Efficient solution of the first passage problem by Path Integration for normal and Poissonian white noise

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A B S T R A C T

In this paper the first passage problem is examined for linear and nonlinear systems driven by Poissonian and normal white noise input. The problem is handled step-by-step accounting for the Markov properties of the response process and then by Chapman–Kolmogorov equation. The final formulation consists just of a sequence of matrix–vector multiplications giving the reliability density function at any time instant. Comparison with Monte Carlo simulation reveals the excellent accuracy of the proposed method.

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1. Introduction

The first passage problem has been investigated in many publications over almost a century because of its relationship to the safety of structural systems under random excitations. The exact solution to the first passage problem is not available because even in the case of a normal white process the Fokker–Planck equation with associated boundary conditions is in general unknown [1]. Many approximate methods have been proposed [2–7], however, the analytical approximation methods are available only for light damping for the stochastic averaging, weak nonlinearity and Gaussian approximations. First passage time for linear systems with stochastic coefficients has been addressed by using Pontryagin–Vitt equations in [8]. A quite different approach for non-linear equations driven by normal white noise has been proposed in [9] by using a generalized cell mapping method. Other relevant contributions on the subject may be found in [10–12].

In order to determine the probability distribution of the first passage time, efficient solution of the Fokker–Planck equation is necessary. Moreover, we need a solution of the problem step-by-step in order to cancel the trajectories that for the first time leave the safe domain (absorbing barrier problem). In order to have such a control on the path of the trajectories the only way is using the so-called Path Integration (PI) method. It mainly consists of using the Chapman–Kolmogorov (CK) equation giving the probability density at a certain time instant as weighted sum of the contributions of the various trajectories that in a previous time instant start with deterministic initial condition. As the interval between the two time instants becomes small, then the so-called short time Gaussian approximation [13] remains still valid and the step-by-step solution technique of the CK equation reverts to the PI method. Many papers have been devoted to this subject for normal [14–19] and Poissonian white noise as well as renewal processes [20–23].

The PI method is versatile and in [24] it has been used for solving the first passage problem. It mainly consists in defining the so-called reliability function which is a function giving the probability that the various trajectories will remain inside the safe barrier conditioned by the fact that each of them never crosses the barrier up to the observation time.

In this paper, by using the concepts exploited in [24] the first passage problem is revisited in the light of the cell mapping method and extended to the case of Poissonian white noise input. It is shown that the reliability function by discretization of the Chapman–Kolmogorov equation may be easily implemented in a computer program as just a sequence of matrix–vector multiplications whose sizes depend on the threshold barriers and the spatial discretization steps. Moreover, as the input is stationary the reliability function is governed by a transition matrix that does not explicitly depend on time so that it can be computed once
beforehand.

The paper is organized as follows. In Section 2 the PI, some well-known concepts of the PI method for both normal and Poissonian white noise are presented for clarity’s sake as well as for introducing appropriate notations. These concepts are framed in the context of a cell-mapping method. In Section 3 the first passage time by using PI method is presented for both normal and Poissonian white noise for the half oscillator. In Section 4 the extension to a single degree-of-freedom oscillator is presented while in Section 5 the numerical applications are presented and the results are compared with those obtained from Monte-Carlo simulation.

2. Path Integration method

In this section some preliminary concepts on Path Integral Solution (PIS) will be introduced for clarity’s sake as well as for introducing appropriate notation. Let any nonlinear system be governed by the equation

$$X + f(X, t) = W(t)$$  \( (1) \)

where \( f(X, t) \) is any non-linear function of the response process \( X(t) \) and \( W(t) \) is a normal white noise characterized by the strength \( Q \). This means that

$$E[W(t_1)W(t_2)] = Qδ(t_1 - t_2)$$  \( (2) \)

where \( E[] \) denotes ensemble average and \( δ(\cdot) \) is Dirac’s delta function. The response process \( X(t) \) is Markovian and the Chapman–Kolmogorov equation

$$p_X(X, t + r) = \int_{-\infty}^{\infty} p_X(X, t + r; y)p_X(y, t)\,dy$$  \( (3) \)

holds true. In Eq. (3) \( p_X(X, t) \) is the probability density function (PDF) of the process \( X(t) \) at time \( t \) and \( p_X(X, t + r; y) \) is the conditional PDF at time \( t + r \) for an assigned (deterministic) initial condition \( y \) at time \( t \). Eq. (3) is valid for any value of \( r \). However, for finding the evolution in time of the PDF of the response process \( X(t) \), the Chapman–Kolmogorov equation is written for a small value of \( r \) that will be denoted as \( r = \Delta t \), and Eq. (3) particularized for \( r = \Delta t \) is usually called Path Integral (PI).

