Target localization based on structured total least squares with hybrid TDOA-AOA measurements

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\textbf{A B S T R A C T}

In this paper, we focus on the target localization problem which finds broad applications in radar, sonar and wireless sensor networks. A pseudolinear overdetermined system of equations is constructed from the nonlinear hybrid TDOA-AOA measurements about target location. Considering the matrix and vector in the constructed pseudolinear system are both contaminated by the measurement noise, a new weighted least squares (WLS) method which is based on the first order Taylor expansions of the noise terms is developed in this paper and it can reduce the estimation bias that arise from the least squares (LS) method. In particular we focus on constructing a localization algorithm to reduce the bias that easily arise from the traditional methods. Thus in addition, a novel structured total least squares (STLS) method is also developed in this paper to further reduce the estimation bias specially when the target is outside the convex hull formed by sensors. Numerical examples show the superiority of the proposed STLS method in estimation accuracy compared with the LS method, total least squares (TLS) method and the proposed WLS method.

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1. Introduction

The problem of target localization is a fundamental theme for many applications in radar, sonar, communications and wireless sensor networks [1–5], etc. It has received considerable attention over the past few years. There are many target localization methods which are based on different types of measurements. All these location schemes can be classified to several categories, such as time of arrival (TOA) [2,6], time difference of arrival (TDOA) [7–9], received signal strength (RSS) [10], angle of arrival (AOA) [11–15] and their combinations [4,16,17].

Every category has its own advantages and limitations. The TOA-based approaches require the localization systems to be accurate time synchronization and these approaches are only used for the localization of a cooperative target. Compared to the TOA-based approaches, the TDOA-based approaches are more general methods because TOA measurement can also be transformed into TDOA measurement, but they require at least four properly located nodes for three-dimensional (3-D) target localization [6,8]. It is known that the strength or equivalently the energy of signal attenuates as a function of distance from the source and this property can be used for target localization whose transmission power is known. However, RSS-based methods are susceptible to the multipath signals, which restrict the application scenario to an open terrain sensor field. AOA can be estimated by exploiting the phase difference measured at received sensors [18] and the AOA schemes [4,15] only require two nodes minimum for localization. But, on the other hand, the AOA-based approaches require every node has an array to estimate the direction of arrival and they are highly range dependent. The localization result is accurate when target is not far away from the nodes. In this paper, we focus on hybrid TDOA and AOA based system in 3-D localization scenario.

There are several advantages with the hybrid TDOA-AOA approach. First of all, it reduces the number of nodes which are required to achieve a desired level of localization performance. Second, it prevents the appearance of the so-called ghost targets which may arise from the individual use of TDOA measurements. The paper [17] presented a hybrid TDOA-AOA location scheme that gives better accuracy than using TDOA alone. Therefore, rather than using only TDOA or AOA measurements, the hybrid TDOA-AOA approach can provide a better performance from fewer sensors [16].
Generally speaking, the target localization problem is nontrivial because of the nonlinear relationships between the position of target and the measurements, which makes target localization problem a challenging estimation problem. The maximum likelihood (ML) estimator is introduced to estimate the location since it is asymptotically efficient. However, it is hard to find a closed-form solution, or a closed-form solution does not exist at all. This necessitates the use of iterative numerical search technique. The numerical method can converge to an optimal solution only if the ML function is convex. If not, it is difficult to guarantee its global convergence and calculating time.

To address the problem, a pseudolinear overdetermined system of equations \( Au = h \) was constructed in [4,11–13,16,17,19] from the nonlinear measurements about target location \( u \), where \( A \) and \( h \) are constructed from the measurements. A two-step LS location estimator was developed in [17] to solve the pseudolinear system of equations and it gives a good localization accuracy. The methods proposed in papers [4,19] were extensions of the weighted least squares (WLS) method. Indeed, most of the existing target localization methods with pseudolinear overdetermined system are based on the traditional LS method. Considering both the matrix \( A \) and vector \( h \) are contaminated by the measurement noise, the traditional LS method may not be appropriate.

The total least squares (TLS) method was introduced by Golub and Van Loan [20,21], which is a natural generalization of the traditional LS method when the data in \( A \) and \( h \) is perturbed. A TLS estimator was considered in [11–13] for target localization with improved accuracy over the LS solutions. They focus on 2-D target localization only with AOA measurement. The TLS method has its limitation which does not utilize the structure of the matrix \( A \). An approach called structured total least squares (STLS) was proposed in [22,23] to address this situation.

This paper focuses on the target localization based on STLS method with hybrid TDOA-AOA measurement. Considering both the matrix \( A \) and vector \( h \) are contaminated by the measurement noise, a new WLS method which is based on the first order Taylor expansions of the noise terms is developed in this paper. The proposed WLS method considers the effect of the noise terms appeared in both \( A \) and \( h \) and the performance of the proposed method is better than that of LS. In addition, we propose to use STLS method to solve the constructed pseudolinear over-determined system of equations and this method can further reduce the impact of noise terms. The Cramér–Rao bound is obtained for theoretical analysis. We use 3 different geometry scenarios to implement numerical simulations and the numerical results demonstrate that both the proposed methods can reduce the bias of the estimate when target is not in the convex hull formed by sensors and the performance of STLS method is generally the best of all mentioned methods.

The main contributions of this work are highlighted as follows:

- A weight least squares (WLS) method which is based on the first order Taylor expansions of the noise terms is developed and it reduces the estimation bias arising from the least squares (LS) method when the target is outside the convex hull formed by sensors.
- A novel localization method based on structured total least squares (STLS) is also developed in this paper to further reduce the estimation bias that easily arise from the traditional methods, while the RMSE of the STLS is comparable to the other methods specially when the target is outside the convex hull formed by sensors.
- The STLS method makes great use of the special sparse structure of the constructed measurement matrix, and thus improves the estimation accuracy.

- The performance improvement achieved by the WLS and STLS methods is demonstrated with respect to the LS method in 3 different geometry scenarios.

