotically stable:
\[ \dot{z}_i(t) = Az_i(t) - \mu_i \dot{\delta}(t) B z_i(t_k), \quad t \in [t_k, t_{k+1}), \quad i = 2, \ldots, N \]
where \( \mu_i \) is nonzero eigenvalue of Laplacian matrix \( L \) and
\[ \delta(t) = \begin{cases} 1, & t \in [t_k, t_k + \theta) \\ 0, & t \in [t_k + \theta, t_{k+1}), \end{cases}, \quad k \in \mathbb{N}. \]

**Proof**: The proof is similar to [30, Corollary 1].

Although some conditions for reaching consensus are presented in Lemma 5 and Corollary 1, it is still not reveal how the control gains, the network structure, the sampling period, and the communication width affect the consensus behavior. Therefore, the following theorems are derived to reflect the relationship among them.

**A. Selection of the Sampling Period**

**Theorem 1**: Under Assumption 1, the second-order consensus in system (1) with protocol (4) can be achieved if and only if
\[ \frac{2\beta}{\alpha} > \theta \]
and
\[ \theta \leq T < \frac{\theta^2(\alpha \theta - 2\beta)^3 \| \mu_i \|^4 + 4\theta R(\mu_i)(2\beta - \alpha \theta)^2 \| \mu_i \|^2}{\alpha \theta^2(2\beta - \alpha \theta)^2 \| \mu_i \|^4 + 16\theta^2 T^2(\mu_i)} \]

where \( \mu_i \) is nonzero eigenvalue of Laplacian matrix \( L \).

**Proof**: From (10), it follows that:
\[ \left( e^{-At} z_i(t) \right)' = -e^{-At} \mu_i \dot{\delta}(t) B z_i(t_k), \quad t \in [t_k, t_{k+1}). \]

Integrating both sides of (13) from \( t_k \) to \( t \) (\( t \leq t_k + \theta \)) and combining with \( e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \), one has
\[ z_i(t_k) = e^{A(t-t_k)} z_i(t_k) = e^{-At} \int_{t_k}^{t} e^{At} \mu_i B z_i(t_k) ds \]
\[ = \begin{pmatrix} 1 - \frac{\alpha \mu_i}{2} (t - t_k)^2 & (t - t_k) - \frac{\beta \mu_i}{2} (t - t_k)^2 \\ -\alpha \mu_i (t - t_k) & 1 - \beta \mu_i (t - t_k)^2 \end{pmatrix} z_i(t_k). \]

Therefore
\[ z_i(t_k + \theta) = \begin{pmatrix} 1 - \frac{\alpha \mu_i}{2} \theta^2 & \theta - \frac{\beta \mu_i}{2} \theta^2 \\ -\alpha \mu_i \theta & 1 - \beta \mu_i \theta \end{pmatrix} z_i(t_k). \]

For \( t \in [t_k + \theta, t_{k+1}) \), one obtains
\[ z_i(t_k) = e^{A(t_k + \theta - t_k)} z_i(t_k + \theta) \]
\[ = \begin{pmatrix} \Xi_1(t_k + \theta - t_k) & \Xi_2(t_k + \theta - t_k) \\ -\alpha \mu_i \theta & 1 - \beta \mu_i \theta \end{pmatrix} z_i(t_k). \]

where \( \Xi_1(t_k + \theta - t_k) = 1 - (\alpha \mu_i/2) (t - t_k) - \alpha \mu_i \theta (t - t_k + \theta), \)
\( \Xi_2(t_k + \theta - t_k) = \theta (\beta \mu_i/2) (t - t_k) + (1 - \beta \mu_i \theta) (t - t_k + \theta) \).

Let
\[ C(\theta) = \begin{pmatrix} 1 - \frac{\alpha \mu_i}{2} \theta^2 & \theta - \frac{\beta \mu_i}{2} \theta^2 \\ -\alpha \mu_i \theta & 1 - \beta \mu_i \theta \end{pmatrix} \]

\( C(\theta) \) is bounded on \([0, \theta]\). Furthermore, let
\[ D(\theta) = \begin{pmatrix} \Xi_1(\theta) & \Xi_2(\theta) \\ -\alpha \mu_i \theta & 1 - \beta \mu_i \theta \end{pmatrix}. \]

It is easy to see that \( D(\theta) \) is also bounded on \([t_k + \theta, t_{k+1}) \).

Hence, for \( 0 = t_0 < t_1 < \ldots < t_k < \ldots \) and \( t_{k+1} - t_k = T \), one has
\[ z_i(t_k) = \begin{pmatrix} C(t - t_k) D^k(T - \theta) z_i(t_0), & t \in [t_k, t_k + \theta) \\ D(t - (t_k + \theta)) D^k(T - \theta) z_i(t_0), & t \in [t_k + \theta, t_{k+1}). \end{pmatrix} \]

Since \( C(t - t_k) \) and \( D(t - (t_k + \theta)) \) are all bounded when \( t \in [t_k, t_k + \theta) \) and \( t \in [t_k + \theta, t_{k+1}) \), respectively, \( z_i(t) \to 0 \) if and only if all eigenvalues \( \lambda \) of \( D(T - \theta) \) satisfy \( |\lambda| < 1 \).

Let \( \lambda J_2 - D(T - \theta) = 0 \), it yields
\[ \lambda^2 - \left( 2 + \frac{\alpha \mu_i}{2} \theta^2 - \alpha \mu_i \theta T - \beta \mu_i \theta \right) \lambda \]
\[ + 1 - \beta \mu_i \theta + \frac{\alpha \mu_i}{2} \theta^2 = 0. \]

Let \( \lambda = [(s + 1)/(s - 1)] \). Then, (17) can be transformed to
\[ s^2 \left( \frac{2\beta}{\alpha T} - \frac{\theta}{T} \right) s + 1 - \frac{4}{\alpha \mu_i \theta T} + \theta - \frac{2\beta}{\alpha T} = 0. \]

As we all know that \( |\lambda| < 1 \) in (17) if and only if \( \Re(s) < 0 \) in (18). Therefore, \( z_i(t) \to 0 \) if and only if all roots in (18) have negative real parts.

Based on Lemma 1, (18) is stable if and only if
\[ \frac{2\beta}{\alpha T} - \frac{\theta}{T} > 0 \]
and
\[ \left( \frac{2\beta}{\alpha T} - \frac{\theta}{T} \right)^2 \left( \frac{\theta}{T} - \frac{2\beta}{\alpha T} \right) - \frac{16\Re(\mu_i)}{\alpha \mu_i \theta T \| \mu_i \|^2} T^2 - 16\Re(\mu_i) \frac{\alpha \mu_i}{2} \theta^2 T^2 \| \mu_i \|^4 > 0. \]
By solving these two inequalities and combining with the condition $0 < \theta \leq T$, one gets (11) and (12). Consequently, $z_i(t) \to 0$ if and only if (11) and (12) are satisfied. According to Corollary 1, second-order consensus in system (1) with protocol (4) can be achieved if and only if (11) and (12) hold.

