Feature selection and approximate reasoning of large-scale set-valued decision tables based on $\alpha$-dominance-based quantitative rough sets

Hong-Ying Zhang*, Shu-Yun Yang

School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an, Shaan’xi 710049, P.R. China

Abstract

Set-valued data are a common type of data for characterizing uncertain and missing information. Traditional dominance-based rough sets can not efficiently deal with large-scale set-valued decision tables and usually neglect the disjunctive semantics of sets. In this paper, we propose a general framework of feature selection and approximate reasoning for large-scale set-valued information tables by integrating quantitative rough sets and dominance-based rough sets. Firstly, we define two new partial orders for set-valued data via the conjunctive and disjunctive semantics of a set. Secondly, based on $\alpha$-disjunctive dominance relation and $\alpha$-conjunctive dominance relation defined by the inclusion measure, we present $\alpha$-dominance-based quantitative rough set models for these two types of set-valued decision tables. Furthermore, we study the issue of feature selection in set-valued decision tables by employing $\alpha$-dominance-based quantitative rough set models and discuss the relationships between the relative reductions and discernibility matrices. We also present approximate reasoning models based on $\alpha$-dominance-based quantitative rough sets. Finally, the application of the approach is illustrated by some real-world data sets.

© 2016 Elsevier Inc. All rights reserved.

1. Introduction

The classical rough set theory, proposed by Pawlak [37,38], defines a pair of lower and upper approximations or equivalently three pair-wise disjoint positive, negative and boundary regions of a given set by using the set-inclusion relation and the nonempty set-overlapping condition. Rough sets evaluate the significance of attributes and derive the decision rules. The approximation operators of a Pawlak rough set ensure that both positive and negative regions are error-free. That is, no acceptance and rejection errors are allowed during the process of decision making. While we should allow some degrees of errors in dealing with large-scale data sets for the sake of flexible and practicability. Several generalized quantitative rough set models, which aim at making an acceptable decision for large-scale data sets, have been proposed based on an inclusion measures. They may be broadly classified into probabilistic and non-probabilistic approaches.

Probabilistic rough sets [39,53,54,60–65,76] quantify the set inclusion relation by a conditional probability, where the rough set approximations are presented by thresholds on the probability. Pawlak et al. [39] and Wong and Ziarko [53,54] first proposed probabilistic rough sets and introduced several models such as a 0.5-model [39,53] and a 0.5-$\beta$-model [55]. To...
overcome the lack of the interpretation and computation of the thresholds, Yao et al. [64,65] presented decision-theoretic rough sets for interpreting and computing a pair of thresholds based on a Bayesian decision theory. Yao [62] also proposed an approach to derive three-way decision rules based on decision-theoretic rough sets. The probabilistic rough sets are more general and flexible, which can be used to treat the objects from much larger universe and more insensitive to noises [63].

Non-probabilistic rough set models [9,28,40,48,58,66,71,72,74] are presented by considering other kinds of inclusion measures. Zhang and Leung [72,74] introduced the notion of inclusion degree to quantify the partial orders, which had been used to study non-probabilistic rough sets [58,71]. Xu et al. [58] investigated that many measures in rough sets, including accuracy measures and measures of dependence of attributes, are inclusion measures. Polkowski and Skowron [40,48] presented rough inclusion as a measure of subshet relation. Gomolińska [9] gave a systematic research on rough-inclusion functions and their relationships with similarity measures and distance metrics. But the aforementioned studies about probabilistic and non-probabilistic approaches have not been unified, meanwhile insufficient attention has been paid to quantitative rough sets of non-probabilistic approaches. To fill this gap, Yao and Deng [66] initiated a study of quantitative rough set model which defines three regions by using inclusion measures with two thresholds and encompasses both probabilistic and non-probabilistic rough sets.

Set-valued information tables (SITs) [2,4,13,26,29,30,42,43], which can well characterize uncertainty information and missing information by using set-valued attributes. An incomplete information table can be regarded as a special kind of SITs [5,12] in which an object with a missing attribute value which can be replaced with the set of all possible values for this attribute. SITs have been analyzed by dominance-based rough sets [6,10,11,17,18,24,50,51,56,75] and fuzzy rough sets [4,5,8,16]. For instance, Shao and Zhang [47] presented a dominance-based rough set approach to reason in incomplete ordered information tables. Combining with fuzzy set theory [67], Dai and Tian [5] defined a fuzzy relation and constructed a fuzzy rough set model for SITs. Considering the ordered relations in SITs, Qian et al. [43,44] presented a rough set approach in set-valued ordered information tables (SOITs). Fan et al. [8] presented a dominance-based fuzzy rough set model for the decision analysis of an ordered uncertain and possibilistic data table. Greco et al. [10,11] presented a new fuzzy rough set approach based on the ordinal properties of fuzzy membership degrees. Incremental approaches of updating approximations and fast algorithms for computing approximations in set-valued approximations were proposed in [26,29,30]. Moreover, Inuiguchi et al. [17] introduced a variable-precision dominance-based rough set approach and studied the attribute reduction. Zhang et al. [70] proposed a general framework for the study of interval-valued decision tables by integrating the variable-precision-dominance-based rough set theory and the inclusion measure theory. However, disjunctive and conjunctive semantics of set-valued data have not been fully explored and the relative quantitative rough set models on SITs have not been studied. In this paper, on the basis of two new partial orders, we propose an $\alpha$-dominance quantitative rough set model of SITs based on inclusion measures.

The rest of this paper is organized as follows. In Section 2, we mainly review the related work about inclusion measures and quantitative rough set model. In Section 3, we present two partial orders based on disjunctive and conjunctive semantics of a set. Furthermore, some inclusion measures on the partially ordered sets are constructed and a ranking model is proposed. In Section 4, we formulate an $\alpha$-dominance-based quantitative rough set approach based on the inclusion measure and further study its properties. In Section 5, we investigate a framework of knowledge acquisition for SITs on the basis of $\alpha$-dominance-based quantitative rough set model. Attribute reduction and the relationship between the reductions and the discernibility matrices are also analyzed. In Section 6, we further discuss approximate reasoning based on quantitative rough sets. In Section 7, we show some numerical experiments to demonstrate the application of the proposed approach. The paper is then ended with conclusions.

2. Preliminaries

In this section, we review some concepts of Pawlak’s rough sets, dominance-based rough sets, inclusion measures and quantitative rough sets. One can refer to [10,37,38,46,66] for details.

2.1. Pawlak’s rough sets

The theory of Pawlak rough sets [37,38] deals with inconsistent problems by separating of certain and doubtful knowledge. Now we present a slightly different formulation of Pawlak lower and upper approximations and the positive, negative and boundary regions by using the set-inclusion relation [62,63,66].

Let $U$ be a finite nonempty set of objects and $R \subseteq U \times U$ be an equivalence relation on $U$, that is, $R$ is reflexive, symmetric and transitive. The equivalence class containing $x$ is defined as $[x]_R = \{y \in U | xEy\}$. The family of all the equivalence classes of $R$ denoted by $U/R$ constitutes a partition of $U$. An information table (IT) is a quadruple $S = (U, At, V, F)$ where $U$ is a finite nonempty set of objects and $At$ is a finite nonempty set of attributes. In $S$, the relation functions between the set of objects and the set of attributes are defined as $f_a: U \rightarrow V_a$ for any $a \in At$ where $V_a$ is called the domain of an attribute $a$.

**Definition 2.1.** Let $S = (U, At, V, F)$ be an IT and $X \subseteq U$. The lower and upper approximations of $X$ are defined by

$$R(X) = \{x \in U | [x] \subseteq X\},$$

$$\bar{R}(X) = \{x \in U | \neg([x] \subseteq X')\}$$

Please cite this article as: H.-Y. Zhang, S.-Y. Yang, Feature selection and approximate reasoning of large-scale set-valued decision tables based on $\alpha$-dominance-based quantitative rough sets, Information Sciences (2016), http://dx.doi.org/10.1016/j.ins.2016.06.028
And the positive, negative and boundary regions of $X$ are defined as follows:

$$\begin{align*}
\text{POS}(X) &= \mathcal{R}(X) = \{ x \in U | [x] \subseteq X \}, \\
\text{NEG}(X) &= (\mathcal{R}(X))^c = \{ x \in U | [x] \subseteq X^c \}, \\
\text{BND}(X) &= (\text{POS}(X) \cup \text{NEG}(X))^c \\
&= (\text{POS}(X))^c \cap (\text{NEG}(X))^c \\
&= \{ x \in U | \neg ([x] \subseteq X) \land \neg ([x] \subseteq X^c) \}.
\end{align*}$$

### 2.2. Dominance-based rough sets

By replacing the equivalent relation with a dominance one and taking the ordering properties of attributes into account, Greco et al. [10,11] proposed the dominance-based rough sets.