We start with a brief introduction to the construction of pseudolinear overdetermined system of equations with TDOA-AOA measurements in Section 2. In Section 3, we describe the traditional least square method and we propose a new WLS method to improve the performance. In Section 4, we explore the total least square solution and propose a new target localization method based on structured total least square. In Section 5, the maximum likelihood estimation and the derived Cramér–Rao bound are used for performance analysis with the proposed method. Then in Section 6, the performances of the mentioned several methods (LS, WLS, TLS, STLS, ML) are explored using simulated experiments in 3 different geometry scenarios. The defined bias and root mean square error (RMSE) are utilized for analyzing the performance of the target location estimation. Finally, conclusions are made in Section 7.

In this paper, we use capital italic bold letters to represent matrices, and lowercase italic bold letters to represent vectors. The symbols \( I \) and \( \Theta \) represent the identity matrix and zero matrix with appropriate dimension. For a given matrix \( A, A^T \) or \( A' \) denotes the transpose of \( A \). \( ||A||_F \) denotes the Frobenius norm of \( A \). For a given vector \( u \), \( ||u||_2 \) is the \( l_2 \) norm and note that for a vector \( u \) we have \( ||u||_2 = ||u||_F \). \( E \{ \cdot \} \) is the mathematical expectation.

2. Problem formulation

In this paper, we consider the TDOA-AOA based localization system which contains \( N \) sensors. The \( N \) sensors are at known positions \( s_k = [s_{k,x}, s_{k,y}, s_{k,z}] \in \mathbb{R}^3 \), \( k = 1, 2, \ldots, N \) and the unknown target position is represented by \( u = [u_x, u_y, u_z] \in \mathbb{R}^3 \). Besides the TDOA measurements, the sensor \( s_k \) is capable of observing the angles (the azimuth \( \theta_k \) and the elevation \( \phi_k \)) of the target, as shown in Fig. 1.

Without loss of generality, the first sensor is supposed to be the reference node. Therefore, \( N − 1 \) TDOA measurements \( r_k \) and \( N \) AOA pairs \( (\theta_1, \varphi_1) \) can be obtained by the system. Finally the measurement model in vector form is expressed as

\[
\hat{m} = m + n
\]

(1)

where \( n \in \mathbb{R}^{3N−1} \) is assumed to be a zero-mean Gaussian noise with covariance \( Q \), and \( m = [\theta_1, \varphi_1, \theta_2, \varphi_2, \ldots, \theta_N, \varphi_N, r_N^T] \in \mathbb{R}^{3N−1} \) are true values of TDOA and AOA pairs. \( r_k = (r_k − r_1)/c, k = 2, \ldots, N \) are TDOA from \( k \)th sensor to the first sensor, and \( r_k = ||u − s_k||_2, k = 1, 2, \ldots, N \) are the actual ranges of the target at \( N \) sensors, and \( c \) is the signal propagation speed.

We should note that the covariance \( Q \) is a Toeplitz matrix. The azimuth \( \theta_k \), the elevation \( \varphi_k \) and the TDOA mea-

Fig. 1. Localization geometry of TDOA-AOA measurements in 3D.
measurements, \( r_{k} \) in \( m \) can be supposed to be independent with each other. \( \theta_{k_{1}} \) and \( \theta_{k_{2}} \) \((k_{1} \neq k_{2})\) are also supposed to be independent with each other as well as \( \psi_{k_{1}} \) and \( \psi_{k_{2}} \) \((k_{1} \neq k_{2})\). However, because of the correlation between TDOA measurements, the covariance matrix \( Q \) is generally not diagonal and actually it is a Toeplitz matrix.

In this paper, the variance of all AOA measurement (the azimuth and the elevation) noise is supposed to be \( \sigma_{A} \). The variance of TDOA measurement noise is supposed to be \( \sigma_{T} \) and the covariance of different TDOA measurement noise will be obtained as \( \sigma_{T}/2 \) according to the correlation between TDOA measurements. Define \( Q = \{q_{ij}\} \) and \( Q \) can be modeled as

\[
q_{ij} = \begin{cases} 
\sigma_{A} & i = j \neq 3k - 1 \\
\sigma_{T} & i = j = 3k - 1 \\
\sigma_{T}/2 & i \neq j, i = 3k_{1} - 1, j = 3k_{2} - 1 \\
0 & \text{otherwise}
\end{cases}
\]

where \( k, k_{1}, \) and \( k_{2} \) satisfy \( k, k_{1}, k_{2} = 2, 3, \ldots, N \).

Our first aim is to estimate the unknown location of the target from the noisy measurements \( \hat{m} \). Considering the nonlinear relationship between measurements \( \hat{m} \) and the target location \( u \), it is hard to solve the problem directly. Hence a linear system of equations about \( u \) should be constructed from the measurements \( \hat{m} \). The nonlinear relationship between the azimuth \( \theta_{k} \) and target location \( u \) can be obtained from Fig. 1, given by

\[
\tan(\theta_{k}) = \frac{\sin(\theta_{k})}{\cos(\theta_{k})} = \frac{u_{y} - s_{k,y}}{u_{x} - s_{k,x}}
\]

(3)

\[
\sin(\theta_{k})(u_{x} - s_{k,x}) - \cos(\theta_{k})(u_{y} - s_{k,y}) = 0
\]

(4)

Fig. 1 also shows the nonlinear relationship between the elevation \( \psi_{k} \) and target location \( u \)

\[
\tan(\psi_{k}) = \frac{\sin(\psi_{k})}{\cos(\psi_{k})} = \frac{u_{y} - s_{k,y}}{\sqrt{(u_{x} - s_{k,x})^{2} + (u_{y} - s_{k,y})^{2}}}
\]

(5)

Because

\[
\sqrt{(u_{x} - s_{k,x})^{2} + (u_{y} - s_{k,y})^{2}} = \cos(\theta_{k})(u_{x} - s_{k,x}) + \sin(\theta_{k})(u_{y} - s_{k,y})
\]

(6)

Eq. (5) can be transformed into

\[
\cos(\theta_{k}) \sin(\psi_{k})(u_{x} - s_{k,x}) + \sin(\theta_{k}) \sin(\psi_{k})(u_{y} - s_{k,y}) - \cos(\psi_{k})(u_{x} - s_{k,x}) = 0
\]

(7)

Let \( G_{k} \in \mathbb{R}^{2 \times 3} \) be

\[
G_{k} = \begin{pmatrix} 
\sin(\theta_{k}) & \cos(\theta_{k}) & 0 \\
\cos(\theta_{k}) \sin(\psi_{k}) & \sin(\psi_{k}) & -\cos(\psi_{k})
\end{pmatrix}
\]

We note that the rows of \( G_{k} \) are an orthonormal basis \([24,25]\).