Remark 2: When $\theta = T$, the transmission of information among all agents are continuous. The conditions in Theorem 1 become

$$\frac{2\beta}{\alpha} > T$$

and

$$\frac{(2\beta}{\alpha T - 1)^2 \left( \frac{4R(m_i)}{\alpha \|m_i\|^2 T^2} - \frac{2\beta}{\alpha T} \right) - \frac{16\tilde{I}(m_i)}{\alpha^2 \|m_i\|^4 T^4} > 0$$

which are same as the ones in [30, Th. 2]. Hence, the results obtained in [30, Th. 2] can be considered as a special case of intermittent sampled data control.

If the network is undirected, simplified result can be derived.

Corollary 2: Under Assumption 2, the second-order consensus in system (1) with protocol (4) can be reached if and only if

$$\frac{2\beta}{\alpha} > \theta$$

and

$$\theta < T < \frac{4}{\alpha \mu_N \theta} - \frac{2\beta}{\alpha}.$$  

(22)

Particularly, when $\theta = T$

$$T < \min \left( \frac{2\beta}{\alpha}, \frac{2}{\beta \mu_N} \right)$$

(23)

where $\mu_N$ is the maximum eigenvalue of Laplacian matrix $L$.

Proof: According to Assumption 2, all eigenvalues of Laplacian matrix are real numbers. Then, $T(m_i) = 0$ and $R(m_i) = \mu_i > 0$ for $i = 2, \ldots, N$. The inequality (20) is equivalent to

$$\left( \frac{2\beta}{\alpha T - \theta} \right)^2 \left( \frac{2}{\alpha T - 1} + \frac{4}{\alpha \theta T \mu_i} \right) > 0, \ i = 2, \ldots, N.$$  

(24)

Combining with the condition (19), one gains that second-order consensus in system (1) can be achieved if and only if conditions (21) and (22) hold. Particularly, when $\theta = T$, one has $T < \min ((2\beta/\alpha), (2/(\beta \mu_N)))$.

Remark 3: It is noted that a upper bound of communication width $\theta$ is given in inequality (11) for reaching consensus, but it is still a challenging work that how to obtain more accurate region of $\theta$ from (12). By simple transformation, (12) can be rewritten as a cubic inequality with respect to $\theta$.

B. Selection of the Communication Width

Hereinafter, the feasible region of communication width $\theta$ is given. Noting that a necessary and sufficient condition to satisfy (12) is

$$\frac{\theta^2 (2\alpha - 2\beta)^2 \|m_i\|^4 + 4\theta R(m_i)(2\beta - \alpha\theta)^2 \|m_i\|^2}{\alpha \theta^2 (2\beta - \alpha\theta)^2 \|m_i\|^4 + 16\alpha \tilde{I}^2(m_i)} > \theta.$$  

(25)

For convenience, we denote

$$g(\theta) = \frac{\theta^2 (2\alpha - 2\beta)^2 \|m_i\|^4 + 4\theta R(m_i)(2\beta - \alpha\theta)^2 \|m_i\|^2}{\alpha \theta^2 (2\beta - \alpha\theta)^2 \|m_i\|^4 + 16\alpha \tilde{I}^2(m_i)}.$$  

Then, by solving (25) and combination with (11), a feasible region of communication width $\theta$ is obtained.

Theorem 2: Under Assumption 1, the second-order consensus in system (1) with protocol (4) can be achieved if and only if one of the following cases holds.

Case 1: $\beta^2/\alpha > \tilde{T}(m_i)/ R(m_i) \|m_i\|^2$ and $\Delta = B^2 - 4\alpha C = 0$

$$0 < \theta < \min \left( \frac{2\beta}{\alpha}, \frac{2\alpha R(m_i) + 4\beta^2 \|m_i\|^2}{3\alpha \beta \|m_i\|^2} \right), \ T \in [\theta, g(\theta)).$$  

(26)

Case 2: $\beta^2/\alpha > \tilde{T}(m_i)/ R(m_i) \|m_i\|^2$ and $\Delta = B^2 - 4\alpha C < 0$

$$\theta \in (0, 2\beta/\alpha) \cap \{(-\infty, \theta_1) \cup (\theta_2, \theta_3)\}, \ T \in [\theta, g(\theta)).$$  

(27)

Case 3: $\Delta = B^2 - 4\alpha C > 0$

$$0 < \theta < \min \left( \frac{2\beta}{\alpha}, \frac{b - B}{a + A} \right), \ \theta \neq \frac{B}{2A}, \ \ T \in [\theta, g(\theta)).$$  

(28)

Case 4: $\Delta = B^2 - 4\alpha C > 0$

$$0 < \theta < \min \left( \frac{2\beta}{\alpha}, \frac{b - B}{a + A} \right), \ \ T \in [\theta, g(\theta)).$$  

(29)

where $a = 2\alpha^2 \beta^2, b = -(4\alpha R(m_i)/ \|m_i\|^2 + 8 \alpha \beta^2), c = 8 \beta^3 + 16 \alpha \beta R(m_i)/ \|m_i\|^2, d = 16 \alpha \tilde{I}(m_i)/ \|m_i\|^4 - 16 \beta^2 R(m_i)/ \|m_i\|^2, A = b^2 - 3ac, B = bc - 9ad, C = c^2 - 3bd, \Delta = B^2 - 4\alpha C, \ \theta_1 = \min \{g(1), c_2, c_3\}, \ \theta_2 = \max \{c_1, c_2, c_3\} , \ \ c_1 = -b - 2\alpha \sqrt{\cos(\varphi(\beta)/3)}(3a), \ c_2 = -b + \sqrt{\lambda} \left( \sqrt{1 + \sqrt{1 + \Delta}} \right)/3(3a), \ \varphi = \arccos \Theta, \ \Theta = (2b - 3ac)/\sqrt{A}, \ (A > 0, 1 < \Theta < 1), \ \theta = -b(\sqrt{Y_1} + \sqrt{Y_2})/(3a)$, and $Y_{1,2} = bA + 3a((-B \pm \sqrt{\Delta})/2)$.