Let $S = (U, A^f, V, F)$ be an IT and $A \subseteq A^f$. The dominance class of an object $x$ with respect to $A$, induced by dominance relation $\geq$, is the set of the objects dominating $x$. That is,

$$[x]^\geq_A = \{ y \in U | f_a(y) \geq f_a(x), \ \forall a \in A \}.$$

For any $X \subseteq U$ and $A \subseteq A^f$, the lower and upper approximations of $X$ with respect to the dominance relation $R^\geq_A$ are defined as follows:

$$\begin{align*}
\overline{R^\geq_A}(X) &= \{ x \in U | [x]^\geq_A \subseteq X \}, \\
\underline{R^\geq_A}(X) &= \{ x \in U | \neg ([x]^\geq_A \subseteq X^c) \}.
\end{align*}$$

In Pawlak rough sets and dominance-based rough sets, the lower and upper approximations are constructed by set-inclusion relation $\subseteq$, which can be quantitatively generalized by an inclusion measure. In what follows, we review some notions about the inclusion measures.

### 2.3. Inclusion measures

Inclusion measures [46], also known as subsheath measures, describe the extent of one element of a poset contained in another one of a partially ordered set (poset). The inclusion measure theory describes the quantification of pair-wise inclusion relation between any two elements of a poset. Many papers have paid attention to the study of inclusion measures. Kitanik [22] first presented four axioms on inclusion measures based on the properties of crisp inclusion relations. Zhang and Leung [72] proposed a generalized quantitative method for inclusion measures to deal with the uncertainty inference in the field of artificial intelligence. Zhang and Zhang [69] proposed a hybrid monotonic inclusion measure (HM inclusion measure) which preserves the monotonicity of two variables in terms of partial orders. Furthermore, the HM inclusion measure is applied in measuring similarity and distance between fuzzy sets. Zhang et al. [70] used HM inclusion measure to present the variable-precision-dominance-based rough set model and further investigated the issues of knowledge acquisition and attribute reduction of interval-valued information tables. Xu et al. [58] proved that many methods in the rough set theory are inclusion measures. Mi et al. [34] studied knowledge acquisition based on a variable-precision rough set model defined by a particular inclusion measure. Based on fuzzy inclusion measures, distribution reduction, maximum distribution reduction and lower and upper approximation reduction of fuzzy information systems were presented in [4]. Let $\leq$ is a partial order on $\mathcal{P}(0, 1)$, where $\mathcal{P}(0, 1)$ denotes the power set of the interval [0, 1]. Then the HM inclusion measure for poset $\mathcal{P}(0, 1)$, $\leq$, is presented in Definition 2.2.

**Definition 2.2** (HM inclusion measure). Let $A$ and $B \in (\mathcal{P}(0, 1), \leq)$. A real number $\text{Inc}(A, B) \in [0, 1]$ is called an HM inclusion measure between $A$ and $B$, if $\text{Inc}(A, B)$ satisfies the following properties:

\begin{align*}
(11) &\quad 0 \leq \text{Inc}(A, B) \leq 1, \\
(12) &\quad \text{if } A \leq B, \text{ then } \text{Inc}(A, B) = 1, \\
(13) &\quad \text{if } A = \{1\} \text{ with } \{1\} \text{ being the maximal element of the poset } (\mathcal{P}(0, 1), \leq), \text{ then } \text{Inc}(A, A^c) = 0, \\
(14) &\quad \text{if } A \leq B, \text{ then for any } C \in \mathcal{P}(0, 1), \text{ we have } \text{Inc}(C, A) \leq \text{Inc}(C, B) \text{ and } \text{Inc}(B, C) \leq \text{Inc}(A, C).
\end{align*}

An inclusion measure $\text{Inc}$ is called a $\mathcal{T}$-transitive inclusion measure with $\mathcal{T}$ being a $t$-norm on $\mathcal{P}(0, 1)$ [49], if $\text{Inc}$ satisfies the $\mathcal{T}$-transitivity, i.e., $\forall A, B, C \in \mathcal{P}(0, 1)$,

$$\sup_{B \in \mathcal{P}(0, 1)} \mathcal{T}(\text{Inc}(A, B), \text{Inc}(B, C)) \leq \text{Inc}(A, C).$$

**Remark 2.1**. An inclusion measure $\text{Inc}$ is called maximal inclusion measure if it satisfies $\text{Inc}(X_1 \cup X_2, Y) = \min\{\text{Inc}(X_1, Y), \text{Inc}(X_2, Y)\}$, $\forall X_1, X_2, Y \in \mathcal{P}(0, 1)$.

**Remark 2.2**. An inclusion measure $\text{Inc}$ is self-dual, if $\text{Inc}(X, Y) + \text{Inc}(X, Y^c) = 1$, for any $X, Y \in \mathcal{P}(0, 1)$. 

---

Please cite this article as: H.-Y. Zhang, S.-Y. Yang, Feature selection and approximate reasoning of large-scale set-valued decision tables based on $\alpha$-dominance-based quantitative rough sets, Information Sciences (2016), http://dx.doi.org/10.1016/j.ins.2016.06.028
2.4. Quantitative rough sets based on inclusion measures

Qualitative representation of Pawlak rough sets ensures that both positive and negative regions are errorfree. Such a rigorous requirement makes the analysis of rough set model easier, which restricts the applicability and flexibility. Quantitative rough sets are effective quantitative generalization of Pawlak rough sets via inclusion measures. The framework of quantitative rough sets is defined as follows.

**Definition 2.3** [66]. Let $\text{Inc} : \mathcal{P}(U) \times \mathcal{P}(U) \rightarrow [0, 1]$ be an inclusion measure. For a pair of thresholds $(p, q)$ satisfying the following property: $\forall x \in U$ and $\emptyset \neq X \in \mathcal{P}(U)$,

$$\text{Inc}([x], X) \geq p \Rightarrow \text{Inc}([x], X^c) < q,$$

then the three regions of quantitative rough set are defined by

$\text{POS}_{(p, q)}(X) = \{ x \in U | \text{Inc}([x], X) \geq p \}$.

$\text{NEG}_{(p, q)}(X) = \{ x \in U | \text{Inc}([x], X^c) \geq q \}$.

$\text{BND}_{(p, q)}(X) = \{ x \in U | \text{Inc}([x], X) < p \land \text{Inc}([x], X^c) < q \}$.

An inclusion measure $\text{Inc}$ is self-dual for $A, B \in \mathcal{P}(U)$ and $A \neq \emptyset$, we have

$$\text{Inc}(A, B) \geq p \Leftrightarrow 1 - \text{Inc}(A, B^c) \geq p \Leftrightarrow \text{Inc}(A, B^c) \leq 1 - p$$

Furthermore, Yao and Deng [66] proposed a framework of quantitative rough sets based on a self-dual inclusion measure as follows:

**Definition 2.4** [66]. Suppose that $\text{Inc} : \mathcal{P}(U) \times \mathcal{P}(U) \rightarrow [0, 1]$ is an inclusion measure satisfying self-duality. For a pair of thresholds $(p, q)$, $0 \leq q < p \leq 1$, $X \subseteq U$, the three regions of $X$ are defined as

$\text{DPOS}_{(p, q)}(X) = \{ x \in U | \text{Inc}([x], X) \geq p \} = \text{POS}_{(p, \cdot)}(X)$.

$\text{DNEG}_{(p, q)}(X) = \{ x \in U | \text{Inc}([x], X^c) \leq q \} = \text{NEG}_{(\cdot, 1-q)}(X)$.

$\text{DBND}_{(p, q)}(X) = \{ x \in U | q < \text{Inc}([x], X) < p \} = \text{BND}_{(p, 1-q)}(X)$

3. $\alpha$-dominance relation in conjunctive and disjunctive set-valued information tables

Let $S = (U, At, V, F)$ be an IT if all objects have set-valued attribute values, it is called a set-valued information table (SIT). A set-valued decision information table (SDIT) $(U, At \cup d, V, F)$ is a special case of a SIT, where $S = (U, At, V, F)$ is a SIT and $d$ is a single-valued decision attribute.

**Example 3.1.** A SIT $S = (U, At, V, F)$ is presented in Table 1, where $U = \{x_1, \ldots, x_{10}\}$ and $At = \{a_1, \ldots, a_5\}$.

3.1. Partial orders on conjunctive and disjunctive set-valued information tables

There are many ways to give a semantic interpretation of SITs [2,3,7,27,35,36]. Here we summarize two types of them [43,44].

**Type 1:** For a SIT $S = (U, At, V, F)$. $x \in U$ and $a \in At$, $a(x)$ is interpreted via the conjunctive semantics of a set. For example, if $a$ is the attribute “speaking a language”, then $a(x) = \{\text{German; Polish; French}\}$ can be interpreted as, $x$ speaking German, Polish, and French. When considering the attribute “feeding habits” of animals, if we denote the attribute value of herbivore as “0” and carnivore as “1”, then the animals possessing attribute value “0; 1” are considered as possessing both herbivorous and carnivorous nature. Let us take blood origin for another example. If we denote the three types of pure blood as “0”, “1” and “2”, then we can denote the mixed-blood as “0; 1” or “1; 2,” etc.