Based on the Eqs. (4) and (7), a linear system of equations about target location can be obtained

\[
G_{k}u = C_{k} s_{k}
\]

(8)

The localization geometry also shows that

\[
u - s_{k} = r_{k} b_{k}
\]

(9)

where \( b_{k} = \cos(\theta_{k}) \cos(\psi_{k}), \sin(\theta_{k}) \cos(\psi_{k}), \sin(\psi_{k}) |^{T} \) is the unit vector of the actual target location with respect to the \( k \)th sensor. Because \( b_{k} \) and \( b_{1} \) are unit norm vector and we can see that \( b_{k} - b_{1} \) and \( b_{k} + b_{1} \) are orthogonal. We then have

\[
r_{k}(b_{k} - b_{1})^{T}(b_{k} + b_{1}) = 0
\]

(10)

Based on Eq. (9), we have

\[
2u = (s_{1} + r_{1} b_{1} + s_{k} + r_{k} b_{k}) = (s_{1} + (r_{k} - c \tau_{b}) b_{1} + s_{k} + r_{k} b_{k})
\]

\[
= (s_{1} + s_{k} - c \tau_{b} b_{1}) + r_{k}(b_{k} + b_{1})
\]

(11)

Combining Eqs. (11) and (10), we have another linear system of equations

\[
2(b_{k} - b_{1})^{T}u = (b_{k} - b_{1})^{T}(s_{1} + s_{k} - c \tau_{b} b_{1})
\]

(12)

Eqs. (8) and (12) can be written in matrix form and finally we get the linear equations

\[
A u = h
\]

(13)

where \( A \in \mathbb{R}^{(3N-1) 	imes 3} \)

\[
A = \begin{pmatrix} 
G_{1} \\
2(b_{2} - b_{1})^{T} \\
\vdots \\
G_{N}
\end{pmatrix}
\]

and \( h \in \mathbb{R}^{3N-1} \)

\[
h = \begin{pmatrix} 
G_{1}s_{1} \\
(b_{2} - b_{1})^{T}(s_{1} + s_{2} - c \tau_{b} b_{1}) \\
G_{2}s_{2} \\
\vdots \\
G_{N}s_{N}
\end{pmatrix}
\]

We need to note that Eq. (13) only holds for the true values of TDOA and AOA pairs \( m \). Given the noisy observations \( \hat{m} \) from the model (1), both the matrix \( A \) and the vector \( h \) are contaminated by noise, and consequently the model in equation can be transformed into

\[
(\hat{A} + E)u = \hat{h} + r
\]

(14)

where \( \hat{A} + E = A, \hat{h} + r = h \). In this framework \( \hat{A} \) and \( \hat{h} \) are known, and the target localization problem can then be stated as finding \( u \) from the TDOA-AOA model (14).

3. The classical least squares solution to TDOA-AOA model

3.1. Least squares method

Let us assume that the measurement noise in TDOA-AOA model is very small, i.e., \( \hat{m} - m \approx 0 \), so that \( G_{k}(\hat{m}) \approx G_{k}(m) \) and \( b_{k}(\hat{m}) \approx b_{k}(m) \) and we can write (14) as the following least squares (LS) model

\[
\hat{A}u = \hat{h} + \eta
\]

(15)

where \( \hat{A} = A(\hat{m}), \hat{h} = h(\hat{m}) \) and \( \eta \) is a residual error. In this simplest form, a LS solution to \( \hat{A}u \approx \hat{h} \) is given by

\[
u_{LS} = \arg \min_{u \in \mathbb{R}^{3}} \|\hat{A}u - \hat{h}\|^{2}_{2}
\]

(16)

which is also called the pseudolinear estimator (PLE). The bias of the LS estimator \( \hat{u}_{LS} \) is defined by

\[
E(\hat{u}_{LS}) - u = -E[(\hat{A}^{T}\hat{A})^{-1}\hat{A}^{T}\eta]
\]

(17)

We can see that the LS estimator is approximated as an unbiased estimator if the noise term \( \eta \) satisfies \( E[\eta] = 0 \) when the measurements noise is negligible.
3.2. Weighted least squares method

The LS estimator discussed above ignores the perturbed error in matrix $A$. In fact, because of the presence of errors in TDOA-AOA measurements, the system matrix $A$ is also corrupted by noise. To improve the accuracy, the perturbed error need to be considered especially in the case of large measurement errors. Next we will use first order Taylor expansion to approximate the noise $E$ and $r$ with a linear expression. Considering the $h$ in (13), we have

$$G_k(m)s_k \approx G_k(\hat{m})s_k + \frac{\partial G_k(\hat{m})}{\partial \theta} \frac{\partial \hat{h}_k}{\partial \theta} + \frac{\partial G_k(\hat{m})}{\partial \phi} \frac{\partial \hat{h}_k}{\partial \phi}$$

and

$$G_k(\hat{m})s_k + \frac{\partial G_k(\hat{m})}{\partial \theta} \frac{\partial \hat{h}_k}{\partial \theta} + \frac{\partial G_k(\hat{m})}{\partial \phi} \frac{\partial \hat{h}_k}{\partial \phi} = G_k(\hat{m})s_k + \frac{\partial G_k(\hat{m})}{\partial m} (m - \hat{m}) \quad (18)$$

Take $h_k(m) = (b_k - b_1)^T (s_k + c_k b_1)$ and we get the Taylor expansion

$$h_k(m) \approx h_k(\hat{m}) + \frac{\partial h_k(\hat{m})}{\partial m} (m - \hat{m}) \quad (19)$$