Proof: By using simple transformation and combining with $\theta > 0$, one can obtain that the equality (25) holds if

$$2\alpha^2 \beta^2 \theta^3 - \left( \frac{4\alpha R(m_i)}{\|m_i\|^2} + 8 \alpha \beta^2 \right) \theta^2 + \left( 8 \beta^3 + \frac{16\alpha \beta R(m_i)}{\|m_i\|^2} \right) \theta + \frac{16 \alpha \tilde{I}(m_i)}{\|m_i\|^4} - \frac{16 \beta^2 R(m_i)}{\|m_i\|^2} < 0.$$  

(30)

Let $a = 2\alpha^2 \beta, b = -(4\alpha R(m_i)/ \|m_i\|^2 + 8 \alpha \beta^2), c = 8 \beta^3 + 16 \alpha \beta R(m_i)/ \|m_i\|^2, d = 16 \alpha \tilde{I}(m_i)/ \|m_i\|^4 - 16 \beta^2 R(m_i)/ \|m_i\|^2$. Then, the inequality (30) can be transformed as

$$a \theta^3 + b \theta^2 + c \theta + d < 0.$$  

(31)

Furthermore, let $A = b^2 - 3ac, B = bc - 9ad, C = c^2 - 3bd$, and $\Delta = B^2 - 4\alpha C$. Based on Lemma 2, one has the following cases.
Case 1: If $A = B = 0$, the equality (31) can be written as $(\theta + b/(3a)) = (c/b) = (3d/c)$. It is easily to see that $a > 0$, $b < 0$ and $c > 0$. Hence, $d < 0$ is a necessary condition for this case. Combination with (11), one obtains that the inequality (31) holds if $$0 < \theta < \min \left\{ \frac{2\beta}{\alpha}, \frac{2aR(\mu_i) + 4\beta^2\|\mu_i\|^2}{3\alpha\beta\|\mu_i\|^2} \right\}.$$

Case 2: If $\Delta = B^2 - 4AC = 0$, the equality (31) can be written as $(\theta + (b/a) - (B/A)) = (\theta + (B/2A))^2 < 0$. Combining with (11), a necessary condition for the inequality (31) with nonempty solution is $d < 0$. Therefore, one has $$0 < \theta < \min \left\{ \frac{2\beta}{\alpha}, -\frac{b}{a} + \frac{B}{A} \right\}, \quad \theta \neq -\frac{B}{2A}.$$

Case 3: If $\Delta = B^2 - 4AC < 0$, then equality (31) can be cast to $(\theta - \epsilon_1)(\theta - \epsilon_2)(\theta - \epsilon_3) < 0$, where $\epsilon_1 = \left(-b - 2\sqrt{A}\cos(\omega/3)/3a\right), \epsilon_2 = \left(-b + 2\sqrt{A}\cos(\omega/3) + 3\sqrt{3}\sin(\omega/3)/3a\right), \epsilon_3 = \arch\cos\theta, \Theta = \left(2b - 3aB\right)/\sqrt{A} \left(A > 0, -1 < \Theta < 1 \right)$. Let $\theta_1$, $\theta_2$, and $\theta_3$ are all roots of equation $(\theta - \epsilon_1)(\theta - \epsilon_2)(\theta - \epsilon_3) = 0$, which satisfy $\theta_1 < \theta_2 < \theta_3$. Combination with (11), one gets $$\theta \in (0, 2\beta/\alpha) \cap \{ -\infty, \theta_1 \} \cup \{ \theta_2, \theta_3 \}.$$ This completes the proof.

For the undirected network, the condition (12) can reduce to a simple form. Let $h(\theta) = \theta + (4/(\alpha\mu N\theta)) - (2\beta/\alpha)$. So, the following corollary is presented.

**Corollary 3:** Under Assumption 2, the second-order consensus in system (1) with protocol (4) can be reached if and only if $$\theta < \min \left\{ \frac{2\beta}{\alpha}, \frac{2}{\beta \mu N} \right\} \quad \text{and} \quad \theta \in [\theta, h(\theta)). \quad (32)$$

In this corollary, the feasible region of $T$ is $T \in [\theta, h(\theta))$. Taking the derivative of $h(\theta)$ with respect to $\theta$, one has $(dh(\theta)/d\theta) = 1 - (4/(\alpha\mu N\theta^2))$. When $0 < \theta < (2/\sqrt{\alpha/\mu N})$, $h(\theta)$ is decreasing respect to $\theta$. Since $\theta < \alpha = \min \left\{ 2\beta/\alpha, 2/(\beta \mu N) \right\}$, the region of sampling period $T$ is decreasing with the increasing of communication width $\theta$.

**Remark 4:** For the undirected network, the feasible region of sampling period decreases with the increasing of communication width. This provides us a guide for design of sampling period. If we want a larger region of sampling period, the communication width should choose smaller. In Theorem 2, all feasible regions of communication width $\theta$ and sampling period $T$ are derived. However, for the directed network, since the feasible region of communication width is relative complex, it is still a open question how does the communication width affect the sampling period.

**Remark 5:** The sampling period $T$ plays an essential role in reaching second-order consensus of multiagent system (6). Consequently, how to choose an appropriate sampling period is very important. The procedure of selecting the sampling period $T$ is given as follows.

Step 1: Choose coupling gains $\alpha$ and $\beta$ randomly.

Step 2: Combining with the eigenvalues of the Laplacian matrix and the discussion of Theorem 2, select a relatively appropriate communication width $\theta$.

Step 3: Calculate the maximum sampling period $g(\theta)$.

Then, one can choose $T \in [\theta, g(\theta))$.

**C. Design of the Coupling Gains**

In the following, a guideline for how to select the coupling gains $\alpha$ and $\beta$ is given. Multiplying $\alpha^2 T^3$ on both sides of inequality (20), then it can be equivalently written as

$$(2\beta - \alpha \theta)^2 \left( \theta \alpha - 2\beta - \alpha T + \frac{4R(\mu_i)}{\theta \|\mu_i\|^2} \right) - \frac{16\alpha T(\mu_i)T}{\theta^2 \|\mu_i\|^4} > 0.$$ \quad (33)

Note that (33) can be transformed to

$$\theta^2 (T - \theta \alpha)^3 + \left(6\beta \theta^2 - 4\beta \theta T - \frac{4\beta R(\mu_i)}{\|\mu_i\|^2} \right) \alpha^2$$

$$+ \frac{\left( 4\beta^2 T - 12\beta^2 + \frac{16\beta R(\mu_i)}{\|\mu_i\|^2} + \frac{16\alpha T(\mu_i)T}{\|\mu_i\|^4} \right) \alpha}{\theta^3 \|\mu_i\|^6}$$

$$+ 8\beta^3 - \frac{16\beta^2 R(\mu_i)}{\|\mu_i\|^2} < 0, \quad i = 2, \ldots, N.$$ \quad (34)

When $\theta = T$, the inequality (25) can be equivalently reduce to

$$(2\beta - \alpha T)^2 \left( \frac{4R(\mu_i)}{\|\mu_i\|^2} - 2\beta T \right) - \frac{16\alpha T(\mu_i)T}{\theta^2 \|\mu_i\|^4} > 0.$$ \quad (35)

**Theorem 3:** Under Assumption 1, the second-order consensus in system (1) with protocol (4) can be reached if and only if one of the following cases holds.

