Doppler tolerance, complementary code sets, and generalised Thue–Morse sequences

Hieu Duc Nguyen¹, Gregory Emmett Coxson²

¹Department of Mathematics, Rowan University, Glassboro, NJ 08012, USA
²Department of Electrical and Computer Engineering (ECE), United States Naval Academy, Annapolis, MD 21402, USA

Abstract: The authors generalise the construction of Doppler-tolerant Golay complementary waveforms by Pezeshki–Calderbank–Moran–Howard to complementary code sets having more than two codes, which they call Doppler-null codes. This is accomplished by exploiting number-theoretic results involving the sum-of-digits function and a generalisation to more than two symbols of the classical two-symbol Thue–Morse sequence. Two approaches are taken to establish higher-order nulls of the composite ambiguity function: one by rewriting it in terms of equal sums of powers (ESP) and the other by factoring it in product form to reveal a higher-order zero, analogous to spectral-null codes. They conclude by describing an application of minimal ESP sets to multiple-input–multiple-output radar.

1 Introduction

A set of \( K \) unimodular codes of length \( N \) is complementary if corresponding sidelobes of the autocorrelations of the separate codes sum to zero. These sets find uses in waveform design for enhanced detection in radar systems [1] and in communication systems [2, 3]. When the code is binary, the set is called a Golay complementary pair, after Marcel Golay who discovered these sets while solving a problem in infrared spectrometry [4]. Complementary code matrices (CCMs) provide a useful matrix formulation for the study of complementary code sets [5–7]. Given a set of \( K \) codes of length \( N \), the corresponding \( N \times K \) CCM has the \( k \) th code as its \( k \) th column, \( k = 1, \ldots, K \).

Complementary code sets have yet to be widely used for radar waveform designs due to certain design challenges. These include sensitivity to Doppler shift due to non-zero relative velocity of a target relative to the radar platform [1, 8]. Complementary code sets may be used in a number of ways in waveform design. Two of these ways are the time-separation approach, where time-separated pulses or subpulses are phase coded using different codes in the set [9, 10], and frequency-separation, where the different codes are used for phase encoding of separate components of a signal and are transmitted concurrently using pulses with different centre frequencies [11, 12]. The time-separation approach is especially sensitive to Doppler shift.

With time-separated pulses encoded using the codes from a complementary set, pulse returns may be match filtered separately and then added to give zero autocorrelation sidelobes, in theory, a desirable result for radar detection. However, target relative velocity yields a phase shift pulse to pulse, and therefore a phase shift of sidelobes, thus preventing zero sidelobe sums in general.

The development in this paper builds on work by Pezeshki, Howard, Moran, Calderbank, Chi, and Searle [4, 9–12]. In particular, Chi et al. [9] considered pulse trains \( T \) in which the pulses are phase coded with binary codes in a Golay complementary pair. They show that for any given \( M, M \) th-order nulls can be created about the zero-Doppler axis of the ambiguity function of \( T \) by mapping the codes to pulses in an order specified by the well-known (two-symbol) Thue–Morse sequence.

The results of Pezeshki et al. [10] for Doppler-tolerant complementary pulse trains based on Golay complementary pairs can be generalised to \( (N, K) \) complementary code sets for \( K \geq 2 \). By using a generalised Prouhet–Thue–Morse (PTM) sequence based on \( m \geq 2 \) symbols to specify the order of the pulses, we show in the first part of our paper that higher-order nulls can again be achieved along the zero-Doppler axis. Our approach makes use of results related to the Prouhet–Tarry–Escott problem [13–15] and number-theoretic entities such as the non-binary digit-sum function [16] and equal sums of (like) powers (ESP) [17]. After obtaining our result in February 2014, we were informed that Howard and Moran [18] had obtained the same result (unpublished) in 2007. Recently, Tang et al. [19] showed that Doppler-tolerant complementary pulse trains can be extended to complete complementary codes (CCC) in an application to multiple-input–multiple-output (MIMO) radar. Their approach again makes use of the generalised PTM sequence and ESP sets to establish Doppler tolerance. We note that the notion of CCC dates back to that of mutually orthogonal complementary sets introduced by Tseng and Liu [20].

The Thue–Morse-ordered complementary-code waveform discussed here works by building aperiodicity into a radar pulse train. The goal of this paper is to show two key points – (i) how to generalise the Thue–Morse ordering for a complementary set of more than two codes and (ii) that in generalising beyond Golay pairs, Doppler resilience is preserved and even enhanced.

An alternative to building aperiodicity into a waveform is to mitigate Doppler sidelobes resulting from periodicity in the waveform by amplitude weighting on receive [1]. An example of a waveform with periodic structure is a pulse Doppler waveform with the same intra-pulse weighting applied to every pulse. While amplitude weighting on receive tends to yield good results for these waveforms, it constitutes mismatch filtering and the incurring of a mismatch loss, evident by a lowering of the autocorrelation peak. This means a departure from optimal signal-to-noise ratio at the filter output, and therefore a loss of sensitivity. The two approaches thus offer an interesting choice between building aperiodicity into a waveform, allowing match filtering to be employed, applying amplitude weighting on receive, and incurring a loss in sensitivity.

It should be noted that applying amplitude weighting only on receive is often the best option when there is a requirement to operate transmitters at saturation, to produce peak power. The amplifier technology to support this mode of operation is either not yet available or impractically costly. As technology develops, the practicality of implementing equal amplitude weighting on receive and transmit will become an option, and will affect the design tradeoff discussed here [1].