---

**Table 1**

A set-valued decision table.

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$[0.5]$</td>
<td>$(0, 0.5)$</td>
<td>$[0]$</td>
<td>$(0.5, 1)$</td>
<td>$(1)$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$(0, 0.5)$</td>
<td>$(1)$</td>
<td>$(0.5, 1)$</td>
<td>$(0)$</td>
<td>$(0)$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$(0)$</td>
<td>$(0.5, 1)$</td>
<td>$(0.5)$</td>
<td>$(0, 0.5)$</td>
<td>$(0)$</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$(0)$</td>
<td>$(0.5)$</td>
<td>$(0.5)$</td>
<td>$(0)$</td>
<td>$(0, 0.5)$</td>
</tr>
<tr>
<td>$x_5$</td>
<td>$(1)$</td>
<td>$(0.5)$</td>
<td>$(0.5)$</td>
<td>$(0)$</td>
<td>$(0, 0.5)$</td>
</tr>
<tr>
<td>$x_6$</td>
<td>$(0, 1)$</td>
<td>$(0.5)$</td>
<td>$(0.5)$</td>
<td>$(0)$</td>
<td>$(0.5)$</td>
</tr>
<tr>
<td>$x_7$</td>
<td>$(0.5)$</td>
<td>$(0, 1)$</td>
<td>$(0.5)$</td>
<td>$(1)$</td>
<td>$(1)$</td>
</tr>
<tr>
<td>$x_8$</td>
<td>$(0)$</td>
<td>$(1)$</td>
<td>$(0.5)$</td>
<td>$(0)$</td>
<td>$(0, 0.5)$</td>
</tr>
<tr>
<td>$x_9$</td>
<td>$(0.5)$</td>
<td>$(0, 0.5)$</td>
<td>$(0)$</td>
<td>$(0.5)$</td>
<td>$(1)$</td>
</tr>
<tr>
<td>$x_{10}$</td>
<td>$(0.5)$</td>
<td>$(5)$</td>
<td>$(0.5)$</td>
<td>$(1)$</td>
<td>$(0, 0.5)$</td>
</tr>
</tbody>
</table>

---

Please cite this article as: H.-Y. Zhang, S.-Y. Yang, Feature selection and approximate reasoning of large-scale set-valued decision tables based on $\alpha$-dominance-based quantitative rough sets, Information Sciences (2016), http://dx.doi.org/10.1016/j.ins.2016.06.028
Type 2: For a SIT $S = (U, At, V, F)$, $x \in U$ and $a \in At$, $a(x)$ is interpreted via the disjunctive semantics of a set. For example, if $a$ is the attribute “speaking a language”, then $a(x) = \{\text{German; Polish; French}\}$ can be interpreted as $x$ speaking either German or Polish or French, i.e., $x$ speaking only one of the languages. Incomplete information systems with some unknown or partially known attribute values [25] are such type of SITs.

In what follows, two partial orders defined by disjunctive and conjunctive semantics, will be proposed. First, we introduce the definition of a partial order.

**Definition 3.1** [41]. A partial order $\leq$ on a set $P$ is a binary relation satisfying

- (po1) reflexivity: for all $x \in P, x \leq x$,
- (po2) antisymmetry: for all $x, y \in P, x \leq y$ and $y \leq x$ imply $x = y$,
- (po3) transitivity: for all $x, y, z \in P, x \leq y$ and $y \leq z$ imply $x \leq z$.

Relaxing the conditions by dropping antisymmetry leads to what is known as a pre-order. Namely, a pre-order is reflexive and transitive.

For all $A \in \mathcal{P}([0, 1])$, the number of its elements or the cardinality of $A$ is denoted as $|A|$.

**Assumption 3.1.** For all $A \in \mathcal{P}([0, 1])$, the $i$th element of $A$ is denoted as $A(i)$ and the values of $A$ is arranged by increasing order, i.e.,

$$
A(1) \leq A(2) \leq \ldots \leq A(|A|).
$$

Obviously, $A(1) = A^{-}$, $A(|A|) = A^{+}$, where $A^{-}$ and $A^{+}$ denote the minimal and maximal element of $A$ respectively. Now we propose a partial order on $\mathcal{P}([0, 1])$ based on the basis of conjunctive and disjunctive semantics.

**Definition 3.2.** For $A, B \in \mathcal{P}([0, 1])$, $A \leq_{\wedge} B$ is defined as follows:

$A \leq_{\wedge} B$ if and only if there exist $B(j_i) \subseteq B$ such that $|B(j_i)| = |A|$ and $A(i) \leq B(j_i)$ for $i = 1, 2, \ldots, |A|$.

**Remark 3.1.** Obviously, $\leq_{\wedge}$ is a partial order which generalizes the inclusion relation of $\subseteq$ [43]. For example, let $A = \{0.2, 0.4, 0.6\}$ and $B = \{0.2, 0.3, 0.4, 0.7\}$. Then $A \leq_{\wedge} B$, but $A \leq B$ does not hold.

To propose disjunctive partial order, we first define $\beta$-normalization and $\gamma$-normalization of $A$ with respect to $B$ as follows:

Let $\beta : \mathcal{P}([0, 1]) \times \mathcal{P}([0, 1]) \to \mathcal{P}([0, 1])$ be the function defined by

$$
\beta(A, B) = \begin{cases} 
A, & \text{if } |A| \leq |B|; \\
n\{A(i) | i \in \{ |A| - |B| + 1, \ldots, |B| \}\}, & \text{otherwise.}
\end{cases}
$$

And $\gamma : \mathcal{P}([0, 1]) \times \mathcal{P}([0, 1]) \to \mathcal{P}([0, 1])$ is defined by

$$
\gamma(A, B) = \{A(i) | i \in \{1, \ldots, \min(|A|, |B|)\}\}.
$$

**Definition 3.3.** For $A, B \in \mathcal{P}([0, 1])$, $A \leq_{\vee} B$ is defined as

$A \leq_{\vee} B$ if

$$
\begin{cases} 
A(i) \leq \gamma(B, A)(i), & i = 1, \ldots, |A|, \\
\beta(A, B)(i) \leq B(i), & i = 1, \ldots, |B|,
\end{cases}
$$

if $|A| \leq |B|$, otherwise.

**Theorem 3.1.** $\leq_{\wedge}$ and $\leq_{\vee}$ are partial orders.

**Proof.** The proof of $\leq_{\wedge}$ being partial order can be found in Theorem 3.3 of [68]. In what follows, we prove $\leq_{\wedge}$ is a partial order.

( po1) and (po3) are obvious from Definitions 3.1 and 3.2.

( po2) $\forall A, B \in \mathcal{P}([0, 1]), A \leq_{\wedge} B \Rightarrow |A| \leq |B|$. Similarly, we have $B \leq_{\wedge} A \Rightarrow |B| \leq |A|$. Therefore $|A| = |B|$. It can be obtained from Definition 3.2 that $A(i) \leq B(i)$ and $B(i) \leq A(i)$, $\forall i = 1, \ldots, |A|$, so that $A = B$. That is to say, $\leq_{\wedge}$ satisfies antisymmetry. So we can conclude that $\leq_{\wedge}$ is a partial order.  

Then $(\mathcal{P}([0, 1]), \leq_{\wedge})$ and $(\mathcal{P}([0, 1]), \leq_{\vee})$ are partially ordered sets.

**Remark 3.2.** Four special disjunctive ordered relations [43,44] are constructed by the boundary-values as follows:

- $\leq_{DU}$: $A \leq_{DU} B$ if $A(1) \leq B(2)$.
- $\leq_{D}$: $A \leq_{D} B$ if $A(1) \leq B(1)$.
- $\leq_{U}$: $A \leq_{U} B$ if $A(|A|) \leq B(|B|)$.
- $\leq_{UD}$: $A \leq_{UD} B$ if $A(|A|) \leq B(1)$.

We give an example to show the difference between $\leq_{\wedge}$ defined in this paper and $\leq_{DU}$ in [43,44]. Let $A = \{2, 4\}, B = \{2, 4, 5\}, C = \{4, 5, 8\}$. Obviously, $A \leq_{\wedge} B \leq_{\wedge} C$. But $A \leq_{DU} B$, $B \leq_{DU} A \neq A = B$. $A \leq_{DU} B$ and $B \leq_{DU} C \Rightarrow A \neq_{DU} C$. That is to say, $\leq_{DU}$ is not a partial order because of the invalidation of antisymmetry and transitivity. In what follows, we present the relationships between the partial orders defined in this paper and the orders in [43,44].
It can be concluded from Fig. 1 that $\leq_{UD}$ is more precise than the other three orders which satisfy transitivity, irreflexivity and antisymmetry. $\leq_{U}$ and $\leq_{D}$ are pre-orders and $\leq_{DU}$ does not satisfy anti-symmetry and transitivity. $\leq_{\vee}$ and $\leq_{\wedge}$ are partial orders which are more exact to describe the ordered relations based on disjunctive and conjunctive semantics respectively.