1) $\theta = T$

$$\alpha < \min \left\{ \frac{2\beta}{T} + \frac{1}{\gamma_i} \sqrt{\gamma_i^2 + \frac{4\beta^2}{T}} \right\}$$ \quad (36)

and

$$\beta < \min \left\{ \frac{2R(\mu_i)}{\|\mu_i\|^2 T} \right\}.$$ \quad (37)

2) $0 < \theta < T$.

**Case 1:** $\beta < \min \left\{ \frac{2R(\mu_i)}{\|\mu_i\|^2 \theta} \right\}$ and $\tilde{A} = \tilde{B} = 0$

$$0 < \alpha < \min \left\{ \frac{2\beta}{\theta}, \frac{\tilde{b}}{3\tilde{a}} \right\}.$$ \quad (38)

**Case 2:** $\beta < \min \left\{ \frac{2R(\mu_i)}{\|\mu_i\|^2 \theta} \right\}$ and $\tilde{A} = \tilde{B}^2 - 4\tilde{A}\tilde{C} = 0$

$$0 < \alpha < \min \left\{ \frac{2\beta}{\theta}, \frac{\tilde{b}}{3\tilde{a}} + \frac{\tilde{B}}{A} \right\}, \quad \alpha \neq \frac{\tilde{B}}{2A}.$$ \quad (39)
Case 3: \( \Delta = \hat{B}^2 - 4\hat{A}\hat{C} < 0 \)
\[
\alpha \in (0, 2\beta/\theta) \cap (-\infty, \alpha_1) \cup (\alpha_2, \alpha_3). \tag{40}
\]

Case 4: \( \Delta = \hat{B}^2 - 4\hat{A}\hat{C} > 0 \)
\[
0 < \alpha < \min \{ 2\beta/\theta, \hat{a} \} \tag{41}
\]

where \( \gamma_i = 4T^2(\mu_i)/\|\mu_i\|^2T(2R(\mu_i) - \beta\|\mu_i\|^2T) \),
\[
i = 2, \ldots, N, \quad \hat{a} = \hat{t}^2(T - \theta), \quad \hat{b} = 6\beta\theta^2 - 4\beta\theta T - 4\beta R(\mu_i)/\|\mu_i\|^2, \]
\[
\hat{c} = 4\beta^2T - 12\beta^2\theta + 16\beta R(\mu_i)/\|\mu_i\|^2 + 16\beta^2(\mu_i/T)^2/\|\mu_i\|^2, \quad \hat{d} = 8\beta^3 - 16\beta^2 R(\mu_i)/\theta/\|\mu_i\|^2.,
\]
\[
\text{Furthermore, let } \alpha = \min \{ \alpha_1, \alpha_2, \alpha_3 \}, \quad \alpha_2 \in \{ \alpha_1, \alpha_2, \alpha_3 \} \cap (\alpha_1, \alpha_3), \]
\[
\omega_1 = (\sqrt{2}\cos(\varphi/3))/3(\hat{a}), \quad \omega_2, \omega_3 = (1/2)(\sqrt[3]{\hat{a} + \sqrt{\hat{a}^2 - 4\hat{a}\hat{C}}}/(\hat{a})), \quad \varphi = \arccos(\hat{\theta}), \quad \hat{\theta} = (2(\hat{a} - 3\hat{b}))/\sqrt{\hat{a}^2}, (\hat{A} > 0, -1 < \hat{\theta} < 1).
\]

Proof: When \( \theta = T \), the proof is similar to [30, Th. 4]. So, the inequalities (36) and (37) can be obtained. When \( 0 < \theta < T \), \( \theta^2(T - \theta) \neq 0 \). Hence, (34) can be regarded as a cubic inequality with respect to \( \alpha \).

Let \( \hat{a} = \hat{t}^2(T - \theta), \quad \hat{b} = 6\beta\theta^2 - 4\beta\theta T - 4\beta R(\mu_i)/\|\mu_i\|^2, \)
\[
\hat{c} = 4\beta^2T - 12\beta^2\theta + 16\beta R(\mu_i)/\|\mu_i\|^2 + 16\beta^2(\mu_i/T)^2/\|\mu_i\|^2, \quad \hat{d} = 8\beta^3 - 16\beta^2 R(\mu_i)/\theta/\|\mu_i\|^2.,
\]
\[
\text{Then, (34) can be cast to the following form:}
\]
\[
\alpha \hat{a}^3 + \hat{b}\hat{a}^2 + \hat{c}\hat{a} + \hat{d} < 0. \tag{42}
\]

Based on Lemma 2, we obtain the following results.

Let \( \Delta = \hat{B}^2 - 3\hat{a}\hat{c}, \quad \hat{B} = \hat{b} - 9\hat{a}\hat{d}, \hat{C} = \hat{c}^2 - 3\hat{b}\hat{d}, \text{ and } \Delta = \hat{B}^2 - 4\hat{A}\hat{C} \).

When \( \Delta = 0 \), the inequality (42) can be transformed as \( \alpha = (\hat{b}/(3\hat{a}^2))^3 < 0 \). Combining with (19), one gets
\[
0 < \alpha < \min \left\{ \frac{2\beta}{\theta}, \frac{-\hat{b}}{3\hat{a}} \right\}. \tag{43}
\]

When \( \Delta = \hat{B}^2 - 4\hat{A}\hat{C} = 0 \), the inequality (42) can be transformed as \( \alpha = (\hat{b}/\hat{\alpha}) - (\hat{B}/\hat{A})(\alpha + (\hat{B}/(2\hat{A}))^2 < 0 \). Based on (19), one has
\[
0 < \alpha < \min \left\{ \frac{2\beta}{\theta}, \frac{-\hat{b}}{\hat{A}} \right\} \cap (\hat{A} \neq -\frac{\hat{B}}{2\hat{A}}). \tag{44}
\]

When \( \Delta = \hat{B}^2 - 4\hat{A}\hat{C} < 0 \), the inequality (42) can be rewritten as
\[
(\alpha - \omega_1)(\alpha - \omega_2)(\alpha - \omega_3) < 0
\]
\[
0 < \alpha < \min \left\{ \frac{2\beta}{\theta}, \frac{-\hat{b}}{\hat{A}} \right\} \cap (\hat{A} \neq -\frac{\hat{B}}{2\hat{A}}). \tag{45}
\]

By the same calculation, we can also get that the asymptotical behavior of system (46) is dominated by the stability of the system as follows:
\[
\hat{z}_i(t) = A\hat{z}_i(t) - \mu_i(t)B\hat{z}_i(t - \tau) \quad t \in [t, k_1 + \theta)
\]
\[
0 < \tau < T - \theta
\]
\[
\text{where } \mu_i \text{ is nonzero eigenvalue of the Laplacian matrix } \mathcal{L}
\]
\[
\delta(t) = \begin{cases} 1, & t \in [t, k_1 + \theta) \quad \text{where } k_1 \text{ is the sampling period, and the network structure.}
\end{cases}
\]

Next, a necessary and sufficient condition is established to reveal the relationship among the time delay, the coupling gains, the communication width, the sampling period, and the network structure.