2.1. Gaussian white noise

In this case the conditional PDF in Eq. (3) is determined from the so-called short time Gaussian approximation [13]. A deeper insight into the concept is necessary in order to clearly understand the use of Eq. (3) particularized for \( r = \Delta t \). The conditional PDF in Eq. (3) is the solution to the Fokker–Planck (FP) equation associated to Eq. (1) with the assigned deterministic initial condition \( x(t_0) = y \) in \( t_0 \). It is obvious that if we know the transient solution of the FP equation for any value of \( y \) we may also solve the FP equation for the original system (1). In order to get the conditional PDF in Eq. (3) we subdivide the \( r \)-axis into small intervals of equal length \( \Delta t \) and rewrite this equation into the form

$$p_X(X, t_k + \Delta t) = \int_{-\infty}^{\infty} p_X(X, t_k + \Delta t; y)p_X(y, t_k)\,dy$$  \( (4) \)

Then we define a new process \( \tilde{X}(t) \) governed by the equation

$$\dot{\tilde{X}} + f(\tilde{X}(t), r) = W(t_k + r); \quad \tilde{X}(0) = y$$  \( (5) \)

This situation is depicted in Fig. 1.

Now in virtue of the short time Gaussian approximation since

$$E[\tilde{X}(\Delta t)] = \mu_\tilde{X}(\Delta t) = y - f(y, t_k)\Delta t$$  \( (6) \)

$$\sigma_\tilde{X}^2(\Delta t) = Q\Delta t$$  \( (6) \)

as we assume that the PDF of the process \( \tilde{X}(t) \) in the interval \( 0 \leq r \leq \Delta t \) is Gaussian, then

$$p_\tilde{X}(X, t_k + \Delta t; y) = \frac{1}{\sqrt{2\pi\sigma_\tilde{X}^2(\Delta t)}} \exp\left(-\frac{(x - \mu_\tilde{X}(\Delta t))^2}{2\sigma_\tilde{X}^2(\Delta t)}\right)$$  \( (7) \)

By inserting Eq. (7) into Eq. (4) the step-by-step-solution may be readily found.

If the system is linear, namely \( f(X, t) = aX \) \( (a > 0) \), then the exact values of the mean and the variance of the process \( X(t) \) is readily found in the form

$$\mu_X(\Delta t) = y \exp(-a\Delta t); \quad \sigma_X^2(\Delta t) = \frac{Q}{2a}(1 - \exp(-2a\Delta t))$$  \( (8) \)

Now we may give an interpretation of Eq. (4) that will be useful for the first passage problem. We have the process \( X(t) \), solution of Eq. (1) from the whole sample functions, some of them lie within the interval \([y, y + dy]\), and this happens with probability \( p_X(y, t)\,dy \). These trajectories generate the process \( \tilde{X}(t) \) that is characterized in \( t_k + \Delta t \) by the conditional PDF given in Eq. (7), then Eq. (4) gives \( p_\tilde{X}(X, t_k + \Delta t) \) as the sum of the contribution of \( p_X(X, t_k + \Delta t) \) weighted by \( p_X(X, t_k) \) (see Fig. 1). This perspective is important for the first passage problem since, as the cell mapping method [9] we have control on the various trajectories.

2.2. Poissonian white noise

For the case of Poissonian white noise the PI has been performed in [17]. Herein this is briefly summarized. Let the equation of motion (1) be driven by a Poisson white noise \( W_p(t) \). It is defined as

$$W_p(t) = \sum_{k=1}^{N(t)} z_k\delta(t - t_k)$$  \( (9) \)

where \( z_k \) is the \( k \)-th realization of a random variable \( Z \) with assigned probability density function \( P_Z(z) \), \( t_k \) is the \( k \)-th realization of a random variable \( T \) distributed in time according to the Poisson law with expected arrival rate \( \lambda \) and \( N(t) \) is the number of spikes within the interval \([0, t]\). A sample function \( W_p(t) \) of such a process is depicted in Fig. 2(a).