3.2. Inclusion measures on $(\mathcal{P}(0,1], \leq_{\wedge})$ and $(\mathcal{P}(0,1], \leq_{\vee})$

In this subsection, some inclusion measures will be constructed under conjunctive and disjunctive semantics respectively. We first introduce the relevant notions of fuzzy logic operators [49]. Triangular norm ($t$-norm for short) is an increasing, associative and commutative mapping $\mathcal{F} : [0,1]^2 \rightarrow [0,1]$ that satisfies the boundary condition: for all $\alpha \in I$, $\mathcal{F}(\alpha, 1) = \alpha$. The most popular continuous $t$-norns include

- the standard min operator: $\mathcal{F}_{M}(\alpha, \beta) = \min(\alpha, \beta)$.
- the algebraic product: $\mathcal{F}_{P}(\alpha, \beta) = \alpha \cdot \beta$.
- the bold intersection (also called the ukasiewicz $t$-norm): $\mathcal{F}_{U}(\alpha, \beta) = \max(0, \alpha + \beta - 1)$.

An implicator is a function $\mathcal{I} : [0,1]^2 \rightarrow [0,1]$ satisfying $\mathcal{I}(1,0) = 0$ and $\mathcal{I}(1,1) = \mathcal{I}(0,1) = \mathcal{I}(0,0) = 1$. An implicator $\mathcal{I}$ is called left monotonic (resp. right monotonic) iff for every $\alpha \in [0,1]$, $\mathcal{I}(\alpha, \cdot)$ is decreasing (resp. $\mathcal{I}(\cdot, \alpha)$ is increasing). If $\mathcal{I}$ is both left monotonic and right monotonic, then it is called hybrid monotonic. For all $x,y \in [0,1]^2$, $\mathcal{I}$ satisfies $x \leq y \Leftrightarrow \mathcal{I}(x,y) = 1$ and then it possesses Confainment Principle (CP Principle).

Several classes of implicators have been studied in the literature. Here we refer the definition of R-implicator (residual implicator). An implicator is an R-implicator based on a left-continuous $t$-norm $\mathcal{F}$ iff for every $x, y \in [0,1]$, $\mathcal{F}(x,y) = \sup[\gamma \in [0,1], \mathcal{F}(x,\gamma) \leq y]$.

The well-known $R$-implicators are

- the Lukasiewicz implicator: $\mathcal{F}_{L}(x,y) = \min(1, 1 - x + y)$,
- the Gödel implicator: $\mathcal{F}_{G}(x,y) = 1$ for $x \leq y$ and $\mathcal{F}_{G}(x,y) = y$ otherwise,
- the Gaines implicator: $\mathcal{F}_{G}(x,y) = 1$ for $x \leq y$ and $\mathcal{F}_{G}(x,y) = \frac{1}{y}$ otherwise.

**Proposition 3.1.** Let $A, B, C \in \mathcal{P}([0,1])$ and $N(A) = \sum_{i=1}^{\lvert A \rvert} A(i)$. Then the following function is an inclusion measure under the partial order relationship $\leq_{\wedge}$:

$$\text{Inc}_{\leq_{\wedge}}(A, B) = \frac{N(B)}{\max\{N(A), N(B)\}}.$$  

**Proof.** It is obvious that $\text{Inc}_{\leq_{\wedge}}(A, B)$ satisfies 11-13 in Definition 2.2. Now we prove it satisfies 14. If $A \leq_{\wedge} B \Rightarrow |A| \leq |B|$, $A(i) \leq B(j)$, then for any $C \in \mathcal{P}([0,1])$,

(i) if $N(A) \leq N(B) \leq N(C)$,

$$\text{Inc}(C, A) = \frac{N(A)}{\max\{N(C), N(A)\}} = \frac{N(A)}{N(C)} \leq \frac{N(B)}{N(C)} = \frac{N(B)}{\max\{N(C), N(B)\}} = \text{Inc}(C, B),$$

and

$$\text{Inc}(B, C) = \frac{N(B)}{\max\{N(C), N(B)\}} = \frac{N(B)}{N(C)} \leq 1 = \frac{N(B)}{\max\{N(C), N(B)\}} = \text{Inc}(C, B),$$

and

(ii) if $N(A) \leq N(C) \leq N(B)$,

$$\text{Inc}(C, A) = \frac{N(A)}{\max\{N(C), N(A)\}} = \frac{N(A)}{N(C)} \leq 1 = \frac{N(B)}{\max\{N(C), N(B)\}} = \text{Inc}(C, B),$$

and

$$\text{Inc}(B, C) = \frac{N(B)}{\max\{N(C), N(B)\}} = \frac{N(B)}{N(C)} \leq 1 = \text{Inc}(A, C).$$

(iii) similar to the above situations, $\text{Inc}(C, A) \leq \text{Inc}(C, B)$ and $\text{Inc}(B, C) \leq \text{Inc}(A, C)$ can be proved when $N(C) \leq N(A) \leq N(B)$.

In conclusion, $\text{Inc}_{\leq_{\wedge}}$ is a HM inclusion measure. □

We further introduce some inclusion measures [68] based on the partial order $\leq_{\vee}$.

**Proposition 3.2 [68].** Let $A, B \in \mathcal{P}([0,1])$, the following three functions are HM inclusion measures under the partial order $\leq_{\vee}$:

$$\text{Inc}_{\leq_{\vee}}(A, B) = \begin{cases} \bigwedge_{i=1}^{\lvert A \rvert} \mathcal{I}(A(i), B(i)), & i = 1, 2, \ldots, \lvert A \rvert, \text{ if } |A| \leq |B|, \\ \bigwedge_{i=1}^{\lvert B \rvert} \mathcal{I}(A(i), B(i)), & i = 1, 2, \ldots, \lvert B \rvert, \text{ else.} \end{cases}$$

Please cite this article as: H.-Y. Zhang, S.-Y. Yang, Feature selection and approximate reasoning of large-scale set-valued decision tables based on $\alpha$-dominance-based quantitative rough sets, Information Sciences (2016), http://dx.doi.org/10.1016/j.ins.2016.06.028
where $\mathcal{I}$ is an implicator satisfying hybrid monotonicity.

Additionally, the induced hybrid inclusion measure can be obtained by $\text{Inc}'(A, B)$ and $\text{Inc}''(A, B)$ on $(\mathcal{P}(R), \leq_{\triangle})$:

(4) $\text{Inc}_{\leq_{\triangle}}(A, B) = \mathcal{I}(\text{Inc}'(A, B), \text{Inc}''(A, B))$. $\mathcal{I}$ is a t-norm.

(5) $\text{Inc}_{\leq_{\triangle}}(A, B) = \alpha \text{Inc}'(A, B) + \beta \text{Inc}''(A, B)$. $\alpha + \beta = 1$.

3.3. $\alpha$-dominance relation in set-valued information tables

In this subsection, we introduce $\alpha$-dominance relations under disjunctive and conjunctive semantics in SITs and show their properties respectively.

**Definition 3.4.** Let $S = (U, \mathcal{A}, V, F)$ be a SIT, $A \subseteq \mathcal{A}$, and $\alpha \in (0.5, 1]$. If $\text{Inc}_{\triangle}$ is a HM inclusion measure defined by the partial orders $\leq_{\triangle}$ and $\leq_{\triangledown}$, then the $\alpha$-dominance relation under disjunctive and conjunctive semantics can be defined as follows respectively:

$$R^{\alpha}_{A} = \{(x, y) \in U \times U | \text{Inc}_{\triangle}(f_{a}(y), f_{a}(x)) \geq \alpha, \forall a \in A \}, \Delta = \wedge, \triangledown.$$

The following properties can be obtained directly.

**Theorem 3.2** [68]. Let $S = (U, \mathcal{A}, V, F)$ be a SIT, $A \subseteq \mathcal{A}$ and $\alpha \in (0.5, 1]$. Then

1. $R^{\alpha}_{A} \subseteq A \subseteq \mathcal{A}$ is reflexive and asymmetric.
2. $R^{\alpha}_{A}$ is $\alpha$-transitive, i.e., $(x, y) \in R^{\alpha}_{A}, (y, z) \in R^{\alpha}_{A} \Rightarrow (x, z) \in R^{\alpha}_{A},$ if the inclusion measure $\text{Inc}_{\triangle}(x, y)$ on $(\mathcal{P}([0, 1]), \leq_{\triangle})$ is $T_{M}$-transitive.