Theorem 4: Under Assumption 1, the second-order consensus in system (1) with protocol (5) can be reached if and only if
\[
\frac{2\beta}{\alpha} = 2\tau > \theta
\]
and \[
T \leq \frac{4\theta R(\mu_i)s^2\|\mu_j\|^2 - \theta^2\|\mu_i\|^2}{\theta^2\|\mu_i\|^2 + 16\sigma i^2(\mu_i)}.
\] (49)

2) \( T - \theta \leq \tau \)

\[
T > \frac{2\tau}{3} - \frac{2\beta}{3\alpha} p_2^2 p_1 - q_1^2 - p_2 p_0 > 0
\] (50)

and

\[
p_2 \det \left( \begin{array}{ccc}
0 & -p_1 & -q_0 \\
p_2 & q_1 & -p_0 \\
1 & 0 & -p_1 - q_0 \\
0 & p_2 & q_1 & -p_0 \\
\end{array} \right) - \det \left( \begin{array}{ccc}
q_1 & -p_0 & 0 \\
p_2 & q_1 & -p_0 \\
1 & 0 & -p_1 - q_0 \\
0 & p_2 & q_1 & -p_0 \\
\end{array} \right) > 0
\] (52)

where \( \sigma = 2\beta - \alpha\theta - 2\alpha\tau, p_2 = (2\beta - 2\alpha\tau)/(\alpha T) + 3, p_1 = 4R(\mu_i)/(\alpha T|\mu_i|^2) + (2\alpha(T - \tau)^2 + 4\beta(T - \tau) - 4\beta\theta + 2\alpha\theta^2 - 4\alpha T - 4\alpha\theta(T - \tau))/(\alpha T), q_1 = 4I(\mu_i)/(\alpha T|\mu_i|^2) + (2\alpha(T - \tau)^2 + 4\beta(T - \tau) - 2\beta\theta - 2\alpha\theta(T - \tau) - \alpha\theta(T - \theta))/(\alpha T), q_0 = 4I(\mu_i)/(\alpha T|\mu_i|^2) \).

**Proof:** From (47), it follows that:

\[
e^{-At}z_i(t) = e^{-At} \mu_i \delta(t) B z_i(t - \tau), \quad t \in [t_k, t_{k+1}).
\] (53)

Integrating both sides of (53) from \( t_k \) to \( t (t \leq t_k + \theta) \), one has

\[
z_i(t) = \begin{pmatrix} 1 & t - t_k \\ 0 & 1 \end{pmatrix} z_i(t_k)
- \begin{pmatrix} \frac{\alpha(t - t_k)^2}{\mu_i} & \frac{\theta(t - t_k)^2}{\mu_i} \\ \alpha(t - t_k)\mu_i & \beta(t - t_k)\mu_i \end{pmatrix} z_i(t_k).
\] (54)

Let

\[
E(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F_i(t) = \begin{pmatrix} \frac{\alpha t^2}{\mu_i} & \frac{\theta t^2}{\mu_i} \\ \alpha t \mu_i & \beta t \mu_i \end{pmatrix}.
\]

It is easy to see that \( E(t) \) and \( F_i(t) \) are both bounded on \([t_k, t_{k+\theta}]\). Hence, for \( 0 = t_0 < t_1 < \ldots < t_k < \ldots, t_{k+1} - t_k = T, \) and \( \tau < t_k \), one has

\[
z_i(t_k + \theta) = E(\theta)z_i(t_k) - F_i(\theta)z_i(t_k - \tau).
\] (55)

For \( t_k + \theta \leq t < t_{k+1} \), one obtains

\[
z_i(t) = \begin{pmatrix} 1 & t - (t_k + \theta) \\ 0 & 1 \end{pmatrix} z_i(t_k + \theta).
\] (56)

Let

\[
H_i(t) = \begin{pmatrix} \frac{\alpha t^2}{\mu_i} + \alpha t \mu_i \theta & \frac{\theta t^2}{\mu_i} + \beta t \mu_i \theta \\ \alpha t \mu_i \theta & \beta t \mu_i \theta \end{pmatrix}
\]

and

\[
G(t) = \begin{pmatrix} 1 & t + \theta \\ 0 & 1 \end{pmatrix}.
\]

Then, combining with (55), (56) can be transformed to

\[
z_i(t) = G(t - (t_k + \theta))z_i(t_k) - H_i(t - (t_k + \theta))z_i(t_k - \tau).
\] (57)

Therefore, for \( t \in [t_k, t_{k+1}) \), one has

\[
z_i(t) = \begin{cases} 
E(t - t_k)z_i(t_k) - F_i(t - t_k)z_i(t_k - \tau) & t \in [t_k, t_k + \theta) \\
G(t - (t_k + \theta))z_i(t_k) - H_i(t - (t_k + \theta)) & t \in [t_k + \theta, t_{k+1}).
\end{cases}
\] (58)

Let \( v_i(t) = (z_i(t), z_i(t - \tau)) \). In order to obtain \( z_i(t_k - \tau) \), we discuss the following cases.

**Case 1:** \( \tau < \min[\theta, T - \tau] \).

For \( t \in [t_k, t_{k+1}) \),

\[
z_i(t) = E(t - t_k)z_i(t_k) - F_i(t - t_k)z_i(t_k - \tau)
\]

\[
z_i(t - \tau) = G(t - t_k - 1 - \theta - \tau)z_i(t_k - 1)
- H_i(t - t_k - 1 - \theta - \tau)z_i(t_k - 1 - \tau).
\]

For \( t \in [t_k + \theta, t_k + \theta + \tau) \),

\[
z_i(t) = E(t - t_k - \theta)z_i(t_k) - F_i(t - t_k - \theta)z_i(t_k - \tau)
\]

\[
z_i(t - \tau) = E(t - t_k - \theta - \tau)z_i(t_k - 1)
- H_i(t - t_k - \theta - \tau)z_i(t_k - 1 - \tau).
\]

For \( t \in [t_k + \theta + \tau, t_{k+1}) \),

\[
z_i(t) = G(t - t_k - \theta)z_i(t_k) - H_i(t - t_k - \theta)z_i(t_k - \tau)
\]

\[
z_i(t - \tau) = E(t - t_k - 1 - \theta - \tau)z_i(t_k - 1)
- H_i(t - t_k - 1 - \theta - \tau)z_i(t_k - 1 - \tau).
\]

Then

\[
z_i(t_{k+1}) = G(T - \theta)z_i(t_k) - H_i(T - \theta)z_i(t_k - \tau)
\]

\[
z_i(t_{k+1} - \tau) = G(T - \theta - \tau)z_i(t_k)
- H_i(T - \theta - \tau)z_i(t_k - \tau).
\]