By integrating Eq. (1) for each sample function we get the
corresponding sample function of the response process \(X(t)\). Now we subdivide the interval \([0, t]\) into small subintervals \(\Delta t\) and because of the Markovianity of the process \(X(t)\) the Chapman–Kolmogorov equation holds true. Then in order to evaluate the conditional probability, that is the kernel in Eq. (3), we need to make some considerations. If we call \(\lambda\) the mean number of impulses per unit time then in the generic interval \([t_k, t_k + \Delta t]\) two different situations in the whole sample functions may happen: (i) no spikes are present within the interval. This situation happens on average \((1 - \lambda \Delta t)\) times (see Fig. 2b); (ii) one spike is present which happens on average \(\lambda \Delta t\) times. If \(\Delta t\) is small then the probability that two spikes occur in \([t_k, t_k + \Delta t]\) is of order \(\Delta t^2\) and therefore negligible. Now we may group the sample functions according to cases (i) and (ii). The first group does not contain any impulse \((1 - \lambda \Delta t)\) times and the second one that contains one impulse with probability density function \(p_x(t, t_k + \Delta t)\). As we linearize the system within the interval \([t_k, t_k + \Delta t]\) the various trajectories \(X(t)\) are governed by the differential equations

\[
\begin{align*}
\dot{X} + f(X(t), t) &= 0; \quad X(0) = y \quad \text{(10a)} \\
\dot{X} + f(X(t), t) &= z_k \delta(t - t_k); \quad X(0) = y \quad \text{(10b)}
\end{align*}
\]

Eq. (10a) is related to the situation in which no impulses are present and Eq. (10b) to the situation in which an impulse arrives at time \(t_k\). As we assume that within a small time interval \(\Delta t\) the linearized equation holds true, we have from Eq. (10a) upon taking expectation

\[
E[\dot{X}(\Delta t)] = y - f(y, t_k) \Delta t = m_y(\Delta t)
\]

(11)

and for such a process the variance is zero. Exactly the same mean value may readily be found from Eq. (10b) as we assume that the mean value of \(Z\) is zero. Now because for \(\Delta t\) small the exact position of the spike is irrelevant we may suppose that they always occur at the end of the interval and then we have in all sample functions in which the impulse is present a jump whose amplitude is \(z_k\). Then the distribution of the trajectories starting in \(t_k\) with the assigned initial condition \(y\) for Eq. (10b) is for the sample in which the impulse is present

\[
p_x(X, \Delta t) = p_x(x - E[\dot{X}(\Delta t)] + m_y(\Delta t))
\]

(12)

while the PDF of the trajectories in which the impulse is not present becomes

\[
p_x(X, \Delta t) = \delta(x - m_y(\Delta t))
\]

(13)

Then the conditional probability for Poisson white noise is

\[
p_x(X, \Delta t) = p(x, t + \Delta t|y, t)) = \lambda \Delta t p_x(x - m_y(\Delta t)) + (1 - \lambda \Delta t) \delta(x - m_y(\Delta t))
\]

(14)

It follows that the PI for nonlinear systems under Poisson white noise is

\[
p_x(X, t_k + \Delta t) = \lambda \Delta t \int_{-\infty}^{\infty} p_x(x - m_y(\Delta t)) p_x(y, t_k)
\]

\[
dy + (1 - \lambda \Delta t) \int_{-\infty}^{\infty} \delta(x - m_y(\Delta t)) p_x(y, t_k) \, dy
\]

(15)

More information on the PI for Poisson white noise may be found in [21,22]. With these informations we can now proceed for solving the first passage problem via PI.

3. First passage time

The first passage problem consists in finding the probability that the various trajectories of the stochastic process \(X(t)\), solution of Eq. (1) crosses for the first time out of a prescribed safe domain within a certain time interval \([0, t]\). In order to solve the fundamental problem we may use a modification of the PI.