The $\alpha$-dominance class of object $x$ is denoted as $[x]^{\alpha}_{A}$, which means the objects in $[x]^{\alpha}_{A}$ dominating $x$ at least by degree $\alpha$:

$$[x]^{\alpha}_{A} = \{y \in U | (y, x) \in R^{\alpha}_{A}, \alpha \}.$$

We conclude the following properties from the definition of $[x]^{\alpha}_{A}$.

**Theorem 3.3** [68]. Let $S = (U, \mathcal{A}, V, F)$ be a SIT, $A \subseteq \mathcal{A}$ and $\alpha \in (0.5, 1]$. Then

1. if $B \subseteq A \subseteq \mathcal{A}$, then $R^{\alpha}_{B} \subseteq R^{\alpha}_{A} \subseteq R^{\alpha}_{A}$,
2. if $B \subseteq A \subseteq \mathcal{A}$, then $[x]^{\alpha}_{B} \subseteq [x]^{\alpha}_{A} \subseteq [x]^{\alpha}_{A}$,
3. if $\alpha_{1} \leq \alpha_{2}$, then $R^{\alpha_{2}}_{A} \subseteq R^{\alpha_{1}}_{A}$,
4. if $x \in [x]^{\alpha_{1}}_{A} \Rightarrow \text{Inc}_{\triangle}(f_{a}(x)) \geq \alpha, \forall a \in A$, then $[x]^{\alpha_{2}}_{A} \subseteq [x]^{\alpha_{1}}_{A}$,
5. if $\text{Inc}_{\triangle}$ is $T_{M}$-transitive, $\forall x \in [x]^{\alpha}_{A}$, then $[x]^{\alpha}_{A} \subseteq [x]^{\alpha}_{A}$ and $[x]^{\alpha}_{A} = \bigcup \{[x]^{\alpha}_{A} : x \in [x]^{\alpha}_{A} \}$.

Qiu et al. [73] proposed a dominance relation for ranking all objects in ordered information systems. Qian et al. [43] and Zhang et al. [68] introduced an $\alpha$-dominance relation in interval-valued ordered and general interval-valued information tables respectively. Now, we present an $\alpha$-dominance to rank all the objects in a SIT.

**Definition 3.5.** Let $S = (U, \mathcal{A}, V, F)$ be a SIT and $A \subseteq \mathcal{A}$. An $\alpha$-dominance degree is defined by

$$D^{\alpha}_{A}(x_{i}, x_{j}) = \frac{|[x_{i}]^{\alpha}_{A} \cup [x_{j}]^{\alpha}_{A}|}{|[x_{i}]^{\alpha}_{A} \cup [x_{j}]^{\alpha}_{A}|}.$$

The $\alpha$-dominance degree $D^{\alpha}_{A}(x_{i}, x_{j})$ measures the degree that the dominance class $[x_{i}]^{\alpha}_{A}$ contained in $[x_{j}]^{\alpha}_{A}$ with respect to $R^{\alpha}_{A}$. Based on this measure, Qian et al. [43] defined an $\alpha$-dominance degree of $x_{i}$ as

$$D^{\alpha}_{A}(x_{i}) = \frac{1}{|U| - 1} \sum_{j \neq i} D^{\alpha}_{A}(x_{i}, x_{j}), \ x_{i} \in U.$$

Obviously, for all $x_{i} \in U$, the higher the value of $D^{\alpha}_{A}(x_{i})$, the better the performance of $x_{i}$ is.

Please cite this article as: H.-Y. Zhang, S.-Y. Yang, Feature selection and approximate reasoning of large-scale set-valued decision tables based on $\alpha$-dominance-based quantitative rough sets, Information Sciences (2016), http://dx.doi.org/10.1016/j.ins.2016.06.028
Example 3.2. Continued Example 3.1. This example demonstrate the classes induced by the dominance relation $R^{\geq,\alpha}_{At}$. If $\alpha = 1$, from Table 1, we obtain

$$U/R^{\geq,\alpha}_{At} = \left\{ [x_1]_{At}^{\geq,\alpha}, \ldots, [x_{10}]_{At}^{\geq,\alpha} \right\},$$

where

$$[x_1]_{At}^{\geq,\alpha} = \{x_1\}, \quad [x_2]_{At}^{\geq,\alpha} = [x_3]_{At}^{\geq,\alpha} = [x_5]_{At}^{\geq,\alpha} = [x_9]_{At}^{\geq,\alpha} = [x_{10}]_{At}^{\geq,\alpha} = \{x_1, x_2, x_7, x_9, x_{10}\},$$

$$[x_4]_{At} = U, \quad [x_6]_{At}^{\geq,\alpha} = \{x_1, x_2, x_4, x_7, x_9, x_{10}\},$$

$$[x_5]_{At}^{\geq,\alpha} = \{x_5, x_6\}, \quad [x_8]_{At}^{\geq,\alpha} = U \setminus \{x_1\}. $$

We conclude that $x_2$, $x_7$, $x_9$, $x_{10}$ and $x_5$, $x_6$ are in the same dominance class in terms of conjunctive semantics when $\alpha = 1$.

4. $\alpha$-dominance-based quantitative rough sets based on inclusion measures

In this section, we propose an $\alpha$-dominance-based quantitative rough set approach to SIT and investigate its properties.

4.1. $\alpha$-dominance-based quantitative rough sets based on inclusion measures

A quantitative rough sets based on $\alpha$-dominance relation to SIT are presented as follows:

**Definition 4.1.** Let $S = (U, At, V, F)$ be a SIT. For any $X \subseteq U$, $A \subseteq At$ and $p, q \in (0, 1]$. The quantitative rough set based on the $\alpha$-dominance relation $R^{\geq,\alpha}_{A}$ are defined as

$$\text{POS}^{\geq,\alpha}_{A,p}(X) = \{x \in U | Inc([x]_{\alpha}^{\geq,\alpha}, X) \geq p\},$$

$$\text{NEG}^{\geq,\alpha}_{A,q}(X) = \{x \in U | Inc([x]_{\alpha}^{\geq,\alpha}, \bar{X}) \leq q\},$$

$$\text{BND}^{\geq,\alpha}_{A,p,q}(X) = \{x \in \text{Inc}([x]_{\alpha}^{\geq,\alpha}, X) < p \land \text{Inc}([x]_{\alpha}^{\geq,\alpha}, \bar{X}) < q\}$$

where $Inc([x]_{\alpha}^{\geq,\alpha}, X) \geq p \Rightarrow Inc([x]_{\alpha}^{\geq,\alpha}, \bar{X}) < q$. We can define equivalently the quantitative rough set as

$$R^{\geq,\alpha}_{A,p}(X) = \{x \in U | Inc([x]_{\alpha}^{\geq,\alpha}, X) \geq p\},$$

$$R^{\geq,\alpha}_{A,q}(X) = \{x \in U | Inc([x]_{\alpha}^{\geq,\alpha}, \bar{X}) < q\}.$$

If the inclusion measure $Inc$ is self-dual, we present the simplified quantitative rough sets based on $\alpha$-dominance relation in SIT as follows

**Definition 4.2.** Let $Inc : \mathcal{P}(U) \times \mathcal{P}(U) \rightarrow [0, 1]$ be an inclusion measure satisfying self-duality. For a pair of thresholds $(p, q)$, $0 \leq q < p \leq 1$, the three regions of quantitative rough sets based on the $\alpha$-dominance relation are defined by

$$D\text{POS}^{\geq,\alpha}_{A,p}(X) = \{x \in U | Inc([x]_{\alpha}^{\geq,\alpha}, X) \geq p\} = \text{POS}^{\geq,\alpha}_{A,p,q}(X),$$

$$D\text{NEG}^{\geq,\alpha}_{A,q}(X) = \{x \in U | Inc([x]_{\alpha}^{\geq,\alpha}, X) \leq q\} = \text{NEG}^{\geq,\alpha}_{A,1,q}(X),$$

$$D\text{BND}^{\geq,\alpha}_{A,p,q}(X) = \{x \in U | q < Inc([x]_{\alpha}^{\geq,\alpha}, X) < p\} = \text{BND}^{\geq,\alpha}_{A,p,1-q}(X).$$

**Theorem 4.1.** For all $X, Y \subseteq U$, $\text{POS}^{\geq,\alpha}_{A,p}(X)$, $\text{NEG}^{\geq,\alpha}_{A,q}(X)$ and $\text{BND}^{\geq,\alpha}_{A,p,q}(X)$ have the following properties:

1. If $Inc([x]_{\alpha}^{\geq,\alpha}, \emptyset) = 0$, then $\text{POS}^{\geq,\alpha}_{A,p,q}(\emptyset) = \text{BND}^{\geq,\alpha}_{A,p,1-q}(\emptyset) = \emptyset$.