Let

\[
M_{0i}(t) = \begin{pmatrix} \Xi_3(t) & \Xi_4(t) \\
G(t - T - \theta - \tau) & -H_i(1 + T - \theta - \tau) \end{pmatrix},
\]

\[
M_{1i}(t) = \begin{pmatrix} E(t) & -F_i(t) \\
E(t) & -F_i(t) \end{pmatrix}
\]

\[
M_{2i}(t) = \begin{pmatrix} G(t) & -H_i(t) \\
G(t - T - \theta - \tau) & -H_i(T - \theta - \tau) \end{pmatrix},
\]

\[
M_{3i}(t) = \begin{pmatrix} G(t - \theta) & -H_i(t - \theta) \\
G(t - \theta - \tau) & -H_i(T - \theta - \tau) \end{pmatrix}
\]

where \( \Xi_3(t) = E(t)G(T - \theta) - F_i(t)G(T - \theta - \tau) \) and \( \Xi_4(t) = -E(t)H_i(T - \theta) + F_i(t)H_i(T - \theta - \tau) \). Then, one has

For \( t \in [t_k, t_k + \tau) \),

\[
v_i(t) = M_{0i}(t - t_k)v_i(t_{k-1}) = M_{0i}(t - t_k)M_{k-1}(T)v_i(t_0).
\]

For \( t \in [t_k + \tau, t_k + \theta) \),

\[
v_i(t) = M_{1i}(t - t_k)v_i(t_k) = M_{1i}(t - t_k)M_{k-1}(T)v_i(t_0).
\]
For $t \in [t_k + \theta, t_k + \theta + \tau)$

$$v_i(t) = M_{t_k}(t - t_k)v_i(t_k) = M_{t_k}(t - t_k)M_{t_k}^k(T)v_j(t_0).$$

For $t \in [t_k + \theta + \tau, t_{k+1} + \theta + \tau)$

$$v_i(t) = M_{t_k}(t - t_k)v_i(t_k) = M_{t_k}(t - t_k)M_{t_k}^k(T)v_j(t_0).$$

Since $M_{t_k}(t-t_k)$ for $i = 2, \ldots, N$, $j = 0, 3, 6, 7$ are all bounded when $t \in [t_k, t_{k+1}]$, one has that $z_i(t) \to 0$ if and only if $v_i(t) \to 0$, or equivalently, all eigenvalues of $M_{t_k}(T)$ satisfy $\|\lambda\| < 1$.

Case 2: $T - \theta \leq t < \theta (T > (T/2))$ By similar discussions, one can also deduce

$$v_i(t) = M_{t_k}(t - t_k)v_i(t_k) = M_{t_k}(t - t_k)M_{t_k}^k(T)v_j(t_0).$$

For $t \in [t_k + \theta + \tau, t_{k+1} + \theta + \tau)$

$$v_i(t) = M_{t_k}(t - t_k)v_i(t_k) = M_{t_k}(t - t_k)M_{t_k}^k(T)v_j(t_0).$$

Since $M_{t_k}(t-t_k)$ for $i = 2, \ldots, N$, $j = 0, 3, 6, 7$ are all bounded when $t \in [t_k, t_{k+1}]$, one has that $z_i(t) \to 0$ if and only if $v_i(t) \to 0$, or equivalently, all eigenvalues of $M_{t_k}(T)$ satisfy $\|\lambda\| < 1$.

Case 3: $\theta < T - \theta (\theta < (T/2))$

For $t \in [t_k, t_k + \theta)$

$$v_i(t) = M_{t_k}(t - t_k)v_i(t_k) = M_{t_k}(t - t_k)M_{t_k}^k(T)v_j(t_0).$$

For $t \in [t_k + \theta, t_k + \tau)$

$$v_i(t) = M_{t_k}(t - t_k)v_i(t_k) = M_{t_k}(t - t_k)M_{t_k}^k(T)v_j(t_0).$$

Since $M_{t_k}(t-t_k)$ for $i = 2, \ldots, N$, $j = 0, 3, 6, 7$ are all bounded when $t \in [t_k, t_{k+1}]$, one has that $z_i(t) \to 0$ if and only if $v_i(t) \to 0$, or equivalently, all eigenvalues of $M_{t_k}(T)$ satisfy $\|\lambda\| < 1$.

Case 4: $\tau > \max[\theta, T - \theta]$

For $t \in [t_k + \theta + \tau, t_{k+1} + \theta + \tau)$

$$v_i(t) = M_{t_k}(t - t_k)v_i(t_k) = M_{t_k}(t - t_k)M_{t_k}^k(T)v_j(t_0).$$

Since $M_{t_k}(t-t_k)$ for $i = 2, \ldots, N$, $j = 0, 3, 6, 7$ are all bounded when $t \in [t_k, t_{k+1}]$, one has that $z_i(t) \to 0$ if and only if $v_i(t) \to 0$, or equivalently, all eigenvalues of $M_{t_k}(T)$ satisfy $\|\lambda\| < 1$.

According to discussion, for cases 1 and 3, one obtains that $z_i(t) \to 0$ if and only if all eigenvalues of $M_{t_k}(T)$ satisfy $\|\lambda\| < 1$. And for cases 2 and 4, one has $z_i(t) \to 0$ if and only if all eigenvalues of $M_{t_k}(T)$ satisfy $\|\lambda\| < 1$.

For cases 1 and 3, let $\lambda I_4 - M_{t_k}(T) = 0$, one has

$$\lambda^4 + \left(\alpha \mu_\theta T - \frac{\alpha}{2} \mu_\theta^2 - \alpha \mu_\theta^3 T + \beta \mu_\theta + 2\right)\lambda^3 + \left(\frac{\alpha}{2} \mu_\theta^3 T^2 + \alpha \mu_\theta T + \beta \mu_\theta + 1\right)\lambda^2 = 0. \tag{59}$$

It is easy to see that, $M_{t_k}(T)$ has two eigenvalues $\lambda_1 = \lambda_2 = 0$. Let $\lambda = (s + 1)/(s - 1)$. One gets

$$s^2 + \left(\frac{2\beta}{\alpha T} \frac{\theta}{T} - \frac{2\tau}{T} + \frac{4}{\alpha T} \frac{\theta}{T} + 1 + \frac{2\tau}{T} - \frac{2\beta}{\alpha T} \frac{\theta}{T} - \frac{2\tau}{T} = 0. \tag{60}$$

It is well known that $\|\lambda\| < 1$ in (59) if and only if $R(s) < 0$ in (60). It implies that $z_i(t) \to 0$ if and only if all roots in (60) have negative real parts.
and
\[
\left( \frac{2\beta}{\alpha T} - \frac{\theta}{T} - \frac{2\tau}{T} \right)^2 \left( \frac{\theta}{T} - \frac{2\beta}{\alpha T} - 1 + \frac{2\tau}{T} + \frac{4R(\mu_i)}{\alpha \theta T \|\mu_i\|^2} \right) < 16I^2(\mu_i) \alpha^2 - \frac{2\beta}{\alpha T} T \|\mu_i\|^2 > 0. \quad (62)
\]

By solving the above two inequalities and combining with the condition \(0 < \theta \leq T\), one has (48) and (49). Consequently, \(z_i(t) \rightarrow 0\) if and only if (48) and (49) are satisfied. According to Corollary 1, second-order consensus in system (1) with protocol (5) can be reached if and only if (48) and (49) hold.