3.1. Normal white noise input

Let us denote as \(\eta, \xi\) the threshold barriers and let us subdivide the interval \([0, t]\) into intervals of length \(\Delta t\) so small that the short time Gaussian approximation remains still valid and that the various trajectories starting in \(t_k = k \Delta t\) with deterministic initial condition \(x = y\) are monotone within the interval \([t_k, t_k + \Delta t]\). On the other hand, as some trajectory crosses for the first time the thresholds \(\eta, \xi\) this trajectory has to be cancelled to avoid that, at a later time instant, it crosses the barrier \(\xi\) (or \(\eta\) again returning to the safe domain. This is the so-called absorbing barrier problem. With these concepts in mind we may proceed by evaluating the probability distribution of the first passage problem via PI.

Let us consider a quiescent system at \(t = 0\) \((P_x(x, 0) = \delta(x))\) and we fix the barrier \(\eta, \xi\). In the first time step, according to Eq. (3) we have

\[
p_x(X, \Delta t) = \int_{-\infty}^{\infty} p_x(x, \Delta t | y, 0) p_x(y, 0) \, dy
\]

\[
= \frac{1}{\sqrt{2\pi \sigma_x(\Delta t)}} \exp \left( -\frac{x^2}{2\sigma_x^2(\Delta t)} \right)
\]

(16)

where \(\sigma_x^2(\Delta t) = Q \Delta t\).

In Fig. 3 the double barrier is depicted. From this figure we

![Fig. 2. Realizations of Poisson white noise and response.](image-url)
realize that at first time instant $\Delta t$ some trajectories cross for the first time the two barriers. Because these trajectories have to be cancelled this produces that in the time step $2\Delta t$ the PI has to be considered only for those trajectories starting inside the two barriers. It follows that we define a new function termed as reliability function $q_{s,t}(x, \Delta t)$ that is
\[
q_{s,t}(x, \Delta t) = U(x - \eta)U(\xi - x)p_x(x, \Delta t)
\]
in which $U[\cdot]$ denotes the unit-step (Heaviside) function. The function $q_{s,t}$ is not a PDF since its area is not unity, nevertheless $q_{s,t}(x, \Delta t)$ gives the probability that the trajectories fall, at time $\Delta t$, in the interval $[x, x + dx]$. Hence, in the subsequent time interval $\Delta t$ the probability density of trajectories giving a contribution to the reliability function is given by $q_{s,t}(x, \Delta t)$. It follows that in the generic time instant $(r + 1)\Delta t$ the reliability function is given as
\[
q_{s,t}(x, tr + \Delta t) = U(x - \eta)U(\xi - x)\int_{x}^{\xi} p(x, t_r + \Delta t|y, t_r) dy
\]
(18)
That represents the modification of the PI for the problem at hands. Once the reliability density function is evaluated its integral in $t_r + \Delta t$ gives the probability that the first passage time is larger than $t_r + \Delta t$. It is obvious that the reliability density function will change at each time instant and also for the case of linear system it is not a Gaussian PDF in $[\eta, \xi]$. As the interval $[\eta, \xi]$ becomes smaller and smaller the probability that the trajectories will remain inside the safe domain diminishes. Moreover, for stationary excitation the following inequality holds true and, as $t_r \to \infty$, the probability inside the threshold barrier approaches zero, and therefore the probability that the first passage time is less than $t_r \to \infty$ approaches one.