Based on Eqs. (18) and (19), the Taylor expansion of $h$ is

$$h(m) \approx h(\hat{m}) + \frac{\partial h(\hat{m})}{\partial m} n \quad (20)$$

where

$$\frac{\partial h(\hat{m})}{\partial m} \in \mathbb{R}^{(3N-1) \times 3}$$

and note that $\frac{\partial h(\hat{m})}{\partial m}$ is a sparse matrix. Furthermore, we consider $A$ in (13). The columns of $G_k(m)$ in $A$ can be Taylor expanded as

$$G_k'(m) \approx G_k'(\hat{m}) + \frac{\partial G_k'(\hat{m})}{\partial \theta} \frac{\partial \hat{h}_k}{\partial \theta} + \frac{\partial G_k'(\hat{m})}{\partial \phi} \frac{\partial \hat{h}_k}{\partial \phi}$$

$$\approx G_k'(\hat{m}) + \Delta G_k'(\hat{m}) \quad (21)$$

where

$$\frac{\partial G_k'(\hat{m})}{\partial \theta} = \begin{bmatrix} \cos(\hat{\theta}_k) \\ \sin(\hat{\theta}_k) \\ 0 \end{bmatrix}$$

$$\frac{\partial G_k'(\hat{m})}{\partial \phi} = \begin{bmatrix} \sin(\hat{\phi}_k) \\ \cos(\hat{\phi}_k) \\ 0 \end{bmatrix}$$

and

$$\frac{\partial G_k'(\hat{m})}{\partial \theta} = \begin{bmatrix} -\sin(\hat{\phi}_k) \sin(\hat{\theta}_k) \\ \cos(\hat{\phi}_k) \cos(\hat{\theta}_k) \\ \cos(\hat{\phi}_k) \sin(\hat{\theta}_k) \end{bmatrix}$$

Take $k_A(m) = 2(b_k - b_1)$ and we get

$$A_k(m) \approx A_k(\hat{m}) + \frac{\partial A_k(\hat{m})}{\partial \theta} (\theta_0 - \hat{\theta}_1) + \frac{\partial A_k(\hat{m})}{\partial \phi} (\phi_0 - \hat{\phi}_1) \quad (22)$$

Combining (22) and (21) we have

$$A(m) = A(\hat{m}) + \Delta A(m) \quad (23)$$

where

$$\Delta A(m) = [\Delta A_1(m), \ldots, \Delta A_N(m), \Delta G_N'(\hat{m})]' \in \mathbb{R}^{(3N-1) \times 3}.$$ Substituting (20) and (23) in (13) and rearranging yields

$$A(\hat{m}) u = h(\hat{m}) - \frac{\partial h(\hat{m})}{\partial m} n - \Delta A(\hat{m}) u \quad (24)$$

Based on the relationship in (9), expression (24) can be approximated as the following WLS model

$$A(\hat{m}) u = h(\hat{m}) + P n \quad (25)$$

where $n$ is the zero-mean Gaussian noise with covariance $Q$ in (1) and $P$ is expressed as in (26), where

$$P = \begin{pmatrix}
P_{11} & 0 & 0 & 0 \\
-r_1 b_1^T P_{01} & P_{22} & 0 & 0 \\
0 & P_{22} & 0 & 0 \\
-r_1 b_1^T P_{02} & 0 & P_{22} & 0 \\
0 & 0 & 0 & P_{22} \\
-\cdots & -r_1 b_1^T P_{0N} & \cdots & \cdots \\
0 & 0 & \cdots & P_{22} \\
0 & 0 & \cdots & 0
\end{pmatrix}$$

$$P_{jk} = -r_k \left( \begin{array}{c}
\cos(\hat{\phi}_k) \\
\sin(\hat{\phi}_k) \\
1
\end{array} \right)$$

$$P_{kk} = \left( \begin{array}{cc}
-\cos(\hat{\phi}_k) \sin(\hat{\theta}_k) & -\sin(\hat{\phi}_k) \cos(\hat{\theta}_k) \\
\sin(\hat{\phi}_k) \cos(\hat{\theta}_k) & -\sin(\hat{\phi}_k) \sin(\hat{\theta}_k) \\
0 & \cos(\hat{\phi}_k)
\end{array} \right)$$

Note that the WLS model in (25) is similar to the LS model in (15) except for their noise term. It follows Eq. (1) that $Pn$ in (25) is zero-mean Gaussian with the covariance matrix $Q_P = PQP^T$. Finally, a weighted least square (WLS) estimate to $u$ is obtained as

$$\hat{u}_{WLS} = \arg \min_{u \in \mathbb{R}^T} \| W - h(\hat{m}) - \hat{A} \hat{u} \|_2^2$$

$$\hat{u}_{WLS} = (\hat{A}^T \hat{W} \hat{A})^{-1} \hat{A}^T \hat{W} h(\hat{m})$$

where $W = Q_p^{-1}$ is the weighted matrix. If $W = I$, then the WLS estimate could degenerate into LS estimate in (16). In fact, the matrix $Q_p$ in (27) is not known in practice because it depends on the range $r_k$ (we only have the prior knowledge of TDOA and AOA measurements). Therefore, we need the actual source position. Here, the LS estimate can be used to provide us an initial range estimate. A possible implementation of the WLS method can be summarized as the steps in the following algorithm.

4. The structured total least squares solution to TDOA-AOA model

4.1. Total least squares method

The LS method in Section 3.1 is unbiased when the measurement noise is negligible. With the noise increasing, the measurement noise $m$ appears in both $A$ and $h$. The estimation bias of LS $-E[(\hat{A}^T \hat{A})^{-1} \hat{A}^T \eta]$ is generally non-zero ($A$ and $h$ are not statistically independent) even if $E[\eta] = 0$ [12]. Although the derivation
of WLS considers the first order term noise, the second order term of noise does exist in both $A$ and $h$. Thus the estimation bias of WLS $E(\hat{u}_{WLS}) - u$ is still non-zero, but it is less than the bias of LS method.