Since the structure of generalised Thue–Morse-ordered complementary-coded waveforms is relatively intricate, it is
conceivable that there exists a yet-undiscovered method compatible with these waveforms that can deliver Doppler sidelobe lowering beyond what is achieved by built-in aperiodicity. It is hoped that this paper will motivate others to investigate this further.

As the cardinality of the complementary code set used in the Thue–Morse-ordered complementary-coded waveform grows, so do the number of pulses in the pulse train, the degree of aperiodicity, and the width of the Doppler null about the zero-Doppler axis. An additional benefit of increasing the number of codes is that code diversity is increased; code diversity is well-recognised option for enhancing performance of a waveform [21, 22]. While longer pulse trains can be difficult to support in many radar applications, there will be applications where the combination of benefits afforded by the waveforms discussed here is attractive, while the sacrifices are acceptable. The benefits include enhanced Doppler tolerance, the option of match filtering, and a linear structure compatible with coherent processing. The application should be one where target kinematics or range rate does not change greatly while the waveform impinges on the target. An example of an application where this might be useful is one where the target moves with low risk of manoeuvre, and with stable kinematics at either great Doppler shift or imprecisely estimated Doppler shift.

In the second part of our paper, we investigate Doppler-tolerant complementary pulse trains from a different perspective, motivated by the observation that these codes bear a striking resemblance to the characterisation of spectral-null codes [23–25]. By demonstrating that the ambiguity function \( \chi \) of a Doppler-tolerant code has a natural product generating function, this proves that \( \chi \) has a higher-order zero. It follows from a well-known result in calculus that the higher-order derivatives of \( \chi \) must vanish at the zero-Doppler axis. Thus, we shall refer to codes that generate Doppler-tolerant complementary pulse trains as Doppler-null codes. This approach utilises ideas based on Lehmer [13] and the first author [26].

Finally, it is shown through the use of minimal ESP sets that the transmission period and the total number of pulses transmitted in Doppler-null codes may be reduced by employing multiple antennas to transmit separate pulse trains staggered in time.

## 2 Notations and terminology

We begin with some standard definitions.

**Definition 1:** A \( p \)-phase matrix \( Q \) is one whose entries are \( p \)th roots of unity, i.e. roots of \( z^p = 1 \).

**Definition 2:** Given a unimodular code \( x \) of length \( N \), the autocorrelation function (ACF) of \( x \) is defined as the sequence of length \( 2N - 1 \)

\[
ACF_x = x \ast \bar{x}
\]

where \( \ast \) represents aperiodic convolution and \( \bar{x} \) means reversal of \( x \). The elements \( ACF_x(k) \) for \( k = 1 - N, \ldots, 1, 0, 1, \ldots, N - 1 \) may be written explicitly as sums of pairwise products of the elements of \( x \). Namely

\[
ACF_x(k) = \sum_{i=1}^{N-k} x[i] \overline{x[i+k]},
\]

(1)

for \( k = 0, 1, \ldots, N - 1 \), where \( x[i] \) denotes the \( i \)th component of \( x \) and \( \overline{x[i]} \) represents complex conjugation. If \( k = 1 - N, \ldots, 1 \), then we define

\[
ACF_x(k) = \overline{ACF_x(-k)}.
\]

- \( |ACF_x(N-1)| = |x[1]\overline{x[N]}| = 1 \).
- When \( k = 0 \), \( ACF_x(k) \) represents the peak of the autocorrelation, which equals

\[
x[1]\overline{x[1]} + \cdots + x[N]\overline{x[N]} = \| x \|^2 = N.
\]

**Definition 3 [5]:** A \( p \)-phase \( N \times K \) matrix \( Q \) consisting of columns \( (x_0, x_1, \ldots, x_{K-1}) \) is said to be a CCM if

\[
ACF_{x_0}(n) + ACF_{x_1}(n) + \cdots + ACF_{x_{K-1}}(n) = NK \delta_n
\]

(2)

for \( n = -(N-1), \ldots, -(K-1), 0, 1, \ldots, (N-1) \) where \( \delta_n \) is the Kronecker delta function.

### 3 Doppler shift in radar

Let \( T = (x_0, x_1, \ldots, x_{K-1}) \) be a modulated pulse train whose composite ambiguity function is given by

\[
g_T(k, \theta) = \sum_{n=0}^{N-1} e^{i\theta n} ACF_{x_n}(k),
\]

(3)

where \( k \) represents range or time delay and \( \theta \) represents Doppler-shift-induced phase advance. We define the \( z \)-transform of a code \( x \) of length \( N \) by

\[
X(z) = x[0] + x[1]z^{-1} + \cdots + x[N-1]z^{-N+1}
\]

Following Pezeshki–Calderbank–Moran–Howard [10], the \( z \)-transform of \( g_T(k, \theta) \) becomes:

\[
G_T(z, \theta) = \sum_{n=0}^{N-1} e^{i\theta n} |X_n(z)|^2,
\]

(4)

where

\[
|X_n(z)|^2 = ACF_{x_n}(0) + \sum_{k=1}^{N-1} ACF_{x_n}(k)z^k + \sum_{k=1}^{N-1} ACF_{x_n}(k)z^{-k}.
\]

(5)

The next lemma relates a CCM \( Q \) with its \( z \)-transform. The proof follows immediately from (2) and (5).