For cases 2 and 4, let \(|\xi_i - M_{i0}(T)| = 0\), one has
\[
\xi^4 + \left( \frac{\alpha}{2} \mu_i (T - \tau)^2 + \mu_i (T - \tau) - 2 \right) \xi^3 \\
+ \left( 1 + \mu_i \theta + \frac{\alpha}{2} \mu_i \theta^2 + \alpha \mu_i \theta (T - \tau) - 2 \beta \mu_i (T - \tau) - \alpha \mu_i (T - \tau)^2 \right) \xi^2 \\
+ \left( \frac{\alpha}{2} \mu_i (T - \tau)^2 + \mu_i (T - \tau) - \alpha \mu_i \theta (T - \tau) \\
- \beta \mu_i \theta + \frac{\alpha}{2} \mu_i \theta^2 \right) \xi = 0. \quad (63)
\]

Note that \(M_{i0}(T)\) has a eigenvalue \(\xi = 0\). Let \(\xi = [(s + 1)/(s - 1)]\), \(a_0 = (\alpha/2) \mu_i (T - \tau)^2 + \mu_i (T - \tau) - \alpha \mu_i \theta (T - \tau) - \beta \mu_i \theta + (\alpha/2) \mu_i \theta^2, a_1 = 1 + \beta \mu_i \theta + (\alpha/2) \mu_i \theta^2 + \alpha \mu_i \theta (T - \tau) - \alpha \mu_i (T - \tau)^2, a_2 = (\alpha/2) \mu_i (T - \tau)^2 + \beta \mu_i (T - \tau) - 2\). One finally gets
\[
(1 + a_2 + a_1 + a_0)s^3 + (3 + a_2 - a_1 - 3a_0)s^2 \\
+ (3 - a_2 - a_1 + 3a_0)s + 1 - a_2 + a_1 - a_0 = 0. \quad (64)
\]

As we all know that \(\|\xi\| < 1\) in (63) if and only if \(R(s) < 0\) in (64). This implies that \(z_i(t) \rightarrow 0\) if and only if all roots in (64) have negative real parts.

Then, (64) can be cast to the following form:
\[
s^3 + (p_2 + q_j)s^2 + (p_1 + q_j)s + p_0 + q_{ij} = 0 \quad (65)
\]

where \(p_2 = (2\beta - 2\alpha \tau)/(\alpha T) + 3, q_2 = 0, p_1 = 4R(\mu_i)/(\alpha \theta T \|\mu_i\|^2), q_1 = 2(\alpha - \alpha \tau) - 4\beta(\alpha - \alpha \tau) + 2\alpha \theta^2 - 4\alpha \theta (T - \tau) - 4\alpha T \|\mu_i\|^2, p_0 = 4R(\mu_i)/(\alpha \theta T \|\mu_i\|^2) - 2(\alpha - \alpha \tau) - 4\beta(\alpha - \alpha \tau) - 2\alpha \theta (T - \tau) - 4\alpha T \|\mu_i\|^2, q_0 = 4I(\mu_i)/(\alpha \theta T \|\mu_i\|^2).
\]

By Lemma 3, (65) is stable if and only if
\[
T > \frac{2\tau}{3} - \frac{2\beta}{3\alpha}, p_2 p_1 - q_1^2 - p_2 p_0 > 0
\]

and
\[
p_2 \text{ det } \begin{pmatrix} 0 & -p_1 & -q_0 \\ p_2 & q_1 & -p_0 \\ 0 & -p_1 & -q_0 \\ 1 & 0 & -p_1 & -q_0 \\ 0 & p_2 & q_1 & -p_0 \end{pmatrix} - \text{ det } \begin{pmatrix} q_1 & -p_0 & 0 & 0 \\ p_2 & q_1 & -p_0 & 0 \\ 1 & 0 & -p_1 & -q_0 \\ 0 & p_2 & q_1 & -p_0 \end{pmatrix} > 0.
\]

**Remark 6:** According to analysis, we obtain that the conditions of reaching consensus mainly dependent on the relationship between the without communication width \(T - \theta\) and the time delay \(\tau\). Therefore, all cases can be divided into two categories, i.e., \(T - \theta \leq \tau\) and \(\tau < T - \theta\). Particularly, for \(0 = \tau < T - \theta\), the conditions (48) and (49) reduce to (11) and (12). Hence, Theorem 1 can be regarded as a special case of Theorem 4. When \(0 < \tau < T - \theta\), the inequality (49) gives the supremum of sampled period \(T\). Then, consensus can be reached if \(T \in (\tau + \theta, T)\). On the other hand, when \(T \leq \tau + \theta\), the infimum of sampled period \(T\) is also derived by solving (50)–(52). Hence, consensus can be achieved if \(T \in (\tau, \tau + \theta]\).

**Corollary 5:** Under Assumption 2, the second-order consensus in system (1) with protocol (5) can be reached if and only if:

1) \(0 < \tau < T - \theta\)
\[
\theta < \frac{2\beta}{\alpha} - 2\tau \quad (66)
\]
\[
T < \theta - \frac{2\beta}{\alpha} + 2\tau + \frac{4}{\alpha \theta \mu_N}. \quad (67)
\]

2) \(0 < \tau < \theta - \tau\)
\[
3 - a_2 - a_1 + 3a_0 > 0
\]
\[
1 - a_2 + a_1 - a_0 > 0
\]
\[
(3 - a_2 - a_1 - 3a_0)(3 - a_2 - a_1 + 3a_0) < (1 + a_2 + a_1 + a_0)(1 - a_2 - a_1 - a_0) > 0 \quad (68)
\]

where \(\mu_N\) is the maximum eigenvalue of Laplacian matrix \(L, a_0 = (\alpha/2) \mu_i (T - \tau)^2 + \mu_i (T - \tau) - \alpha \mu_i \theta (T - \tau) - \beta \mu_i \theta + (\alpha/2) \mu_i \theta^2, a_1 = 1 + \beta \mu_i \theta + (\alpha/2) \mu_i \theta^2 + \alpha \mu_i \theta (T - \tau) - \alpha \mu_i (T - \tau)^2, a_2 = (\alpha/2) \mu_i (T - \tau)^2 + \beta \mu_i (T - \tau) - 2\).

**Remark 7:** In [42], the pulse-modulated intermittent control was given to study the consensus of multiagent systems. When \(\tilde{a}(t) = 1\), the control strategy (5) in [42] is same as the protocol (4) proposed in this paper. By comparison with [42], the main contributions of this paper are summarized as follows.