The total least square (TLS) method was introduced by Golub and Van Loan [20,21], which is a natural generalization of the traditional LS method. As we all know, the LS estimate (the TLS model is similar to LS model expect for their noise term) is obtained as a solution of the optimization problem expressed as

$$\arg\min_{h} \|\eta\|_F \quad \text{s.t.} \quad \hat{A}u = \hat{h} + \eta$$

(28)

From (28) we can see that the basic principle of LS is to correct the right-hand side $\eta$ in Frobenius norm. Nevertheless, the existence of error in the elements of $A$ is not corrected in LS or not corrected completely in WLS. The TLS method seeks the minimal corrections of $E$ and $r$ in Frobenius norm. Eventually, the TLS estimate $\hat{u}_{TLS}$ is given by the following constrained optimization problem

$$\arg\min_{u_{TLS}, r_{TLS}, E_{TLS}} \|E - r\|_F \quad \text{s.t.} \quad (\hat{A} + E)u = \hat{h} + r$$

(29)

The TLS method shows that it involves finding a perturbation matrix $E$ more than just estimating the right-hand side error. Singular value decomposition (SVD) is used for the augmented matrix $[\hat{A}, -\hat{h}]$ to implement TLS method.

Let $C = [\hat{A}, -\hat{h}]$ and let the SVD of $C$ be

$$C = U\Sigma V^T$$

where $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_4)$ and $\sigma_1 \geq \ldots \geq \sigma_4$ are the singular values of $C$. Define the partitioning

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

The theorem in [23] proves that a TLS solution exists if and only if $V_{22}$ is non-singular. In that case the TLS estimate is given by

$$\hat{u}_{TLS} = -V_{12}V_{22}^{-1}$$

Finally, we get the TLS algorithm for target localization.

4.2. Structured total least squares method

In this application of localization with TDOA-AOA model, the matrix $A$ has a special structure, i.e., a sparse structure with zero elements (there are at least $N$ zero elements in $A$). Because $A$ has zero elements and errors occur only in some non-zero elements of $A$, the structure of error matrix $E$ is similar to that of $A$. For the application where the matrix $A$ is sparse, the SVD-based TLS method may not be appropriate. As a matter of fact, using the SVD-based TLS method all the elements of matrix $E$ will often be non-zeros even for the zero elements in $A$. It seems sometimes irrational and obviously deteriorates the performance of localization.

An approach called structured total least squares (STLS) method is introduced to address this situation. The STLS will assure that the error matrix $E$ will have the same structure as $A$. Specifically, those elements in $E$ which represent possible errors in $A$ are approved to be non-zero. The formulation for solving the STLS problem makes great use of the special structure of the matrix $A$. Next we will give the details.

Suppose there are $q (q < 3(3N - 1))$ non-zero elements in $A$. A vector $\alpha$ is used to represent the corresponding non-zero elements of the error matrix $E$. That is to say, given matrix $E$, $\alpha$ is known, and vice versa. The vector $\alpha$ and the matrix $E$ are equivalent. Based on the definition of $\alpha$, the TLS problem (29) can be transformed into STLS problem.

$$\arg\min_{u_{STLS}, r_{STLS}, E_{STLS}} \|r^T \alpha^T\|_F \quad \text{s.t.} \quad (\hat{A} + E)u = \hat{h} + r$$

(31)

The mentioned STLS problem can be solved by an iterative algorithm [22]. First, the term $Eu$ should be represented in terms of $\alpha$. Define a matrix $U \in \mathbb{R}^{(3N-1) \times q}$ such that

$$U\alpha = Eu$$

(32)

where the matrix $U$ consists of all the elements of $u$. Or rather, it is a suitable repetition of the vector $u$. For example, if

$$E = \begin{pmatrix} e_{11} & e_{12} & 0 \\ e_{21} & e_{22} & e_{22} \end{pmatrix}$$

we have

$$U = \begin{pmatrix} u_1 & u_2 & 0 & 0 & 0 \\ 0 & u_3 & u_4 & u_5 \end{pmatrix}$$

with $\alpha = [e_{11}, e_{12}, e_{21}, e_{22}, e_{22}]^T$.

The key focus of this STLS method is to construct an iterative formula. A linear approximation is adopted to construct the iterative formula. Take $\Delta u$ and $\Delta E$ represent a small change in $u$ and $E$ respectively. we have

$$U(\Delta u) = (\Delta E)u$$

(33)

where $\Delta \alpha$ is the corresponding small change of $\alpha$ and we get $r + \Delta r = (\hat{A} + E)u + \Delta Eu - \hat{h}$. Neglecting the second-order terms we have

$$r + \Delta r \approx (\hat{A} + E)u + U\Delta \alpha + (\hat{A} + E)\Delta u - \hat{h} = r + U\Delta \alpha + (\hat{A} + E)\Delta u$$

Now, based on (34), the STLS optimization problem (31) can be linearly approximated as

$$\arg\min_{\Delta \alpha, \Delta u} \left\| \begin{pmatrix} U & \hat{A} + E \end{pmatrix} \begin{pmatrix} \Delta \alpha \\ \Delta u \end{pmatrix} + \begin{pmatrix} r \\ \alpha \end{pmatrix} \right\|_2$$

(35)

Note that $\Delta \alpha$ and $\Delta u$ can be estimated from $E, U, \alpha$ and $r$, while $E, U, \alpha$ and $r$ can be recomputed by the updated $u$ and $\alpha$ with $\Delta \alpha$ and $\Delta u$. This suggests an iterative algorithm, which iterates between (32) and (35), until a convergence criterion has been satisfied. Combining these with above we have the following STLS iterative algorithm for localization.

5. Performance analysis

5.1. Maximum likelihood estimation

We need to find a minimum variance unbiased (MVU) estimate for reference. However, the MVU estimate is always hard to find or does not exist any more, and thus the maximum likelihood (ML) estimate which is asymptotically efficient is used as an alternative. The likelihood function of the measurement model (1) can be written as

$$f(\hat{m}|u) = \frac{1}{(2\pi)^{\frac{3}{2}}|Q|^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2} (\hat{m} - m(u))^T Q^{-1} (\hat{m} - m(u)) \right\}$$

(36)

where $m(u)$ is the vector of length $3N - 1$ as a function of $u = [u_1, u_2, u_3]^T$. The ML method needs to maximize the likelihood function over $u$. Taking logarithm of the likelihood function and neglecting the constant term, we get the ML objective function $J_{ML}(u)$ expressed as

$$J_{ML}(u) = -\frac{1}{2} (\hat{m} - m(u))^T Q^{-1} (\hat{m} - m(u))$$

Finally we get the following estimate

$$\hat{u}_{ML} = \arg\max_{u \in \mathbb{R}^3} J_{ML}(u)$$

(37)
The above optimization problem is obviously a nonlinear problem and a closed-form solution does not exist. Hence we have to solve the problem by numerical techniques.