**Lemma 1:** Let \( Q = (x_0, x_1, \ldots, x_{K-1}) \) be a \( p \)-phase \( N \times K \) CCM. Then

\[
\sum_{l=0}^{K-1} X_l(z) \overline{X}(z) = |X_l(z)|^2 + \cdots + |X_{K-1}(z)|^2 = NK
\]

Next, consider the Taylor expansions of \( g_T(k, \theta) \) and \( G_T(z, \theta) \) about \( \theta = 0 \)

\[
g_T(k, \theta) = \sum_{m=0}^{\infty} c_m(k)(\theta^m)!
\]

(6)

\[
G_T(z, \theta) = \sum_{m=0}^{\infty} C_m(z)(\theta^m)!
\]

(7)

Here, the Taylor coefficients \( c_m(k) \) and \( C_m(z) \) are given by

\[
c_m(k) = \sum_{n=0}^{K-1} n^m ACF_{x_n}(k)
\]

(8)

\[
C_m(z) = \sum_{n=0}^{K-1} n^m |X_n(z)|^2.
\]

(9)

The following theorem demonstrates an equivalence in terms of the ‘vanishing’ of the Taylor coefficients \( c_m(k) \) and \( C_m(z) \).

**Theorem 1:** Let \( m \) be a non-negative integer. Then, \( c_m(k) = 0 \) for all non-zero \( k \) if and only if \( C_m(z) \) is constant and independent of \( z \).
Proof: Assume \( c_n(k) = 0 \) for all non-zero \( k \). It follows from (2) that 
(see equation below) Next, we reverse the order of the two summations to obtain (see equation below) This proves that \( C_n(z) \) is constant and independent of \( z \). Conversely, assume \( C_n(z) \) is constant and independent of \( z \). Then, from the previous calculation we have
\[
C_m(z) = \sum_{n=0}^{N-1} n^m ACF_x(0) + \sum_{k=1}^{N-1} n^m C_m(k) x^k + \sum_{k=1}^{N-1} C_m^{-1}(k) x^{-k}.
\]
It follows that \( C_m(k) = 0 \) for all non-zero \( k \) since \( C_m(z) \) is independent of \( z \). □

4 Generalised PTM sequences

Denote by \( S(L) = \{0, 1, \ldots, L-1\} \) to be the set consisting of the first \( L \) non-negative integers.

Definition 4: Let \( n = n_1, n_2, \ldots, n_k \) be the base-\( p \) representation of a non-negative integer \( n \), where \( n_i \in \{0, 1, \ldots, p-1\} \) for \( i=1, \ldots, k \). We define \( v_p(n) \in \mathbb{Z} \cap \{0, p-1\} \) to be the least positive residue of the sum of the digits \( n_i \) modulo \( p \), that is
\[
v_p(n) \equiv \left( \sum_{i=1}^{k} n_i \right) \mod p.
\]
Note that \( v_p(n) = n \) if \( 0 \leq n < p \).

Definition 5 [14]: Let \( p \) to be a positive integer. We define the mod-\( p \) PTM sequence \( P = (a_0, a_1, \ldots) \) to be such that
\[
a_n = v_p(n).
\]

Example 1: Examples of \( P \) for \( p = 2, 3, 4 \) are given below. Observe that for \( p = 2 \), \( P \) reduces to the classical PTM sequence [27].

\( p = 2 \):
\[
P = \{0, 1, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, \ldots\}
\]
\( p = 3 \):
\[
P = \{0, 1, 2, 1, 2, 0, 2, 0, 1, 1, 2, 0, 2, 0, 1, 0, 1, 2, 0, \ldots\}
\]
\( p = 4 \):
\[
P = \{0, 1, 2, 3, 1, 2, 3, 0, 2, 3, 0, 1, 3, 0, 1, 2, 1, 2, 3, 0, \ldots\}
\]

Definition 6: Let \( p \) and \( M \) be positive integers and set \( L = pM+1 \). We define \( \{S_0, S_1, \ldots, S_{p-1}\} \) to be a PTM \( p \)-block partition of \( S(L) = \{0, 1, \ldots, L-1\} \) as follows: if \( v_p(n) = i \), then:
\[
n \in S_i.
\]

Example 2: Examples of PTM block partitions are given below
\[
p = 2, M = 3, L = 16
\]
\[
S_0 = \{0, 3, 5, 6, 9, 10, 12, 15\}
\]
\[
S_1 = \{1, 2, 4, 7, 8, 11, 13, 14\}
\]
\[
p = 3, M = 2, L = 27
\]
\[
S_0 = \{0, 5, 7, 11, 13, 15, 19, 21, 26\}
\]
\[
S_1 = \{1, 3, 8, 9, 14, 16, 20, 22, 24\}
\]
\[
S_2 = \{2, 4, 6, 10, 12, 17, 18, 23, 25\}
\]
\[
p = 4, M = 2, L = 64 \text{ (see equation below)}
\]

The next theorem establishes the well-known result that PTM \( p \)-block partitions have equal sums of like powers.

Theorem 2 [13–15]: Let \( p \) and \( M \) be positive integers and set \( L = pM+1 \). Define \( \{S_0, S_1, \ldots, S_{p-1}\} \) to be a PTM \( p \)-block partition of \( S(L) = \{0, 1, \ldots, L-1\} \). Then
\[
\sum_{n \in S_0} n^m = \sum_{n \in S_1} n^m = \cdots = \sum_{n \in S_{p-1}} n^m
\]
for \( m = 1, \ldots, M \).

It will be convenient to define \( P_m := P_m(p, M) = \sum_{n \in S_0} n^m \) to be the \( m \)th Prouhet sum corresponding to \( p \) and \( M \). Let \( (A_0, A_1, \ldots) \) be a sequence of elements satisfying the aperiodic property
\[
A_n = A_{v_p(n)}.
\]

We shall define an orthogonal set of sequences \( w_0(n) \) whose values are given by the Rademacher functions [26]. These sequences will be used to define a transformation of elements \( (A_0, A_1, \ldots, A_{p-1}) \).
whose invertibility provides a useful decomposition for isolating sidelobes in the total autocorrelation of a train of coded pulses.

