Variational approach to differential equations with not instantaneous impulses

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1. Introduction

Non-instantaneous impulsive differential equations were introduced by Hernández & O’Regan in [1], motivated by a simplified situation concerning the hemodynamical equilibrium of a person. This type of equations is a generalization of the classical theory of impulsive differential equations. For some general and recent works on the theory of impulsive differential equations we refer the readers to [2–10].

The existence of solutions of non-instantaneous impulsive problem has been studied via some approaches, such as fixed point theory and theory of analytic semigroup, see, for example, [1,11–15]. The variational structure of general non-instantaneous impulsive problem has not been yet developed and critical point theory, to the best of our knowledge, has not been used to consider this kind of problems.

In this note we present the variational structure associated to the following linear problem with not instantaneous impulses
\[
\begin{aligned}
\begin{cases}
-u''(t) = \sigma_i(t), & t \in (s_i, t_{i+1}], i = 0, 1, \ldots, N, \\
u'(t) = \alpha_i, & t \in (t_i, s_i], i = 1, 2, \ldots, N, \\
u'(s_i^+) = u'(s_i^-), & i = 1, 2, \ldots, N, \\
u(0) = u(T) = 0, & u'(0) = \alpha_0,
\end{cases}
\end{aligned}
\]

where \(0 = s_0 < t_1 < s_1 < t_2 < s_2 < \cdots < t_N < s_N < t_{N+1} = T\), the impulses start abruptly at the points \(t_i\) and keep the derivative constant on a finite time interval \((t_i, s_i]\), \(\sigma_i \in L^2((s_i, t_{i+1}), \mathbb{R})\) and \(\alpha_i\) are given constants. Here \(u'(s_i^+) = \lim_{s \to s_i^+} u'(s)\).

2. Preliminaries

Let \(H^1_0(0, T)\) be the Sobolev space endowed with the inner product

\[
(u, v) = \int_0^T u'(t)v'(t)dt
\]

and the corresponding norm

\[
\|u\| = \left(\int_0^T |u'(t)|^2 dt\right)^{1/2}.
\]

Obviously, \(H^1_0(0, T)\) is a Hilbert space. It is a consequence of Poincaré’s inequality that

\[
\int_0^T |u(t)|^2 dt \leq \frac{1}{\lambda_1} \int_0^T |u'(t)|^2 dt,
\]

where \(\lambda_1 = \pi^2/T^2\). Let \(\|u\|_\infty = \max_{t \in [0, T]} |u(t)|\), then

\[
\|u\|_\infty \leq \beta \|u\|, \quad \text{for every } u \in H^1_0(0, T),
\]

where \(\beta = (T\lambda_1)^{-1/2} + T^{1/2}\). In fact, it follows from the mean value theorem that \(\frac{1}{T} \int_0^T u(s)ds = u(\tau)\) for some \(\tau \in (0, T)\). Hence, for \(t \in [0, T]\), using Hölder inequality,

\[
|u(t)| = \left| u(\tau) + \int_\tau^t u'(s)ds \right| \\
\leq \frac{1}{T} \left| \int_0^T u(s)ds \right| + \int_0^T |u'(t)|dt \leq T^{-1/2}\|u\|_{L^2} + T^{1/2}\|u'\|_{L^2}.
\]

**Lemma 1** ([16, Theorem 3.2, Lax–Milgram]). Let \(H\) be a real Hilbert space. Let \(a : H \times H \to \mathbb{R}\) be a bounded bilinear form. If \(a\) is coercive, i.e., there exists \(\delta > 0\) such that \(a(u, u) \geq \delta|u|^2\) for every \(u \in H\), then for any \(f \in H'\) (the dual of \(H\)) there exists a unique \(u \in H\) such that

\[
a(u, v) = \langle f, v \rangle, \quad \text{for every } v \in H.
\]

Moreover, if \(a\) is also symmetric, then the functional \(\varphi : H \to \mathbb{R}\) defined by

\[
\varphi(v) = \frac{1}{2}a(v, v) - \langle f, v \rangle
\]

attains its minimum at \(u\).
3. Main result

Following the ideas of the variational approach to impulsive differential equations of [3,4], for each \( v \in H^1_0(0,T) \), we have

\[
\int_0^T u''(t)v(t)dt = \int_0^{t_1} u''(t)v(t)dt + \sum_{i=1}^{N} \int_{t_i}^{s_i} u''(t)v(t)dt + \sum_{i=1}^{N-1} \int_{s_i}^{t_{i+1}} u''(t)v(t)dt + \int_{s_N}^{T} u''(t)v(t)dt
\]

\[
= -\int_0^T u'(t)v'(t)dt + \sum_{i=1}^{N} \left[ u'(t^-_i) - u'(t^+_i) \right] v(t_i) + \sum_{i=1}^{N} \left[ u'(s^-_i) - u'(s^+_i) \right] v(s_i),
\]

which combined with (1) yields to

\[
\int_0^T u''(t)v(t)dt = -\int_0^T u'(t)v'(t)dt + \sum_{i=1}^{N} (\alpha_{i-1} - \alpha_i) v(t_i) - \sum_{i=0}^{N-1} \int_{s_i}^{t_{i+1}} \sigma_i(t)dv(t_{i+1}).
\]