2. $p_1 \leq p_2 \Rightarrow \text{POS}^{\geq,\alpha}_{A,p_2,q}(X) \subseteq \text{POS}^{\geq,\alpha}_{A,p_1,q}(X)$,

$q_1 \leq q_2 \Rightarrow \text{NEG}^{\geq,\alpha}_{A,q_2}(X) \subseteq \text{NEG}^{\geq,\alpha}_{A,q_1}(X)$,

$p_1 \leq p_2 \land q_1 \leq q_2 \Rightarrow \text{NEG}^{\geq,\alpha}_{A,p_1,q_1}(X) \subseteq \text{NEG}^{\geq,\alpha}_{A,p_2,q_2}(X)$.

3. If $X, Y \subseteq U$, $Inc$ is a HM inclusion measure, then

$\text{POS}^{\geq,\alpha}_{A,p}(X \cap Y) \subseteq \text{POS}^{\geq,\alpha}_{A,p}(X) \cap \text{POS}^{\geq,\alpha}_{A,p}(Y)$,

$\text{NEG}^{\geq,\alpha}_{A,q}(X \cap Y) \supseteq \text{NEG}^{\geq,\alpha}_{A,q}(X) \cap \text{NEG}^{\geq,\alpha}_{A,q}(Y)$,

$\text{POS}^{\geq,\alpha}_{A,p}(X) \cup \text{POS}^{\geq,\alpha}_{A,p}(Y) \subseteq \text{POS}^{\geq,\alpha}_{A,p}(X \cup Y)$,

$\text{NEG}^{\geq,\alpha}_{A,q}(X) \cup \text{NEG}^{\geq,\alpha}_{A,q}(Y) \supseteq \text{NEG}^{\geq,\alpha}_{A,q}(X \cup Y)$.
(4) If $X \subseteq Y \subseteq U$, then
\[\text{POS}_{A|q)}^\alpha (X) \subseteq \text{POS}_{A|p)}^\alpha (Y), \quad \text{NEG}_{A|q)}^\alpha (Y) \subseteq \text{NEG}_{A|p)}^\alpha (X).\]

The properties of the upper and lower approximation of quantitative rough sets are summarized in the following theorem.

**Theorem 4.2.** The quantitative lower and upper approximation operators of $X$, $R^\alpha_{A|p)}^\Delta (X)$ and $R^\alpha_{A|q)}^\Delta (X)$ have the following properties:

1. $R^\alpha_{A|p)}^\Delta (\emptyset) = R^\alpha_{A|q)}^\Delta (\emptyset) = \emptyset$, $R^\alpha_{A|p)}^\Delta (U) = R^\alpha_{A|q)}^\Delta (U) = U$.
2. $R^\alpha_{A|p)}^\Delta (X) \subseteq R^\alpha_{A|q)}^\Delta (X)$ (approximately $\sim$), $R^\alpha_{A|p)}^\Delta (X) \sim R^\alpha_{A|q)}^\Delta (X)$ if $p = q$ and $\forall X \subseteq U$.
3. $R^\alpha_{A|p)}^\Delta (X) \subseteq R^\alpha_{A|q)}^\Delta (X)$.
4. If $X, Y \subseteq U$ and Inc is a HM inclusion measure, then
   \[R^\alpha_{A|p)}^\Delta (X \cap Y) \subseteq R^\alpha_{A|p)}^\Delta (X) \cap R^\alpha_{A|p)}^\Delta (Y), \quad R^\alpha_{A|q)}^\Delta (X \cap Y) \subseteq R^\alpha_{A|q)}^\Delta (X) \cap R^\alpha_{A|q)}^\Delta (Y),\]
   \[R^\alpha_{A|p)}^\Delta (X \cup Y) \geq R^\alpha_{A|p)}^\Delta (X) \cup R^\alpha_{A|p)}^\Delta (Y), \quad R^\alpha_{A|q)}^\Delta (X \cup Y) \geq R^\alpha_{A|q)}^\Delta (X) \cup R^\alpha_{A|q)}^\Delta (Y),\]
   \[R^\alpha_{A|p)}^\Delta (X) \subseteq R^\alpha_{A|q)}^\Delta (Y), \quad \text{if } X \subseteq Y \subseteq U, \text{ then } R^\alpha_{A|p)}^\Delta (X) \subseteq R^\alpha_{A|p)}^\Delta (Y) \subseteq R^\alpha_{A|q)}^\Delta (X).

**Proof.** The proofs are straightforward according to Definition 4.1. □

**Theorem 4.3** [66]. Let the inclusion measure Inc satisfy self-duality. The three regions have the following additional properties:

1. $\text{DPOS}_{A|p)}^\alpha (X) = \text{DNEG}_{A|p)}^\alpha (X^c)$.
2. $\text{DNEG}_{A|q)}^\alpha (X) = \text{DPOS}_{A|q)}^\alpha (X^c)$.
3. $\text{DBND}_{A|p|q)}^\alpha (X) = \text{DBND}_{A|q|p)}^\alpha (X^c)$.

4.2. Attribute reduction based on the $\alpha$-dominance-based quantitative rough set in set-valued information tables

Attribute reduction aims to find out a minimal subset of the conditional attributes that can preserve a particular property of a given IS. In this subsection, we study the issue of attribute reduction in STIs derived by $\alpha$-dominance-based quantitative rough sets.

**Definition 4.3.** Let $S = (U, At, V, F)$ be a SIT, $A \subseteq At$ and $\triangle = \land, \lor, \forall$. $A$ is called an $\alpha$-dominance consistent set of $S$, if $R^\alpha_{A|p)} = R^\alpha_{A|q)}$. If $A$ is an $\alpha$-dominance consistent set and $\forall B \subset A$, $R^\alpha_{B|p)} \neq R^\alpha_{B|q)}$, then $A$ is referred to an $\alpha$-attribute reduction of $S$. An attribute $a \in At$ is dispensable if $R^\alpha_{A\setminus a|p)} = R^\alpha_{A|p)}$ and the set of all indispensable attributes, denoted by $\text{Core}_{A|p|q)}^\alpha (At)$, is called the $\alpha$-dominance core.

For $x_i \neq x_j \in U, 0.5 < \alpha \leq 1$, to distinguish two objects $x_i$ and $x_j$ based on $R^\alpha_{A|p)}$, Zhang et al. [70] defined $\alpha$-discernibility set as
\[D^\alpha_{A|p)}(x_i, x_j) = \{a_k \in At | \text{Inc}_a\{f_{x_i}(a_k), f_{x_j}(a_k)\} < \alpha\} \text{ where } \triangle = \land, \lor.
\]
Then for $\alpha_1 \leq \alpha_2$, we have $D^\alpha_{A|p)} \subseteq D^{\alpha_2}_{A|p)}$. If $A \subseteq At$ is an $\alpha$-dominance consistent set with respect to $R^\alpha_{A|p)}$, then $D^\alpha_{A|p)}(x_i) = D^\alpha_{A|p)}(x_j), \forall x_i \in U$.

Moreover, the $\alpha$-discernibility matrix is defined as $D^\alpha_{A|p)} = \{D^\alpha_{ij} \in At\}$ and $D^\alpha_{A|p)} = \{D^\alpha_{ij} : D^\alpha_{ij} \notin \emptyset\}$ where $\triangle = \land, \lor$.

The following two theorems illustrate that an $\alpha$-dominance consistent set can be determined by the $\alpha$-dominance matrix with respect to $R^\alpha_{A|p)}$, which can be proved by the similar way to Theorem 4 and 5 in [70].

**Theorem 4.4.** Let $S = (U, At, V, F)$ be a SIT and $\alpha \in (0.5, 1]$. $A \subseteq At$ is an $\alpha$-dominance consistent set in $S$, i.e., $R^\alpha_{A|p)} = R^\alpha_{A|q)}$ if $A$ and $D^\alpha_{A|p)} \neq \emptyset$, $\forall D^\alpha_{A|p)} \subseteq D^\alpha_{A|q)}$ where $\triangle = \land, \lor$.

It can be inferred for $\alpha_1 \leq \alpha_2$, we have $D^\alpha_{A|p)} \subseteq D^{\alpha_2}_{A|p)}$ that the $\alpha_2$-dominance reduction is contained in the $\alpha_1$-dominance reduction by **Theorem 4.3**, $A \subseteq At$ is an $\alpha$-dominance reduction if $A$ is the minimal set satisfying $A \cap D^{\alpha_1}_{A|p)} = \emptyset, \forall D^{\alpha_2}_{A|p)} \subseteq D^{\alpha_1}_{A|p)}$.