**Definition 7:** Let

\[ i = d^{(1)}_o 2^{p-1} + d^{(2)}_o 2^{p-2} + \cdots + d^{(p)}_o 2^0 \]

be the binary expansion of \( i \), where \( i \) is a non-negative integer with \( 0 \leq i \leq 2^p - 1 \). Define \( w_0(n), w_1(n), \ldots, w_{2^p-1}(n) \) to be binary ±1-sequences

\[ w_j(n) = (−1)^{d^{(j)}_o 2^i p(n)} \]

for \( n = 0, 1, \ldots \).

**Theorem 3** [26]: Define

\[ B_j = \sum_{n=0}^{2^p-1} w_j(n)A_n \]

for \( i = 0, 1, \ldots, 2^p - 1 \). Then

\[ A_n = \left( 1/2^{p-1} \right) \sum_{i=0}^{2^p-1-1} w_i(n)B_i \]

for \( n = 0, 1, \ldots \).

Owing to Theorem 3, we shall call \( w_0(n), w_1(n), \ldots, w_{2^p-1}(n) \) the PTM weights of \( A_n \) with respect to \( (B_0, B_1, \ldots, B_{2^p-1}) \).

**Example 3:** Examples illustrating Theorem 3 are given below.

1. \( p = 2 \)
   - \( B_0 = A_0 + A_1; A_0 = \frac{1}{2}(B_0 + B_1) \)
   - \( B_1 = A_0 - A_1; A_1 = \frac{1}{2}(B_0 - B_1) \)

2. \( p = 3 \)
   - \( B_0 = A_0 + A_1 + A_2; A_0 = \frac{1}{4}(B_0 + B_1 + B_2 + B_3) \)
   - \( B_1 = A_0 + A_1 - A_2; A_1 = \frac{1}{4}(B_0 + B_1 - B_2 - B_3) \)
   - \( B_2 = A_0 - A_1 + A_2; A_2 = \frac{1}{4}(B_0 - B_1 + B_2 - B_3) \)
   - \( B_3 = A_0 - A_1 - A_2 \)

**Theorem 4** [26]: Suppose \( L = p^{M-1} \) where \( M \) is a non-negative integer. Write

\[ A_n = \left( 1/2^{p-1} \right) w_0(n)B_0 + \left( 1/2^{p-1} \right) S_p(n) \]

(10)

where

\[ S_p(n) = \sum_{i=1}^{2^p-1} w_i(n)B_i \]

Then

\[ \sum_{n=0}^{l-1} n^m S_p(n) = N_m B_0 \]

(11)

for \( m = 1, \ldots, M \) where

\[ N_m = 2^p - 1 \sum_{n=0}^{l-1} n^m. \]

### 5 Doppler-tolerant CCM waveforms

In this section, we generalise the results in [9, 12] by constructing Doppler-tolerant CCM waveforms. We were later informed that Howard and Moran [18] obtained the same results as ours in 2007, but this work was not published.

**Definition 8:** We define a mod-p PTM pulse train \( T = (x_0, x_1, \ldots, x_{L-1}) \) to be a sequence satisfying

\[ x_n = x_{n+p(n)}. \]

Let \( A_p(k) \) represent sidelobe \( k \) for the autocorrelation \( ACF_x \) of code \( x_n \). It follows that \( A_p(k) = A_p(d_p(k)) \). At times, the sidelobe index \( k \) will be suppressed, when the property being discussed applies regardless of the particular sidelobe.

We now use the results from the previous section to isolate the sidelobe term given by (9) in the ambiguity function \( g(k, \theta) \).

Suppose \( L = p^{M-1} \) where \( M \) is a non-negative integer. It follows from (2) and (8) that:

\[ g_p(\theta) = g_p(k, \theta) \]

\[ = \sum_{n=0}^{l-1} A_{p(n)} e^{i\theta n} \]

\[ = \sum_{n=0}^{l-1} \left( \left( (1/2^{p-1}) w_0(n)B_0 + (1/2^{p-1}) S_p(n) \right) e^{i\theta n} \right) \]

\[ = \left( 1/2^{p-1} \right) B_0 \sum_{n=0}^{l-1} e^{i\theta n} + \left( 1/2^{p-1} \right) \sum_{n=0}^{l-1} S_p(n) e^{i\theta n}. \]

The argument uses the fact that \( w_0(n) = 1 \).

**Example 4:** Let \( p = 2 \). Then \( g_2(\theta) \) reduces to equation (11) in [9]

\[ g_2(\theta) = (1/2) B_0 \sum_{n=0}^{l-1} e^{i\theta n} + (1/2) \sum_{n=0}^{l-1} S_p(n) e^{i\theta n} \]

\[ = \left( 1/2 \right) (A_0 + A_1) \sum_{n=0}^{l-1} e^{i\theta n} \]

\[ + (1/2)(A_0 - A_1) \sum_{n=0}^{l-1} w_0(n) e^{i\theta n}, \]

where \( w_0(n) = p_0 \) is the classical PTM sequence defined by the recurrence \( p_0 = 1, p(2n) = p(n), \) and \( p(2n+1) = -p(n) \).