In general, there are two main different numerical methods to solve the problem. One is the Newton–Raphson method and the other is the grid-based search method. The Newton–Raphson algorithm is an iterative method and it needs an initial value which is close to true value. Therefore, it is difficult to guarantee its convergence and calculating time. Although the grid-based search algorithm is time consuming, its robustness and convergence make the grid-based search algorithm suitable to be a reference algorithm.

The norm of estimation bias (Bias) and root mean square error (RMSE) are used to evaluate the performance of these methods. Define

\[
\text{Bias} = \| E[\hat{\mathbf{u}}] - \mathbf{u} \|_2 \\
\text{RMSE} = \sqrt{E[\| \hat{\mathbf{u}} - \mathbf{u} \|_2^2]} \tag{38}
\]

where $\hat{\mathbf{u}}$ denotes a position estimate of a given method.

5.2. Cramér–Rao bound

The Cramér–Rao low bound (CRLB) establish a lower bound on the error covariance matrix for any unbiased estimator. We would like to derive the CRLB of the location vector $\mathbf{u}$. Assume that the TDOA-AOA measurement vector $\hat{\mathbf{m}}$ satisfy the measurement model in (1) and considering that $\hat{\mathbf{u}}$ is an unbiased estimate of $\mathbf{u}$. Defining the CRLB matrix of an estimate of the location vector is $\mathbf{C}_0$. The CRLB matrix can be expressed as

\[
\mathbf{C}_0 = (\mathbf{J}_m \mathbf{Q}^{-1} \mathbf{J}_m^T)^{-1} \tag{39}
\]

where $\mathbf{J}_m$ can be concretely expressed as

\[
\mathbf{J}_m = \frac{\partial \mathbf{m}^T(\mathbf{u})}{\partial \mathbf{u}} = [\partial_1 \mathbf{J}_{21}; \partial_2 \mathbf{J}_{22}; \ldots; \partial_N \mathbf{J}_{2N}]
\]

Fig. 2. This figure illustrates the distribution output of the STLS method and LS method respectively in the scenario case 1. The three-dimensional localization results are given by x-o-z plane and x-o-y plane. The 2 sensors represented by • are located at x-axis and the red circle ○ is the true location of target. ○ represents STLS estimate and * represents LS estimate.

Fig. 3. Scenario case 1: Bias of TDOA-AOA target localization estimators with $\sigma_i$ fixed at 10 m and $\sigma_A$ varying from 0.5° to 3°.

where $\mathbf{f}_{ak} \in \mathbb{R}^{3 \times 2}, k = 1, 2, \ldots, N$ expressed as

\[
\mathbf{J}_{ak} = \begin{pmatrix}
- (u_y - s_{ky}) & -(u_x - s_{kx})(u_z - s_{kz}) \\
\frac{l_k^2}{k} & \frac{r_k^2}{k}^2 \\
\frac{l_k^2}{k} & \frac{r_k^2}{k}^2 \\
0 & l_k^2/r_k^2 \\
\end{pmatrix}
\]

and $\mathbf{J}_{bk} \in \mathbb{R}^{3 \times 4}, k = 1, 2, \ldots, N$ expressed as

\[
\mathbf{J}_{bk} = \begin{pmatrix}
\frac{u - s_k}{r_k} & \frac{u - s_1}{r_1} \\
\end{pmatrix}
\]
\( l_k = \sqrt{(x_k - s_{k,x})^2 + (y_k - s_{k,y})^2} \)

and

\( r_k = \| \mathbf{u} - \mathbf{s}_k \|_2 \).

From Eq. (39), we can see that the CRLB of target localization is relevant to the TDOA-AOA measurement noise and the position of every sensor. Thus for a given scenario of localization, there is a unique CRLB which is a function of the covariance matrix \( Q \) of TDOA-AOA measurement noise.

The trace of the CRLB matrix \( C_\theta \) is the lower bound of the mean-square error for the unbiased estimate \( \hat{\theta} \). Although most of the existing localization algorithms are typically biased estimate, the CRLB matrix is still an important benchmark for performance in terms of mean-square error. The detailed derivations of the CRLB matrix \( C_\theta \) is given in Appendix.

6. Numerical simulation

In this section, we provide 3 different geometrical scenarios to demonstrate the performance of the proposed STLS method and the proposed WLS method in comparison with other mentioned methods in terms of Bias and RMSE defined in (38). The results of the grid-based ML estimate are also included to obtain favorable performance of the ML estimate for reference purpose. The performance of target localization will be investigated using hybrid TDOA-AOA measurements contaminated by noise of different levels.

It is widely known that the accuracy of localization depends greatly on the target and sensor geometry. Therefore, different scenarios of geometry are needed to test and verify the proposed methods. We first consider a localization scenario with only 2 sensors which are the minimum number in hybrid TDOA-AOA localization. Secondly, we assume a wireless sensor networks with 4 sensors forming a convex hull, while the target is outside the convex hull. This geometry is bad for target localization which can test the performance of all the methods. Finally, the target is placed inside the same convex hull formed by 4 sensors.

6.1. Scenario case 1 with only 2 sensors

Let us consider the scenario that the localization system has only two sensors. The target is located at \( \mathbf{u} = [100, 100, 100]^T \) m and the locations of the two sensors are \( \mathbf{s}_1 = [-50, 0, 0]^T \) m and \( \mathbf{s}_2 = [50, 0, 0]^T \) m respectively. Apparently the measurement \( \hat{\mathbf{m}} \) have 5 elements and \( \mathbf{Q} \) is modeled as \( \mathbf{Q} = \text{diag}(\sigma_k, \sigma_k, \sigma_k, \sigma_k, \sigma_k) \).

We directly display the localization results of LS method and STLS method to illustrate the distribution output of the two compared algorithms. Fig. 2 gives 100 localization results of the two methods. The red circle in figure is the true location of target. \( \mathbf{x} \) and \( \hat{x} \) in figure are STLS estimate and LS estimate respectively. In this part, \( \sigma_k = 10 \circ C \) \( \mathbf{s} \) and \( \sigma_k = 2 \circ C \). We take \( \epsilon = 10^{-14} \) in STLS. We can see that both the two methods have several extreme cases because of the high level noise. However, the results of STLS gather around the true location more closely, while the center of the points obtained by LS is clearly away from the true value. Thus LS estimate has a severe bias and the localization performance of STLS is generally better than that of LS.