If the interval $[\eta, \xi]$ is discretized into $n$ values $x_k, k = 1...n$, equally spaced at an interval $\Delta x$, then Eq. (18) can be rewritten as
\[
q_{s,t}(x_k, tr + \Delta t) = U(x - \eta)U(\xi - x)\sum_{j=1}^{n} p(x_k, t_r + \Delta t|y_j, t_r)
\]
(19)
For convenience here it has been assumed that the discretization over $x$ at time $t_r + \Delta t$ is the same as that for $y$ at time $t_r$. It can be seen that this equation is just the componentwise notation of a matrix–vector multiplication. This means that for computation we can arrange the values $q_{s,t}(x_k, tr + \Delta t)$ into a vector $\mathbf{q}(t_r + \Delta t)$ and the values of $q_{s,t}(y_j, t_r)$ into a vector $\mathbf{q}(t_r)$ to obtain
\[
\mathbf{q}(t_r + \Delta t) = \mathbf{T}(t_r + \Delta t, t_r)\mathbf{q}(t_r)
\]
in which the entries $T_{ij}$ of the matrix $\mathbf{T}$ are given by (cf. Eqs. (6) and (7))
\[
T_{ij}(t_r, t_r + \Delta t) = p(x_k, t_r + \Delta t|y_j, t_r)\Delta x = \frac{1}{\sqrt{4\pi Q\Delta t}} \exp \left( \frac{(x_k - y_j - f(y_j)\Delta t)^2}{2Q\Delta t} \right) \Delta x
\]
(22)
If the excitation process is stationary, then the matrix $\mathbf{T}$ does not depend on $t$ explicitly (i.e., only on $\Delta t$) so that it can be computed once before hand. This makes the entire path integral solution just a sequence of matrix–vector multiplications of very moderate size ($n \approx 500$). The final equation, for steady state input, is
\[
\mathbf{q}(tr + \Delta t) = \mathbf{T}(\Delta t)\mathbf{q}(tr)
\]
(23)
The discretization scheme is shown schematically in Fig. 4. As the two barriers become larger and larger then the first excursion probability becomes negligible and then Eq. (23) may be used to determine the PDF at each time instant.

### 3.2. Poissonian white noise input

In the case of Poissonian input the PI is given in Eq. (15). Also in this case the modification of the PI in order to find the reliability density function is done just as for Gaussian white noise, i.e. according to Eq. (18). The required conditional probability is now given by Eq. (14). It follows that the transition matrix $\mathbf{T}(\Delta t)$ already defined in Eq. (23) is for Poissonian input
\[
T_{ij}(\Delta t) = p(y, t_r + \Delta t|y_j, t_r)\Delta x
\]
\[
= \left( 1 - \Delta t p_2(x_k - y_j - f(y_j)\Delta t) \right)
\]
\[
+(1 - \Delta t)\delta(x_k - y_j - f(y_j)\Delta t)\Delta x
\]
(24)
Once the transition matrix is constructed, Eq. (23) remains still valid and then this case may be treated for linear and nonlinear systems using the same procedure as already described in Section 3.1. So the only difference between normal and Poissonian white noise is that in the former case the various entries of the transition matrix are constructed by Eq. (22), while in the latter case the entries are constructed as given in Eq. (24).

### 4. Extension for higher-order systems

The extension for higher-order systems is straightforward. In this section, the case of an SDOF oscillator (with two state variables) will be briefly discussed. The equation of motion of an oscillator with Duffing-type nonlinearity $f(x)$ under white noise is given by
\[
\ddot{X} + \alpha \dot{X} + \gamma X + f(X) = W(t)
\]
(25)
which can be re-cast in first-order form by introducing the state variables $V_1 = X \quad \text{and} \quad V_2 = \dot{X}$
\[ V_1 = V_2 \]
\[ V_2 = -c_k V_1 - f(V_1) - c_n V_2 + W(t) \]  
(26)

Again, \( W(t) \) denotes Gaussian white noise with strength \( Q \). The state vector \( \mathbf{V} = [V_1, V_2]^T \) has Markovian property and therefore satisfies the Chapman–Kolmogorov equation

\[ p_{y_1}(y_2, y_2, t + \Delta t) = \int_{-\infty}^{\infty} p_{y_1}(y_1, y_1, t) p_{y_2}(y_2, t + \Delta t) dy_1 dy_2 \]  
(27)

Assuming that we are only interested in the barrier crossing of the displacement \( V_1 \), the reliability density function is governed by the transition equation

\[ q_{y_1}(y_1, t + \Delta t) = U(x - \eta) U(\xi - x) \int_{-\infty}^{\infty} p_{y_1}(y_1, t + \Delta t) dy_1 \]  
(28)