Define

\[ h_p(\theta) = \left( 1/2^{p-1} \right) \sum_{n=0}^{l-1} S_p(n) e^{i\theta n} \]

so that

\[ g_p(\theta) = (1/2) B_0 \sum_{n=0}^{l-1} e^{i\theta n} + h_p(\theta). \]

If \( Q = (x_0, x_1, \ldots, x_{K-1}) \) is a unimodular \( N \times K \) CCM, then \( h_p(\theta) \) represents the sidelobes of \( g_p(\theta) \) since \( B_0 = A_0 + A_1 + \cdots + A_{K-1} \) vanishes for all non-zero \( k \), being the sum of the ACFs of \( x_0, x_1, \ldots, x_{K-1} \). Expanding \( h_p(\theta) \) in a Taylor series about \( \theta = 0 \) yields

\[ h_p(\theta) = \left( 1/2^{p-1} \right) \sum_{n=0}^{l-1} s_n((\theta)^m/m!) \]

where
The following result generalises Theorem 2 in [9].

**Theorem 5:** Let $\mathbf{Q}$ be a unimodular $N \times K$ CCM consisting of columns $(x_0, x_1, \ldots, x_{K-1})$ and $M$ a positive integer. Set $L = KM+1$ and extend $\mathbf{Q}$ to a pulse train $T = (x_0, x_1, \ldots, x_{K-1}, x_{K}, \ldots, x_{L-1})$ where

$$x_n = x_{v_n(n)}$$

for all $n = 0, 1, \ldots, L - 1$. Then, the Taylor coefficients $s_m$ of $h_k(\theta)$ vanish up to order $M$, namely

$$s_m = 0$$

for $m = 1, \ldots, M$.

**Proof:** Set $p = k$. It follows from (11) that:

$$s_m = N_m \mathcal{B}_0 = N_m (A_0 + A_1 + \cdots + A_{K-1}) = N_m (ACF_{x_0}(k) + ACF_{x_1}(k) + \cdots + ACF_{x_{K-1}}(k)) = 0$$

for all non-zero $k$. □

Next, we move to the $z$-domain and prove an equivalent version of Theorem 5 by generalising Theorem 2 in [10], which constructs Doppler-tolerant pulse trains in the $z$-domain.

**Theorem 6 [18, 19]:** Let $\mathbf{Q}$ be a unimodular $N \times K$ CCM consisting of columns $(x_0, x_1, \ldots, x_{K-1})$ and $M$ a positive integer. Set $L = K^{M+1}$ and extend $\mathbf{Q}$ to a pulse train $T = (x_0, x_1, \ldots, x_{K-1}, x_{K}, \ldots, x_{L-1})$ where

$$x_n = x_{v_n(n)}$$

for all $n = 0, 1, \ldots, L - 1$. Then, the Taylor coefficients $C_m(z)$ are independent of $z$ up to order $M$, namely

$$C_m(z) = NK^2 P_m$$

for $m = 1, \ldots, M$, where $P_m$ is the $m$th Prouhet sum corresponding to $K$ and $M$. As in [10], we call $T$ a mod-$K$ PTM pulse train of length $L$.

**Proof:** Let $\{S_0, S_1, \ldots, S_{K-1}\}$ be a PTM $K$-block partition of $S = \{0, 1, \ldots, L - 1\}$. It follows from Theorem 2 and Lemma 1 that:

$$C_m(z) = \sum_{n = 0}^{L - 1} n^m |x_n(z)|^2 = \sum_{n \in S_0} n^m |x_{v_0(n)}(z)|^2 + \sum_{n \in S_1} n^m |x_{v_1(n)}(z)|^2 + \cdots + \sum_{n \in S_{K-1}} n^m |x_{v_{K-1}(n)}(z)|^2 = |X_0(z)|^2 \sum_{n \in S_0} n^m + |X_1(z)|^2 \sum_{n \in S_1} n^m + \cdots + |X_{K-1}(z)|^2 \sum_{n \in S_{K-1}} n^m = (|X_0(z)|^2 + |X_1(z)|^2 + \cdots + |X_{K-1}(z)|^2) P_m = NK^2 P_m$$

for $m = 1, 2, \ldots, M$. □

**Example 5:** Examples of PTM pulse trains are given below.

i. Let $K = 2$, $M = 3$, and $(x_0, x_1)$ be a binary $N \times 2$ CCM (Golay pair). Then, the following is a mod-2 PTM pulse train of length $L = 2^3 = 16$:

$$T = (x_0, x_1, x_1, x_0, x_1, x_0, x_1, x_1, x_0, x_1, x_0, x_1, x_1, x_0, x_1, x_0)$$

ii. Let $K = 3$, $M = 2$, and $(x_0, x_1, x_2)$ be a tri-phase $N \times 3$ CCM. Then, the following is a mod-3 PTM pulse train of length $L = 3^3 = 27$:

$$T = (x_0, x_1, x_1, x_0, x_1, x_0, x_1, x_1, x_0, x_1, x_0, x_1, x_1, x_0, x_1, x_0, x_1)$$

iii. Let $K = 4$, $M = 2$, and $(x_0, x_1, x_2, x_3)$ be a unimodular $N \times 4$ CCM. Then, the following is a mod-4 PTM pulse train of length $L = 4^3 = 64$:

$$T = (x_0, x_1, x_1, x_0, x_1, x_0, x_1, x_1, x_0, x_1, x_0, x_1, x_1, x_0, x_1, x_0, x_1)$$

### Doppler-null codes

In this section, we study Doppler-tolerant complementary pulse trains from a different perspective, motivated by the observation that the characterisation of Doppler-null codes bears a striking resemblance to the characterisation of spectral-null codes [23–25].

By demonstrating that Doppler-null codes have higher-order zeros, it follows from a well-known result in calculus that the higher-order derivatives in their ambiguity functions must vanish. This explains our rationale for referring to such codes as Doppler-null codes.