1) The communication width and the coupling gains are carefully designed. The main results are presented in Theorems 2 and 3, respectively.

2) The time delay is considered in the communication process, and the distributed protocol (5) is proposed in this paper.

Through an elaborate analysis about the relationship among the time delay, the communication width, the sampling period, and the network structure, a necessary and sufficient condition is obtained for achieving consensus. It is also found that the protocol (4) can be regarded as a special case of protocol (5). Hence, the protocol (5) is more practical and general.

**Remark 8:** In fact, it has been widely recognized that most practical industrial systems are nonlinear by nature, and many control schemes, such as [43]–[45] have been proposed to ensure system stability. Therefore, it is more meaningful to study the consensus of multiagent systems with nonlinear dynamics by using the intermittent sampled data control proposed in this paper.
V. NUMERICAL EXAMPLE

In this section, we give two numerical examples to demonstrate the effectiveness of the theoretical results.

Example 1: Consider the multiagent system (1) consisting of four agents under the protocol (4) with directed network, where $a_{12} = a_{32} = a_{41} = a_{43} = 1$, $a_{13} = a_{21} = 0.5$, $a_{24} = 2$, and $a_{34} = 1.5$. By calculation, one has $\mu_1 = 0$, $\mu_2 = 2$, $\mu_3 = 3.2263 - 1.0629i$, $\mu_4 = 3.2263 + 1.0629i$. We choose
\( \theta = 0.4 \) and \( T = 1.5 \).

\( \theta = 0.4 \) and \( T = 1.58 \).

\( \alpha = 0.5 \) and \( \beta = 1 \). When \( \theta = 0.5 \), according to Theorem 1, one obtains that \( T < 0.817 \). Thus, the consensus of multiagent system (1) can be reached if and only if \( 0.5 \leq T < 0.817 \). The position and velocity states of all agents are presented in Figs. 1–6, where the consensus can be achieved if \( T = 0.6 \) and \( T = 0.81 \), while it cannot be reached if \( T = 0.82 \). When \( \theta = 0.4 \), the consensus can be reached if and only if \( 0.4 \leq T < 1.57 \). All agents states about the position and velocity are shown in Figs. 7–12. The consensus can be achieved if \( T = 1 \) and \( T = 1.5 \), while it cannot be reached if \( T = 1.58 \).

In Example 1, it can be seen that the sampling period plays an essential role to the consensus. We define the supremum of sampling period \( T \) as the threshold. The communication width is vital to the threshold. Compared Figs. 1–6 with Figs. 7–12, one can amazing gain that when \( \theta = 0.4 \), the threshold of
the sampling period is greater than the one when $\theta = 0.5$. Thus, it is desirable to investigate the relationship between the communication width and the sampling period.

In Fig. 13, the relationship between the communication width and sampling period is presented. It is found that the feasible region of sampling period is increasing when the communication width is relatively small, while it is decreasing when the communication width more than some critical value. Figs. 14 and 15 show the effect of coupling gains $\alpha$ and $\beta$ on the feasible regions of communication width and sampling period. In Fig. 14, with the increasing of coupling gain $\alpha$, the feasible regions of communication width and sampling
period are decreasing. Fig. 15 presents that the feasible region of communication width is decreasing with the increasing of coupling gain $\beta$, while the feasible region of sampling period is increasing for given feasible communication width.

Example 2: Consider the multiagent system (1) consisting of four agents under the protocol (5) with directed network, where the elements of weighted adjacency matrix and the coupling gains are same as the ones in example 1.
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**Fig. 32.** $\tau = 0.6$, $\theta = 0.4$, and $T = 2$.

**Fig. 33.** $\tau = 0.6$, $\theta = 0.4$, and $T = 2.02$.

**Fig. 34.** $\tau = 0.6$, $\theta = 0.4$, and $T = 2.02$.

**Fig. 35.** Feasible region with $\tau = 0.6$, $\alpha = 0.5$, and $\beta = 1$.

**Case 1:** $\tau < \min\{\theta, T - \theta\}$. Choose $\tau = 0.2$ and $\theta = 0.55$. By Theorem 4, system (1) under protocol (5) can achieve consensus if and only if $T < 0.853$. It can be shown from Figs. 16–21 that consensus can be reached if $T = 0.8$ and $T = 0.85$, while it cannot be achieved if $T = 0.86$. The feasible regions of $T$ and $\theta$ are shown in Fig. 22.

**Case 2:** $T - \theta \leq \tau < \theta$. Choose $\tau = 0.2$ and $\theta = 0.55$. Based on Theorem 4, the consensus can be achieved if and only if $T > 0.652$. It can be shown from Figs. 23–28 that consensus cannot be reached if $T = 0.645$ and $T = 0.65$, while it can be achieved if $T = 0.655$.

**Case 3:** $\theta < \tau < T - \theta$. Choose $\tau = 0.6$ and $\theta = 0.4$. According to Theorem 4, the consensus can be achieved if and only if $T < 2.01$. It can be seen from Figs. 29–34 that consensus can be reached if $T = 1.9$ and $T = 2$, while it cannot be achieved if $T = 2.02$. The feasible regions of $T$ and $\theta$ are shown in Fig. 35.

**Case 4:** $\tau > \max\{\theta, T - \theta\}$. Choose $\tau = 0.6$ and $\theta = 0.4$. From Theorem 4, system (1) under protocol (5) can achieve consensus if and only if $T > 0.92$. It can be shown from Figs. 36–41 that consensus cannot be reached if $T = 0.91$ and $T = 0.92$, while it can be achieved if $T = 0.93$.

In Example 2, we discuss the consensus of multiagent system (1) under delayed protocol (5). Four cases about the
time delay, communication width and sampling period are considered. Combining with cases 1 and 2, one obtains $T = 0.853$ and $\bar{T} = 0.652$ for given $\tau = 0.2$ and $\theta = 0.55$. For cases 2 and 4, similar results can also be achieved. It is amazing discovered that consensus cannot be reached for very small or very large communication width and sampling period.

VI. CONCLUSION

In this paper, the second-order consensus of multiagent systems via intermittent sampled data control is investigated. First, a without delayed protocol is proposed. A necessary and sufficient condition, which is related to the coupling gains, the sampling period, the communication width, and the structure of the networks, is obtained for achieving consensus. Moreover, the feasible region of communication width is provided and the coupling gains are carefully designed. Besides, the time delay is considered in the process of information transmission. Then, a time delayed protocol is proposed. Four cases about the relationship of the time delay, the sampling period, and the communication width are discussed, respectively. It is found that the consensus can be achieved if and only if the sampling period is bounded by two critical values for given time delay and communication width. In the future, the consensus of multiagent systems with nonlinear dynamics will be considered by using intermittent sampled data control.

REFERENCES