For small time intervals \( \tau = \Delta t \), the conditional probability in the above equation can again be approximated by a Gaussian density function [13]. Specifically, the conditional means of \( z_1, z_2 \) of the displacement and velocity when starting at \( y_1, y_2 \) are given by

\[ E[V_1, t + \Delta t | y_1, y_2] = y_1 + y_2 \Delta t = m_1 \]
\[ E[V_2, t + \Delta t | y_1, y_2] = y_2 - (c_k y_1 + f(y_1) + c_n y_2) \Delta t = m_2 \]  
(29)

and the conditional variances are

\[ \sigma_1^2(t + \Delta t | y_1, y_2) = 0 \]
\[ \sigma_2^2(t + \Delta t | y_1, y_2) = Q \Delta t = \sigma_2^2 \]  
(30)

Therefore, the short time Gaussian approximation for the transition probability density function becomes

\[ p_{V_1V_2}(V_1, V_2, t + \Delta t | y_1, y_2) = \frac{1}{\sqrt{2\pi \sigma_2^2}} \exp \left( -\frac{(V_2 - m_2)^2}{2\sigma_2^2} \right) \delta(V_1 - m_1) \]  
(31)

Upon introducing a suitable discretization of the displacement and velocity axes at times \( t \) and \( t + \Delta t \), the procedure is carried out as outlined above. Of course, the final size of the transition matrix \( T \) is substantially larger. However, due to the presence of the Delta type singularity it contains a very large number of zero entries. Therefore sparse matrix numerical techniques may be fully exploited.

For the case of Poissonian noise as an input, similar considerations as outlined in Eq. (15) lead to the formulation of the short-time transition probability function [21]

\[ p_{V_1V_2}(V_1, V_2, t + \Delta t | y_1, y_2) = \left[ (1 - \lambda \Delta t) \delta(V_2 - m_2) + \lambda \Delta t p_{Z}(V_2 - m_2) \right] \delta(V_1 - m_1) \]  
(32)

in which \( \lambda \) is the mean arrival rate and \( p_{Z}(z) \) denotes the probability density function of the impulse height.

5. Numerical Examples

5.1. Gaussian white noise

Consider a first-order system with cubic nonlinearity described by the lto equation:

\[ dX = - (c_k X + f(X)) dt + \sqrt{Q} dB(t) \]  
(33)

where \( B(t) \) is the Brownian motion process whose (formal) derivative is the unitary white noise. We analyze two cases for the specific nonlinearity \( f(X) = c_n X^3 \). These are Case 1 with \( c_k = 1, c_n = 0 \) (a linear case) and Case 2 with \( c_k = -1, c_n = 0.3 \). Case 2 has a

![Fig. 5. Probability densities of remaining in the interval I for Case 1. Left figure: \( I = [\pm 1, 1] \), Right figure: \( I = [\pm 1, 4] \).](image)

![Fig. 6. Probability densities of remaining in the interval I for Case 2. Left figure: \( I = [\pm 1, 1] \), Right figure: \( I = [\pm 1, 4] \).](image)
Fig. 7. Transition matrices for Cases 1 (left) and 2 (right). Black color indicates values close to zero.

Fig. 8. First passage probabilities for system under white noise. Left: Case 1 \((k = 1, c = 0)\), Right: Case 2 \((k = -1, c = 0.3)\). Dots indicate Monte Carlo results from simulation with 2,000,000 samples.

Fig. 9. Probability densities of remaining in the interval \(I\) for Case 1. Left figure: \(I = [-5, 1]\), Right figure: \(I = [-5, 3]\).

Fig. 10. Probability densities of remaining in the interval \(I\) for Case 2. Left figure: \(I = [-5, 1]\), Right figure: \(I = [-5, 3]\).
bi-modal stationary probability density. In all cases, the initial value was chosen as \( X(0) = 0 \) \( (p_0(x, 0) = \delta(x)) \). The probability density of staying inside the interval \([\eta, \xi]\) with \( \eta = -5 \) is shown in Fig. 5 (left) for \( \xi = 1 \), and in Fig. 5 (right) for \( \xi = 4 \).

Similarly, for case 2 the probability density of staying inside the interval \([\eta, \xi]\) with \( \eta = -5 \) is shown in Fig. 6 (left) for \( \xi = 1 \), and in Fig. 6 (right) for \( \xi = 4 \).

It is quite remarkable that for the case of low threshold values \( (\xi = 1) \) there is a clearly pronounced asymmetry in the reliability density function \( q_2(x, t) \) which develops over time. The first passage probabilities for a total time \( T = 2 \) and different values of the threshold \( \xi \) are shown in Fig. 8 for a linear and a nonlinear case, respectively. In both cases, the results are compared to Monte Carlo simulation with 2,000,000 samples.