We begin by recalling the classic product generating function formula for the associated Thue–Morse sequence $w(n) = (-1)^{\delta(n)} [27]$:

$$\prod_{k = 0}^{M} (1 - x^k) = \sum_{n = 0}^{M - 1} w(n) x^n. \quad (12)$$

Our goal is to extend (12) to zero-sum sets, which generalises Lehmer’s [13] product generating function formula for polynomials whose coefficients are of roots of unity.

Let $p$ be a positive integer with $p \ge 2$ and $A = \{a_0, a_1, \ldots, a_{p-1}\}$ be a set whose elements sum to zero: $a_0 + a_1 + \cdots + a_{p-1} = 0$. We shall call $A$ a zero-sum set.

**Lemma 2:** Let $A = \{a_0, a_1, \ldots, a_{p-1}\}$ be a zero-sum set. Define the polynomial

$$F_A(x) = a_x + a_p x + a_{2p} x^2 + \cdots + a_{p-1} x^{p-1}.$$ Then

$$F_A(x) = (1 - x) \sum_{k = 0}^{p-1} a_k P_{a,k}(x),$$

where

$$P_{a,k}(x) = x^k \sum_{m = 0}^{M} x^m$$

for $0 \le k \le p - 2$ and $P_{a,p-1}(x) = 0$. 

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Proof: Using the relation \(a_{p-1} = -(a_0 + \cdots + a_{p-2})\), we have
\[
F_n(x) = a_0 + \cdots + a_{p-2}x^{p-2} - (a_0 + a_1 + \cdots + a_{p-2})x^{p-1} \\
= a_1(x^{p-1} + \cdots + a_{p-2}x^{p-2}) + a_0(x + x^2 + \cdots + x^{p-2}) \\
= (1-x)(a_1 + x + \cdots + x^{p-2}) \\
+ a_0(x + \cdots + x^{p-2}) \\
= (1-x)^{\frac{p-1}{2}}(a_1 \sum_{k=0}^{p-2} x^k + a_{p-1}) \\
= (1-x)^k \sum_{k=0}^{p-1} a_k P_{k,k}(x)
\]
where \(k = k_p \) mod \(p\).

Definition 9: Let \(p \geq 2\). Define \(P_{n-1}(x) = 0\) for all \(n \geq 1\) and initialise
\[
P_{k,k}(x) = x^k \sum_{m=0}^{k-2} x^m
\]
for \(0 \leq k \leq p-2\). Then for \(n \geq 2\) and \(0 \leq k \leq p-2\), define the polynomials \(P_{n,k}(x)\) by the recurrence
\[
P_{n,k} = \sum_{j=0}^{k-1} P_{n-1,k-j}(x^{p^{n-j}}) - P_{n-1,p-1,k}(x^{p^{n-1}})P_{n-1,k}(x).
\]
We denote \(A_k = \{a_{k+1}, a_{k+2}, \ldots, a_{k+p-1}\}\) to be the left cyclic shift of \(A\) by \(k\) positions, where \(k = 0, 1, \ldots, p-1\). The following theorem generalises (12). An existence proof was initially given by the first author [28]. However, we shall give a different proof that makes the product expansion (13) explicit: namely, as a linear combination of the elements in the zero-sum set \(A\) in terms of the polynomials \(P_{n,k}\).

Theorem 7: Let \(n\) be a positive integer and \(A = \{a_0, a_1, \ldots, a_{p-1}\}\) a zero-sum set. Define the PTM polynomial \(F(x)\) of degree \(L-1\) to be
\[
F_n(x; A) := a_0 + a_1x + a_2x^2 + \cdots + a_{L-1}x^{L-1}
\]
where \(a_m = a_{(p^m)\text{mod}L}\) and \(L = p^{n}\). Then
\[
F_n(x; A) = Q_n(x)[a_0P_{n,0}(x) + \cdots + a_{p-1}P_{n,p-1}(x)]
\]
(13)
where
\[
Q_n(x) = \sum_{m=0}^{n-1} (1-x^m)
\]
and the polynomials \(P_{n,0}(x), P_{n,1}(x), \ldots, P_{n,p-1}(x)\) are given in Definition 9.