In this scenario, we test the localization accuracy of all the methods. We also include the grid-based ML method and we take the grid from \(-150 \text{ m} \) to \(150 \text{ m} \) with step size \( 2 \text{ m} \) to perform the ML method. For every testing point we run over 5000 iterations of Monte Carlo simulations. The other settings are the same as above.

Figs. 3 and 4 show the performance of the proposed methods as \( \sigma_k \) varies from 0.5° to 3°, where \( \sigma_{\theta_c} \) is fixed at 10 m. Fig. 3 gives the Bias results. We can see that the estimation Bias increase as the noise level of AOA measurements increase. Both the WLS method and the STLS method perform better than the other method. The proposed STLS method has the best performance especially when the noise level is large. Bias of ML method is large with a small number of the measurements although the ML estimator is asymptotically efficient. The RMSE is shown in Fig. 4. All these methods could not reach the CRLB. The performance of the methods is very close, except TLS has a large level of RMSE. The RMSE of ML estimator is much closer to CRLB when the noise level is large.

Figs. 5 and 6 show the performance of the proposed methods as \( \sigma_{\theta_c} \) varies from 5 m to 30 m, where \( \sigma_k \) is fixed at 1°. Fig. 5 shows that there is a notable difference in Bias of all the methods. The proposed STLS method has the best Bias. In contrast to the AOA measurements in localization, the change of TDOA measurements has little effect on the performance of localization. Fig. 6 shows that the RMSE of these methods is very close, except TLS has a large level of RMSE. Although all these methods could not reach
the CRLB, the grid-based ML method has the best RMSE which is closed to the CRLB.

There are three primary reasons why all these methods even ML method could hardly reach the CRLB. The first reason is that the TDOA-AOA measurement noise is too large that the gap between RMSE and CRLB becomes widening. The second reason is that this localization scenario is bad for target localization, which may deteriorate the RMSE. The third reason is that the localization scenario contains only 2 sensors and a lack of TDOA-AOA measurements increases the RMSE.

6.2. Scenario case 2 with target outside the convex hull formed by 4 sensors

In this scenario, we construct a simple convex hull with 4 sensors and the target is outside the convex hull. The target is located at \( u = [100, 100, 100]' \) m and the locations of the 4 sensors are deployed at \( s_1 = [-50, 0, 0]' \) m, \( s_2 = [50, 0, 0]' \) m, \( s_3 = [0, 50, 0]' \) m and \( s_4 = [0, 0, 50]' \) m. There are 11 elements in the measurement vector \( \hat{m} \) and \( Q \) is actually a Toeplitz matrix because of correlation between different TDOA measurements. The other settings are the same as above and we test the localization accuracy of all the methods in scenario case 2.

Figs. 7 and 8 show the performance of the proposed methods as \( \sigma_A \) varies from 0.5° to 3°, where \( \sigma_{1c} \) is fixed at 10 m. Fig. 7 gives the Bias results. We can see that the estimation Bias increase as the noise level of AOA measurements increase. Again, it is clear that the proposed STLS method has the best performance regardless of the noise level. The TLS and ML achieve about the same Bias, which can have a large difference with the proposed STLS. Although the proposed WLS has a better performance than LS in terms of Bias, they both have the worst Bias in this case. The RMSE is shown in Fig. 8. Again, all these methods could not reach the CRLB. The performance of the methods is very close, except TLS has a large level of RMSE. However, the proposed STLS and WLS have the best RMSE which is much closer to the CRLB.

Figs. 9 and 10 show the performance of the proposed methods as \( \sigma_{1c} \) varies from 5 m to 30 m, where \( \sigma_A \) is fixed at 1°. Fig. 9 shows that besides the same performance in TLS and ML, there is a notable difference in terms of Bias of the other methods. The proposed STLS method has the best Bias. In contrast to the AOA measurements in localization, the change of TDOA measurements has little effect on the performance of localization.
Fig. 10 shows that although all these methods could not reach the CRLB, the proposed WLS and STLS methods have the best RMSE which is closed to the CRLB.

In scenario case 2, the simulations show that the proposed WLS is better than LS and they also demonstrate the superiority of the proposed STLS in terms of Bias and RMSE compared with the other mentioned methods.

6.3. Scenario case 3 with target inside the convex hull formed by 4 sensors

In this scenario, the target is inside the convex hull formed by 4 sensors. The target is located at $u = [20, 20, 20]^T$ m and the locations of 4 sensors are deployed at the same locations in the scenario case 2. The grid-based ML method is also included and we take the grid from $-100$ m to $200$ m with step size $2$ m to perform the ML method. The other settings are the same as above and we test the localization accuracy of all the methods in scenario case 3.

Figs. 11 and 12 show the performance of the proposed methods as $\sigma_A$ varies from $0.5^\circ$ to $3^\circ$, where $\sigma_T$ is fixed at $10$ m.

Fig. 11 gives the Bias results. We can see that the estimation Bias of all the methods seem to remain steady although the noise level of AOA measurements increase, which is different from the mentioned 2 scenarios. It is clear that the grid-based ML method has the best performance, while the other method have a large Bias difference with the ML method. The proposed STLS and LS have almost the same performance. The RMSE is shown in Fig. 12. Only the grid-based ML method could reach the CRLB, while the other methods still could not reach the CRLB. In addition, The proposed STLS and LS have almost the same performance and the proposed WLS has the worst RMSE.

Figs. 13 and 14 show the performance of the proposed methods as $\sigma_T$ varies from $5$ m to $30$ m, where $\sigma_A$ is fixed at $1^\circ$. Fig. 13 shows that the Bias of the methods remain steady, except WLS has an abnormal Bias that the Bias decrease as the noise level of TDOA measurements increase. It is clear that the grid-based ML method still has the best performance and the proposed STLS and LS have almost the same performance. The RMSE is shown in Fig. 14. Again, only the grid-based ML method could reach the CRLB, while the other methods still could not reach the CRLB. In addition, The proposed STLS and LS have almost the same perfor-
Algorithm 1 WLS algorithm for localization.