The numerical analysis was carried out using 200 time steps of size \( \Delta t = 0.01 \). For such a small time step and a noise level of \( Q = 2 \), the standard deviation \( \sigma \) as referred to in Eq. (7) is \( \sigma = 0.1 \). This means that the transition probability density will be numerically zero along most of the \( x \)-axis except for a region close to the previous value \( y \).

This can be seen in the transition matrices in Fig. 7. Note that in the right figure (Case 2), there is a very strong drift visible in the upper left corner of the matrix (corresponding to values of \( x \) and \( y \) near \( \eta = -6 \)). This is due to the severe nonlinearity which acts in a way very similar to a reflecting boundary.

5.2. Poissonian noise

We consider a system of the form

\[
dx = -(c_3X + f(X)) \, dt + dC(t)
\]

in which \( C(t) \) denotes the compound Poisson process whose formal derivative is the Poissonian white noise. The impulses arrive with a mean rate of \( \lambda \) and the impulse heights are uniformly distributed in the interval \([ -s_R, s_R] \). Numerical values for the Poissonian noise are \( \lambda = 4 \) and \( s_R = 1 \). The nonlinear function is chosen as \( f(X) = c_4X^3 \). The analysis of this excitation is done using the same two cases as previously in the section on white noise, i.e. Case 1 with \( c_3 = 1, c_4 = 0 \) and Case 2 with \( c_3 = -1, c_4 = 0.3 \).

The reliability density functions of the interval \([\eta, \xi]\) for \( \eta = -5 \) and for \( \xi = 1 \) and \( \xi = 3 \) are shown in Fig. 9 for Case 1 and in Fig. 10 for Case 2.

Again, as in the case of white noise, it can be seen that for low threshold values, the reliability density function \( q_3(x, t) \) develops a marked asymmetry, whereas for higher threshold values the reliability density function is essentially a scaled-down version of the unconditional probability density function \( p(x, t) \). The first passage probabilities for a total time \( T = 2 \) and different values of the threshold \( \xi \) are shown in Fig. 11 for a linear and a nonlinear case, respectively. In both cases, the results are compared to Monte Carlo simulation with 1,000,000 samples.

5.3. Duffing oscillator

As a last example, we consider a nonlinear single-degree-of-freedom oscillator under normal white noise as described in Eq. (25). The nonlinear function was chosen as \( f(X) = c_6X^3 \). Numerical values chosen are \( c_6 = 0.5, c_4 = -1, c_5 = 0.3 \) and \( Q = 2 \). For a fixed time duration \( T = 5 \), the first passage probabilities for different threshold values \( \xi \) are computed and shown in Fig. 12 (left). The lower threshold was fixed at \( \eta = -10 \). Also, the case of Poissonian white noise with a mean impulse arrival rate of \( \lambda = 8 \), and a uniform distribution of the impulse height in the interval \([ -0.5, 0.5] \) was analyzed. The results are shown in Fig. 12 (right). The PI solution matches the Monte-Carlo results very well for both cases in the range of threshold levels considered.

6. Conclusions

In this paper the first passage problem for a double barrier is formulated in terms of the Path Integration by using the Markov
properties of the response process of a linear or nonlinear system driven by white noise. The method takes full advantage that for the first passage problem when some trajectories inside the barriers will touch the barriers these trajectories have to be cancelled (absorbing barriers). This produces a reduction of the total probability and a distortion of the probability distribution of the trajectories that survive within the assigned barriers. With this concept a modified PI is proposed giving the reliability function in the generic time instant given the reliability function in the previous one.

The procedure is robust and works for linear and non-linear systems, for normal and Poissonian white noise as well. A numerical procedure for finding the reliability function step-by-step at any time instant has been also presented. This procedure is just a sequence of matrix-vector multiplications in which the transition matrix, for stationary input, does not depend on the current time, and therefore may be computed once beforehand.

Results have been reported for linear and nonlinear half and single oscillators, Poissonian and normal white noise and compared with those obtained by Monte Carlo simulation.

References