**Theorem 4.5.** Let $S = (U, At, V, F)$ be a SIT and $\alpha \in (0.5, 1]$. Then $a_k \in At$ is an element of the $\text{Core}_{A|p|q)}^\alpha$ if there exists $D^\alpha_{ij} \in D^\alpha_{A|p)}$ such that $D^\alpha_{ij} \subseteq \{a_k\}$ where $\triangle = \land, \lor$.
Table 2
The 1-discriminability matrix for SIT in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
<th>$x_9$</th>
<th>$x_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$\emptyset$</td>
<td>${a_1}$</td>
<td>${a_2}$</td>
<td>${a_3}$</td>
<td>${a_4}$</td>
<td>${a_5}$</td>
<td>${a_6}$</td>
<td>${a_7}$</td>
<td>${a_8}$</td>
<td>${a_9}$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>${a_2, a_4, a_5}$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>${a_3}$</td>
<td>${a_4}$</td>
<td>${a_5}$</td>
<td>${a_6}$</td>
<td>${a_7}$</td>
<td>${a_8}$</td>
<td>${a_9}$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$\emptyset$</td>
<td>${a_3}$</td>
<td>${a_4}$</td>
<td>${a_5}$</td>
<td>${a_6}$</td>
<td>${a_7}$</td>
<td>${a_8}$</td>
<td>${a_9}$</td>
<td>${a_10}$</td>
<td></td>
</tr>
</tbody>
</table>

Example 4.1. Continued Table 1. We present all 1-discriminability matrix in terms of conjunctive semantic meaning.

It is known from Table 2 that 1-core of the SIT $S$ is $\text{At}([a_4])$ according to Theorem 4.5. 1-dominance reduction set in $S$ with respect to $R_{\alpha}^A$ is $\text{At}([a_4])$ which is equal to $\text{Core}_{\alpha}^A$.

5. Attribute reduction of a set-valued decision tables based on $\alpha$-dominance-based quantitative rough sets

We first introduce some concepts of attribute reduction on a SDT based on the $\alpha$-dominance-based quantitative rough sets.

Definition 5.1. Let $S=(U, At \cup d, V, F)$ be a SDT, where $d$ is the decision attribute and $R_d^\geq = \{(x, y) \in U \times U | d(x) \geq d(y)\}$. If $R_{\alpha}^A \subseteq R_d^\geq$, then we say that $(U, At \cup d)$ is consistent; otherwise it is inconsistent.

Assume that the decision attribute $d$ partitions $U$ into a finite number of ordered classes which we denote by $U/d = \{D_1, D_2, \ldots, D_r\}$ and $D_j = \{x \in U, d(x) = j, 1 \leq j \leq r\}$. For all $1 \leq i, j \leq r$, if $i \geq j$, then the objects from $D_i$ are preferred to the objects from $D_j$.

The sets to be approximated are an upward union and a downward union of decision classes, which are defined respectively as:

$$D_i^\geq = \bigcup_{j \geq i} D_j, \ \ D_i^\leq = \bigcup_{j \leq i} D_j.$$  

Then the relative decision dominance classes are

$$[x]_d^\geq = \bigcup_{j \geq d(x)} D_j, \ \ [x]_d^\leq = \bigcup_{j \leq d(x)} D_j.$$  

For simplicity and without loss of generality, we just discuss the approximation operators of upward union.

Therefore, $(D_1^\geq, \ldots, D_r^\geq)$ is the set of decision dominance classes. According to Definition 2.3, we can define the three regions of $D_i^\geq$, denoted by $\text{POS}_{\alpha}^A(D_i^\geq)$, $\text{NEG}_{\alpha}^A(D_i^\geq)$ and $\text{BND}_{\alpha}^A(D_i^\geq)$ as follows:

$$\text{Inc}([x]_A^\geq, D_i^\geq) \geq p \Rightarrow \text{Inc}([x]_A^\geq, D_i^\leq) < q,$$

and

$$\text{POS}_{\alpha}^A(D_i^\geq) = \{x \in U | \text{Inc}([x]_A^\geq, D_i^\geq) \geq p\},$$

$$\text{NEG}_{\alpha}^A(D_i^\geq) = \{x \in U | \text{Inc}([x]_A^\geq, D_i^\leq) < q\},$$

$$\text{BND}_{\alpha}^A(D_i^\geq) = \{x \in U | \text{Inc}([x]_A^\geq, D_i^\geq) \geq p \land \text{Inc}([x]_A^\geq, D_i^\leq) < q\}.$$  

Or equivalently,

$$\text{Inc}([x]_A^\geq, D_i^\leq) \geq p \Rightarrow \text{Inc}([x]_A^\geq, D_i^\leq) < q,$$

and

$$\text{R}_{\alpha}^A(D_i^\geq) = \{x \in U | \text{Inc}([x]_A^\geq, D_i^\leq) \geq p\},$$

$$\text{R}_{\alpha}^A(D_i^\geq) = \{x \in U | \text{Inc}([x]_A^\leq, D_i^\leq) < q\}.$$  

In particular, if the inclusion measure $\text{Inc}$ is self-dual, we can present the framework of $\alpha$-dominance-based quantitative rough sets in Definition 5.2.
Definition 5.2. Let Inc: $2^U \times 2^U \to [0, 1]$ be an inclusion measure satisfying self-duality. For a pair of thresholds $(p, q)$, $0 \leq q < p \leq 1$, the three regions of $D_i^{\wedge}$ based on the $\alpha$-dominance relation in SDT are defined as follows:

$$D_{\alpha,p}^\wedge(D_i^{\wedge}) = \{x \in U | \text{Inc}([x]_{A}^{\alpha,p}, D_i^{\wedge}) \geq p\}$$

$$= \text{POS}_{\alpha,p}^\wedge(D_i^{\wedge}),$$

$$\text{DNEG}_{\alpha,q}^\wedge(D_i^{\wedge}) = \{x \in U | \text{Inc}([x]_{A}^{\alpha,q}, D_i^{\wedge}) \leq q\}$$

$$= \text{NEG}_{\alpha,q}^\wedge(D_i^{\wedge}),$$

$$\text{DBND}_{\alpha,q}^\wedge(D_i^{\wedge}) = \{x \in U | q < \text{Inc}([x]_{A}^{\alpha,q}, D_i^{\wedge}) < p\}$$

$$= \text{BND}_{\alpha,q}^\wedge(D_i^{\wedge}).$$

Zhang et al. [70] presented some concepts of attribute reductions based on the variable-precision-dominance-based rough set models in an interval-valued decision table and showed the relationships between those reduction approaches.

Definition 5.3 [70]. Let $(U, At \cup d, V, F)$ be a SDT and $A \subseteq At$.

$$\text{Inc}([x]_{A}^{\alpha,p}, D_i^{\wedge}) \geq p \Rightarrow \text{Inc}([x]_{A}^{\alpha,q}, D_i^{\wedge}) < q,$$

and

$$\mu_A(x) = (\text{Inc}([x]_{A}^{\alpha,p}, D_i^{\wedge}), \ldots, \text{Inc}([x]_{A}^{\alpha,q}, D_i^{\wedge})),$$

$$\psi_A(x) = \{D_k^\wedge : \text{Inc}([x]_{A}^{\alpha,p}, D_k^{\wedge}) = \max\{\text{Inc}([x]_{A}^{\alpha,q}, D_k^{\wedge})\}\}.$$

(1) $A$ is called a distribution consistent set of $(U, At \cup d, V, F)$ when $\mu_A(x) = \mu_A(x)$. If $A$ is a distribution consistent set and no proper subset of $A$ is a distribution consistent, then $A$ is called a distribution reduction of $(U, At \cup d, V, F)$.

(2) Similarly, we can define the maximum distribution consistent set and maximum distribution reduction set.

Zhang [70] defined a possible distribution consistent set by refining the condition that $\text{Inc}(A, B) > 0$, where $A, B \in \mathcal{P}([0, 1])$. If $\text{Inc}$ is self-dual, we have $\text{Inc}(A, B) > 0 \Leftrightarrow \text{Inc}(A, B^c) < 1$. In what follows, we propose the possible distribution consistent set by the condition $\text{Inc}(A, B^c) < 1$. Moreover, the concepts of knowledge reduction based on the $\alpha$-dominance-based quantitative rough set model in SDT are also proposed.