Input: A group of measurements \( \hat{m} \) and an estimated covariance matrix \( Q \) of the Gaussian noise

Output: An estimate \( \hat{u}_{WLS} \) of the target location

1: Compute \( \hat{A} \) and \( \hat{h} \) from \( \hat{m} \)
2: Compute an initial position estimate \( \hat{u}_S \) from LS estimator
3: Compute the covariance matrix \( Q_s \) and its condition number
4: if condition number is more than a given threshold level then
5: \( \hat{u}_{WLS} = \hat{u}_S \)
6: else
7: Compute \( \hat{u}_{WLS} \) from WLS estimator
8: end if
9: return result

Algorithm 2 TLS algorithm for localization.

Input: A group of measurements \( \hat{m} \)

Output: An estimate \( \hat{u}_{TLS} \) of the target location

1: Compute \( \hat{A} \) and \( \hat{h} \) from \( \hat{m} \)
2: Compute the SVD \( [\hat{A}, -\hat{h}] = U \Sigma V^T \) and the condition number of \( V_2 \)
3: if condition number is more than a given threshold level then
4: Compute \( \hat{u}_{TLS} \) from LS estimator
5: else
6: Compute \( \hat{u}_{TLS} \) from Eq. (30)
7: end if
8: return result

Table 1

<table>
<thead>
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<th>( \sigma_C (m) )</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
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<tbody>
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<td>1389</td>
<td>959</td>
<td>710</td>
<td>577</td>
<td>462</td>
</tr>
</tbody>
</table>

In scenario case 3, the simulations show that the grid-based ML method has the best performance in terms of Bias and RMSE. The proposed STLS and LS have almost the same performance in all simulations. However, the proposed WLS has an abnormal and worst performance. The reason is that the condition number of the covariance matrix \( Q_s \) in WLS. Table 1 gives the average condition number of WLS in scenario case 3. We can see that the condition number decrease as the noise level of TDOA measurements increase, which means that the WLS method is more sensitive to noise in lower TDOA noise of scenario 3.

7. Conclusion

In this paper, we explored the least square methods for target localization with TDOA-AOA measurements. We start with a framework to the signal model of the target localization. A pseudolinear overdetermined system of equations is constructed from the TDOA-AOA measurements. A novel weighted least square method which is based on the first order Taylor expansions of the noise terms is proposed in this paper. The proposed WLS method considers the effect of the noise terms appeared in both \( A \) and \( h \) and the performance is better than that of LS when the target is outside the convex hull formed by sensors. In addition, we proposed to use STLS method to solve the pseudolinear overdetermined system of equations and this method can further reduce the impact of noise terms. We use 3 different geometry scenarios to implement numerical simulations and the numerical results demonstrate that the proposed STLS can reduce the bias of the estimate specially when target is outside the convex hull formed by sensors.

The ML method for TDOA-AOA measurements was also considered to find a suitable estimate for reference. The ML estimator does not have a closed-form solution. Hence we proposed to use grid-based algorithm to implement the ML method. The CRLB of target localization with TDOA-AOA measurements was derived in the end. Numerical results demonstrate that: (1) the proposed WLS method and the STLS method can reduce the estimation bias comparable to that of the other method in both 2 scenarios apart from the scenario with target inside the convex hull formed sensors; (2) only ML method could reach the CRLB when target is inside the convex hull, while in the other 2 scenarios all the mentioned method could not reach the CRLB; (3) the performance of the proposed WLS method and the STLS method could not show much superiority in the scenario with target inside the convex hull; (4) the proposed WLS is easily affected by the condition number when target is inside the convex hull.

This paper introduce the STLS method to the target localization with TDOA-AOA measurements. We only consider the single target...
in this paper. Future work can extend the work here to multiple target localization with TDOA-AOA measurements. The proposed STLS algorithm is a recursive numerical method and it is time consuming in some extreme conditions. Thus further improvement of STLS algorithm should be studied in future.

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Appendix A

Assume that the TDOA-AOA measurement vector \( \hat{m} \) satisfy the measurement model in (1), an unbiased estimate of the parameter \( m \) exists and the CRLB matrix of the estimate is \( Q \). Next we will give the details. The logarithm of the likelihood function of \( m \) is

\[
\ln f(\hat{m}|m) = \frac{1}{2} \ln((2\pi)^{N-1}|Q|) - \frac{1}{2} (\hat{m} - m)^T Q^{-1} (\hat{m} - m)
\]

Because

\[
E\left[ \frac{\partial \ln f(\hat{m}|m)}{\partial m} \right] = \int_{-\infty}^{\infty} f(\hat{m}|m)Q^{-1}(\hat{m} - m)d\hat{m} = E(Q^{-1} \hat{m}) - Q^{-1} m = 0
\]

The Fisher information matrix \( I(m) \) is expressed as

\[
I(m) = -E\left[ \frac{\partial^2 \ln f(\hat{m}|m)}{\partial m \partial m^T} \right] = Q^{-1}
\]

We have the CRLB matrix \( C_\hat{m} \) of an estimate of the parameter \( m \)

\[
C_\hat{m} = I^{-1}(m) = Q
\]

Based on the above mentioned result, the CRLB matrix of an estimate of the location vector \( u \) will be derived. Define the CRLB matrix of an estimate of the location vector is \( C_\hat{u} \). Based on the equation in Kay’s book [26], we have

\[
C_\hat{u} = \left( \frac{\partial \hat{m}^T(u)}{\partial u} \right)^T C_\hat{m} \left( \frac{\partial \hat{m}(u)}{\partial u} \right)
\]

Let \( J_m = \frac{\partial \hat{m}(u)}{\partial u} \) and the CRLB matrix \( C_\hat{u} \) is obtained as

\[
C_\hat{u} = (J_m J_m^T)^{-1} J_m C_\hat{m} J_m^T (J_m J_m^T)^{-1}
\]

Because the CRLB matrix \( C_\hat{m} \) of \( m \) is \( Q \), finally combining the property of Moore–Penrose inverse of matrix with above, we have

\[
C_\hat{u} = (J_m Q^{-1} J_m^T)^{-1}
\]

References