Proof: We first decompose \(F_n(x; A)\) as follows:
\[
F_n(x; A) = \sum_{m=0}^{p^{n-1}-1} a_m x^m + \sum_{m=p^{n-1}}^{2p^{n-1}-1} a_m x^m + \cdots + \sum_{m=(p-1)p^{n-1}}^{p^2p^{n-1}-1} a_m x^m + \sum_{m=(p-1)p^{n-1}}^{p^p-1} a_m x^m
\]
This may be rewritten in terms of lower-order polynomials \(F_{n-1}\)
\[
F_n(x; A) = F_{n-1}(x; A) + x^{p^{n-1}}F_{n-1}(x; A) + \cdots + x^{(p-1)p^{n-1}}F_{n-1}(x; A)
\]
Then by induction on \(n\), we have because of Lemma 2 (which also establishes the base case \(n=1\))
\[
F_n(x; A) = Q_{n-1}(x)[a_0P_{n-1,0}(x) + \cdots + a_{p-1}P_{n-1,p-1}(x)]
\]
\[
+ x^{p^{n-1}}[a_0P_{n-1,0}(x) + \cdots + a_{p-1}P_{n-1,p-1}(x)]
\]
\[
+ \cdots + x^{(p-1)p^{n-1}}[a_0P_{n-1,0}(x) + \cdots + a_{p-1}P_{n-1,p-1}(x)].
\]
Next, we rearrange terms so that
\[
F_n(x; A) = Q_{n-1}(x)[(a_0 + \cdots + a_{p-1}x^{p^{n-1}})P_{n-1,0}(x)]
\]
\[
+ (a_0 + \cdots + a_{p-1}x^{p^{n-1}})P_{n-1,0}(x)
\]
\[
+ \cdots + (a_0 + \cdots + a_{p-1}x^{p^{n-1}})P_{n-1,0}(x)
\]
\[
= Q_{n-1}(x)[F_{n-1}(x^{p^{n-1}}; A_0)P_{n-1,0}(x)]
\]
\[
+ F_{n-1}(x^{p^{n-1}}; A_0)P_{n-1,0}(x)
\]
\[
= Q_{n-1}(x) \cdot R(x),
\]
where
\[
R(x) = \left[\sum_{k=0}^{p-1} a_k P_{k,k}(x^{p^{n-1}})\right]P_{n-1,0}(x) + \cdots + \left[\sum_{k=0}^{p-1} a_k P_{k,k}(x^{p^{n-1}})\right]P_{n-1,0}(x)
\]
and \((k_p) = k \mod p\). We then simplify \(R(x)\) by replacing every \(a_{p-1}\) that appears in it by \(-(a_0 + a_1 + \cdots + a_{p-2})\) and regrouping terms to express \(R(x)\) as a linear combination of \(a_0, \ldots, a_{p-1}\) as follows: (see equation below) Thus
\[
F_n(x; A) = Q_n(x)[a_0P_{n,0}(x) + \cdots + a_{p-1}P_{n,p-1}(x)]
\]
as desired. □

We now use Theorem 7 to reach our goal of showing that the ambiguity function \(g_d(\theta)\) defined in Section 5 has higher-order Doppler nulls.
**Theorem 8:** Let \( T = (X_0, X_1, \ldots, X_{L-1}) \) be a complementary PTM pulse train of length \( L = 2^M + 1 \). Then, \( g_k(\theta) \) has Doppler nulls of order \( M \), i.e. the derivatives

\[
g^{(m)}_k(0) = 0
\]

for \( m = 0, 1, \ldots, M \).

**Proof:** We use Theorem 7 with \( x = e^{j\theta} \) to expand \( g_k(\theta) = g_T(k, \theta) \) as

\[
g_k(\theta) = \sum_{n=0}^{L-1} A_{e^{j\theta}} e^{j\theta n} = \sum_{n=0}^{L-1} A_{n} e^{j\theta n} = \sum_{n=0}^{L-1} a_{\theta} P_{n,\theta}(e^{j\theta}) + \cdots + a_{n-1} P_{n-1,\theta}(e^{j\theta})
\]

Since \( Q_s(x) \) has a zero of order \( M + 1 \) at \( x = 0 \), it follows that \( g_k(\theta) \) has a zero of order \( M + 1 \) at \( \theta = 0 \). Thus

\[
g^{(m)}_k(0) = 0
\]

for \( m = 0, 1, \ldots, M \). \( \square \)

### 7 ESP staggered pulse trains

In this section, we introduce pulse trains, called ESP staggered pulse trains, that provide the same Doppler tolerance as PTM pulse trains, but are generally shorter in length, by using multiple antennas to transmit separate pulse trains staggered in time. We begin with definitions of delayed pulse trains and partitions of arbitrary sets of non-negative integers (not necessarily consecutive as with PTM partitions) having ESPs.

**Definition 10:** We define a delayed pulse train

\[
T(d) = (x_0, x_d, \ldots, x_{L-d})
\]

of length \( L \) as one having a delay of \( d \) pulses in the sense that its ambiguity function has the form

\[
g_T(k, \theta, d) = \sum_{n=0}^{L-1} A_{e^{j\theta}} e^{j\theta (n + d)\theta}
\]

**Definition 11:** Let \( S \) be a set of non-negative integers and \( P = \{S_0, S_1, \ldots, S_{p-1}\} \) be a \( p \)-block partition of \( S \). We say that \( P \) has ESP of degree \( M \) if

\[
\sum_{n \in S_0} n^m = \sum_{n \in S_1} n^m = \cdots = \sum_{n \in S_{p-1}} n^m
\]

for \( m = 1, \ldots, M \). In that case, we define

\[
P_m := P_m(C) = \sum_{n \in S_0} n^m.
\]

The following examples demonstrate our concept of using MIMO radar to transmit ESP pulse trains whose overall transmission period is shorter than PTM pulse trains.

**Example 6 (second-order nulls):** Let \( S = \{0, 1, 2, 4, 5, 6\} \) and consider the 2-block partition \( \mathcal{P} = (S_0, S_1) \) of \( S \), where \( S_0 = \{0, 4, 5\} \) and \( S_1 = \{1, 2, 6\} \). Then \( \mathcal{P} \) has ESP of degree 2 since

\[
0^0 + 4^0 + 5^0 = 1 + 2 + 6
\]

\[
0^1 + 4^1 + 5^1 = 1^1 + 2^1 + 6^1
\]

Observe that this partition consists of only six values (skipping the value \( 3 \)) and is smaller in size than the 2-block PTM partition of \( \{0, 1, \ldots, 7\} \). Then, given a Golay pair of codes \((x_0, x_1)\), we can of course construct a single pulse train based on the partition above by inserting a gap or fill pulse for the value at position 3:

\[
T = (x_0, x_1, \ldots, x_0, x_0, x_0, x_0)
\]

This approach however is impractical in terms of transmission. On the other hand, we can modify the partition \( \mathcal{P} \) so that it includes the value 3 in both sets

\[
S_0 = \{0, 3, 4, 5\}
\]

\[
S_1 = \{1, 2, 3, 6\}
\]