(3)

On the other hand,

\[
\int_0^T u''(t)v(t)dt = -\sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} \sigma_i(t)v(t)dt + \sum_{i=1}^{N} \int_{t_i}^{s_i} \frac{d}{dt} [\alpha_i]v(t)dt
\]

\[
= -\sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} \sigma_i(t)v(t)dt.
\]

(4)

Thus, in view of (3), (4) and \( v(t_{N+1}) = v(T) = 0 \), we have

\[
\int_0^T u'(t)v'(t)dt = \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} \sigma_i(t)(v(t) - v(t_{i+1}))dt + \sum_{i=1}^{N} (\alpha_{i-1} - \alpha_i) v(t_i).
\]

(5)

Considering the aforementioned equality, we introduce the following concept of weak solution for (1).

**Definition 1.** A function \( u \in H^1_0(0,T) \) is a weak solution of (1) if (5) holds for any \( v \in H^1_0(0,T) \).

Consider the functional \( \Phi : H^1_0 \rightarrow \mathbb{R} \) defined by

\[
\Phi(u) = \frac{1}{2} \int_0^T |u'(t)|^2 dt - \sum_{i=1}^{N} (\alpha_{i-1} - \alpha_i) u(t_i) - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} \sigma_i(t)(u(t) - u(t_{i+1}))dt.
\]

It is clear that \( \Phi \in C^1(H^1_0, \mathbb{R}) \) and

\[
\langle \Phi'(u), v \rangle = \int_0^T u'(t)v'(t)dt - \sum_{i=1}^{N} (\alpha_{i-1} - \alpha_i) v(t_i) - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} \sigma_i(t)(v(t) - v(t_{i+1}))dt.
\]

Thus critical points of \( \Phi \) correspond to weak solutions of the problem (1).
Defining

\[ a : H^1_0(0, T) \times H^1_0(0, T) \to \mathbb{R}, \quad a(u, v) = \int_0^T u'(t)v'(t)dt \]

and \( l : H^1_0(0, T) \to \mathbb{R}, \)

\[ l(v) = \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} \sigma_i(t)(v(t) - v(t_{i+1}))dt + \sum_{i=1}^{N} (\alpha_{i-1} - \alpha_i) v(t_i), \]

we see that finding weak solutions of (1) is equivalent to the problem of finding \( u \in H^1_0(0, T) \) such that

\[ a(u, v) = l(v), \quad \text{for every } v \in H^1_0(0, T). \]

It is evident that \( a \) is coercive, bilinear, bounded and symmetric, and that \( l \) is linear and bounded. In fact, using Hölder inequality and (2), we have

\[ |l(v)| \leq \sum_{i=0}^{N} \|\sigma_i\|_{L^2(s_i, t_{i+1})} \left( \int_{s_i}^{t_{i+1}} |v(t) - v(t_{i+1})|^2dt \right)^{\frac{1}{2}} + \sum_{i=1}^{N} |\alpha_{i-1} - \alpha_i| \|v(t_i)\| \]

\[ \leq 2 \sum_{i=0}^{N} \|\sigma_i\|_{L^2(s_i, t_{i+1})} \left( \int_{s_i}^{t_{i+1}} |v(t)|^2dt \right)^{\frac{1}{2}} + \sum_{i=1}^{N} |\alpha_{i-1} - \alpha_i| \|v\|_{\infty} \]

\[ \leq \beta \left[ 2 \sum_{i=0}^{N} \|\sigma_i\|_{L^2(s_i, t_{i+1})} \frac{1}{2} + \sum_{i=1}^{N} |\alpha_{i-1} - \alpha_i| \right] \|v\|. \]

Thus, by the Lax–Milgram theorem, we have

**Theorem 1.** Non-instantaneous impulsive problem (1) has a unique weak solution \( u \in H^1_0(0, T) \) for any \( \sigma_i \in L^2(s_i, t_{i+1}) \). Moreover, \( u \) minimizes the functional \( \Phi(u) \).

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**References**

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