Note that \( \mathcal{P} \) is no longer a collection of mutually disjoint sets, but continues to have ESP of degree 2. Suppose we then transmit two separate pulse trains of length 4, \( T_0 \) and \( T_1 \) (each from a separate antenna), but staggered in the sense that we delay the transmission of \( T_1 \) by 3 pulses as follows:

\[
T_0 = (x_0, x_0, x_0, x_0)
\]

\[
T_1(3) = (x_0, x_0, x_0, x_0)
\]

Here, \( T_0 \) transmits pulses corresponding to the first two values of \( S_0 \) (positions 0 and 3) and the first two values of \( S_1 \) (positions 1 and 2). Similarly for \( T_1(3) \), but corresponding to the last two values of \( S_0 \) and \( S_1 \). If we sum the composite ACFs of both pulse trains, then we obtain

\[
g(k, \theta) = g_{T_0}(k, \theta) + g_{T_1}(k, \theta, 3) = ACF_{S_0}(k) + ACF_{S_1}(k) + ACF_{S_0}(k) e^{j\theta} + ACF_{S_1}(k) e^{j\theta}
\]

To show that \( g(k, \theta) \) has Doppler nulls of order 2 at \( \theta = 0 \), we compute its Doppler (Taylor) coefficients

\[
c_m(k) = g^{(m)}(k, 0) = (0^m + 3^m + 4^m + 5^m)ACF_{S_0}(k) + (1^n + 2^m + 3^m + 6^m)ACF_{S_1}(k)
\]

\[
= 2P_m(ACF_{S_0}(k) + ACF_{S_1}(k)) = 2NP_m\delta_k
\]

for \( m = 0, 1, 2 \). This demonstrates that we can achieve the same Doppler tolerance as with a single PTM pulse train of length 8 by using instead two staggered (but overlapping) pulse trains of length 4 to reduce the total transmission time from 8 pulses down to 7 pulses. Note however that the total number of pulses transmitted is the same: namely 8, in both cases.

**Example 7 (third-order nulls):** Consider the following 2-block partition \( \mathcal{P} = (S_0, S_1) \), where:

\[
S_0 = \{0, 4, 7, 11\}
\]

\[
S_1 = \{1, 2, 9, 10\}
\]

which has ESP of degree 3: namely

\[
0^0 + 4^0 + 7^0 + 11^0 = 1^m + 2^m + 9^m + 10^m
\]

for \( m = 0, 1, 2, 3 \). As in the previous example, we modify this partition so that both sets \( S_0 \) and \( S_1 \) contain each of the values 3, 5, 6, and 8
\[ S_0 = (0, 3, 4, 5, 6, 7, 8, 11), S_1 = (1, 2, 3, 5, 6, 8, 9, 10) \]

We now transmit 4 pulse trains \( T_{10}, T_{11}(3), T_{12}(5), T_{13}(8) \) on separate antennas having delays 0, 3, 5, 8, respectively.

\[
T_0 = (x_0, x_0, x_0, \ldots) \\
T_{11}(3) = (x_0, x_0, x_0, x_0) \\
T_{12}(5) = (x_0, x_0, x_0, x_0) \\
T_{13}(8) = (x_0, x_0, x_0, x_0)
\]

Then, it can be shown that the Doppler coefficients of the composite ambiguity function \( g(k, \theta) \) has Doppler nulls of order 3:

\[
c_m(k) = (0^m + 3^m + 4^m + 5^m + 6^m + 7^m + 8^m + 11^m)ACF_{x_0}(k) + (1^m + 2^m + 3^m + 5^m + 6^m + 8^m + 9^m + 10^m)ACF_{x_0}(k) \\
= P_m(ACF_{x_0}(k) + ACF_{x_0}(k)) = 2N P_{F_k}
\]

for \( m = 0, 1, 2, 3 \). Thus, we have reduced the total transmission time from 16 pulses (for a single PTM pulse train of length 16 having the same Doppler tolerance) down to 12 by using instead 4 pulse trains transmitted separately. Again, note that the total number of pulses transmitted is the same (16) in both cases.

**Example 8 (fifth-order nulls):** Consider the following 2-block partition \( \mathcal{P} = (S_0, S_1) \) which has ESP of degree 5:

\[
S_0 = (0, 5, 6, 16, 17, 22) \\
S_1 = (1, 2, 10, 12, 20, 21)
\]

We again modify this partition to include the values \( \{3, 4, 7, 8, 9, 11, 13, 14, 15, 18, 19\} \) without changing its degree.

\[
S_0 = (0, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15, 16, 17, 18, 19, 22) \\
S_1 = (1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 20, 21)
\]

We then transmit 7 pulse trains \( T_{10}, T_{11}(3), T_{12}(7), T_{13}(8), T_{13}(11), T_{13}(13), \) and \( T_{13}(18) \) having delays 0, 3, 7, 8, 11, 13, and 18, respectively.

Again it can be shown that the Doppler coefficients of the composite ambiguity function \( g(k, \theta) \) have Doppler nulls of order 5. Thus, we have reduced the total transmission time from 64 pulses (for a single PTM pulse train of length 64 having the same Doppler tolerance) down to 23 by using instead 7 pulse trains transmitted by separate antennas. Unlike Examples 6 and 7, the total number of pulses transmitted for all 7 staggered pulse trains is only 34 in comparison to 64 for a single PTM pulse train. We observe that the three pulse trains \( T_3(7), T_3(8), \) and \( T_3(13) \) are constant in value.

### 8 References


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