Definition 5.4. Let $(U, At \cup d, V, F)$ be a SDT and $A \subseteq At$. $\text{POS}_{\alpha,p}^\wedge(D_i^{\wedge})$, $\text{BND}_{\alpha,q}^\wedge(D_i^{\wedge})$ and $\text{NEG}_{\alpha,q}^\wedge(D_i^{\wedge})$ are three regions of attribute set $A$ with respect to the decision dominance class $D_i^{\wedge}$. Let

$$\text{Inc}([x]_{A}^{\alpha,p}, D_i^{\wedge}) \geq p \Rightarrow \text{Inc}([x]_{A}^{\alpha,q}, D_i^{\wedge}) < q,$$

and

$$PA_{\alpha,q} = \frac{\sum \{|\text{POS}_{\alpha,p}^\wedge(D_i^{\wedge})| : i \leq r\}}{|U|},$$

$$BA_{\alpha,q} = \frac{\sum \{|\text{BND}_{\alpha,q}^\wedge(D_i^{\wedge})| : i \leq r\}}{|U|},$$

$$NA_{\alpha,q} = \frac{\sum \{|\text{NEG}_{\alpha,q}^\wedge(D_i^{\wedge})| : i \leq r\}}{|U|},$$

$$PD_{\alpha,q} = (\text{POS}_{\alpha,p}^\wedge(D_1^{\wedge}), \ldots, \text{POS}_{\alpha,p}^\wedge(D_r^{\wedge})),$$

$$BD_{\alpha,q} = (\text{BND}_{\alpha,q}^\wedge(D_1^{\wedge}), \ldots, \text{BND}_{\alpha,q}^\wedge(D_r^{\wedge})),$$

$$ND_{\alpha,q} = (\text{NEG}_{\alpha,q}^\wedge(D_1^{\wedge}), \ldots, \text{NEG}_{\alpha,q}^\wedge(D_r^{\wedge})),$$

$$\delta_A(x) = \{D_i^\wedge | \text{Inc}([x]_{A}^{\alpha,p}, D_i^{\wedge}) > 0\}, \quad i = 1, \ldots, r.$$

(1) $A$ is called a $p$-positive approximate consistent set of $(U, At \cup d, V, F)$ when $PA_{\alpha,q} = PA_{\alpha,q}$. If $A$ is a $p$-positive approximate consistent set and no proper subset of $A$ is a $p$-positive approximate consistent set, then $A$ is called a $p$-positive approximate reduction of $(U, At \cup d, V, F)$.

(2) Similarly, we can propose the following concepts: $q$-negative approximate consistent set and $q$-negative approximate reduction; $(p, q)$-boundary approximate consistent set and $(p, q)$-boundary approximate reduction; $p$-positive distribution consistent set and $p$-positive distribution reduction set; $(p, q)$-boundary distribution consistent set and $(p, q)$-boundary distribution reduction set; $q$-negative distribution consistent set and $q$-negative distribution reduction set, possible consistent set and possible reduction set.
From Definitions 5.2 and 5.3, we have the following theorems.

**Theorem 5.1.** Let \( S = (U, A \cup d, V, F) \) be a SDT, \( \alpha \in [0.5, 1] \) and \( p, q \in (0, 1] \). Then \( p \)-positive distribution consistent set is a \( p \)-positive approximate consistent set; \( (p, q) \)-boundary distribution consistent set is a \( (p, q) \)-boundary approximate consistent set and \( q \)-negative distribution consistent set is a \( q \)-negative approximate consistent set.

**Theorem 5.1** can be proved directly from Definitions 5.3 and 5.4.

**Theorem 5.2.** Let \( S = (U, A \cup d, V, F) \) be a SDT, \( \alpha \in [0.5, 1] \) and \( p, q \in (0, 1] \). We can conclude that

1. If \( A \) is a distribution consistent set, then \( A \) is a \( p \)-positive distribution consistent set,
2. If \( A \) is a distribution consistent set, then \( A \) is a \((1 - q)\)-negative distribution consistent set where \( 0 \leq q < p \leq 1 \) and \( Inc \) is self-dual.
3. If \( A \) is a distribution consistent set, then \( A \) is a \((p, q)\)-boundary distribution consistent set where \( 0 \leq q < p \leq 1 \) and \( Inc \) is self-dual.

**Proof.** It is obvious to obtain the conclusion (1).

(2) If \( Inc \) is self-dual and \( 0 \leq q < p \leq 1 \), we conclude from Definition 5.2 that \( DNEG(\alpha, q) := \{ x \in U | Inc(\alpha, x) \subseteq \alpha \} \leq q \). If \( A \) is a distribution consistent set, then \( \mu_A(x) = \mu_A(x), \forall x \in U \). \( \forall i \leq r \), yielding \( Inc(\alpha, x) \subseteq \alpha \) if and only if \( y \in DNEG(\alpha, q) \). That is to say, \( A \) is a \((1 - q)\)-negative distribution consistent set.

Similarly, (3) can be proved directly from Definitions 5.2, 5.3 and 5.4. \( \square \)

With respect to the quantitative rough set model, we conclude that the bigger the threshold \( p \) is, the finer the results of classification are.

**Theorem 5.3.** Let \( S = (U, A \cup d, V, F) \) be a SDT, \( \alpha \in [0.5, 1] \) and \( A \subseteq A \). We denote \( \theta(\alpha, q) = \min \{ Inc(\alpha, x) \subseteq \alpha \} ; x \in U \). If \( Inc \) is a CM inclusion measure, we have the following properties:

1. If \( q \in (0, \theta(\alpha, q)) \), then \( A \) is a possible consistent set, then \( A \) is a \( q \)-negative distribution consistent set,
2. If \( q \in (0, \theta(\alpha, q)) \), then \( A \) is a \( q \)-negative distribution consistent set, then \( A \) is a possible consistent set,
3. If \( q \in (0, \theta(\alpha, q)) \), then \( A \) is a possible reduction set if \( A \) is a \( q \)-negative distribution reduction set.

**Proof.**

(1) If \( Inc \) is a CM inclusion measure, we have \( NEG(\alpha, q) \subseteq \theta(\alpha, q) \), so we only need to prove \( NEG(\alpha, q) \subseteq \theta(\alpha, q) \), \( \forall i \in 1, \ldots, r \). If \( A \) is a possible consistent set, we have \( \delta_A(x) = \delta_A(x), \forall x \in U \). So

\[
x \in \theta(\alpha, q) \Rightarrow Inc(\alpha, x) \subseteq \alpha \Rightarrow q \geq q > 0
\]

Then \( NEG(\alpha, q) \subseteq \theta(\alpha, q) \), which means that \( A \) is a \( q \)-negative consistent set.

(2) If \( A \) is a \( q \)-negative consistent set, \( q \in \{ \theta(\alpha, q) \}, 1 \), we have \( NEG(\alpha, q) \subseteq \theta(\alpha, q) \), \( \forall i \in 1, \ldots, r \). For any \( x \in U \),

\[
x \in \theta(\alpha, q) \Rightarrow Inc(\alpha, x) \subseteq \alpha \Rightarrow q \geq q > 0 \Rightarrow \delta_A(x) \subseteq \delta_A(x).
\]

That is to say \( \delta_A(x) \subseteq \delta_A(x), \forall x \in U \). On the other hand, combined with the properties of CM inclusion measures, we have \( \delta_A(x) \subseteq \delta_A(x), \forall x \in U \). Therefore \( \delta_A(x) = \delta_A(x), \forall x \in U \). That is, \( A \) is a possible consistent.

(3) Combining (1) with (2), \( A \) is a possible reduction set if \( A \) is a \( q \)-negative distribution reduction set. \( \square \)
Similarly, the relationship between the positive distribution consistent set and maximum distribution consistent set is presented in the following Theorem 5.4, which can be proved by the similar way to Theorem 8 in [70].

**Theorem 5.4.** Let \( S = (U, A, d, V, F) \) be a SDT, \( \alpha \in (0.5, 1], p \in (0.5, 1) \) and \( A \subseteq A_t \). We denote
\[
\begin{align*}
p^\alpha_A &= \min \{ \text{max Inc}(x)^A \alpha, D^\alpha_j \} : x \in U, \quad p^\alpha_0 &= \min \{ p^\alpha_{A_t}, p^\alpha_A \} .
\end{align*}
\]
If \( p^\alpha_0 > 0.5 \), \( \{ D_j : \text{Inc}(x)^A \alpha, D^\alpha_j > 0.5 \} \) is a singleton and \( \text{Inc} \) is a HM inclusion measure, then
1. for \( p \in (0.5, p^\alpha_0) \), if \( A \) is a \( p \)-positive distribution consistent set, \( A \) is a maximum distribution consistent set,
2. for \( p \in (0.5, p^\alpha_0] \) and \( \text{Inc} \) is a maximal inclusion measure, if \( A \) is a maximum distribution consistent set, \( A \) is a \( p \)-positive distribution consistent set,
3. for \( p \in (0.5, p^\alpha_0) \) and \( \text{Inc} \) is a maximal inclusion measure, if \( A \) is a maximum distribution reduction, \( A \) is a \( p \)-positive distribution reduction.

We further discuss the approaches to obtain the positive, negative and boundary distribution reductions in SDT based on the \( \alpha \)-dominance-based quantitative rough set model. The equivalent descriptions for the three kinds of consistent sets are presented as follows:

**Theorem 5.5.** Let \( S = (U, A, d, V, F) \) be a SDT, \( \alpha \in (0.5, 1], p, q \in (0, 1) \) and \( A \subseteq A_t \).
\[
\begin{align*}
p^\alpha_{A_{(p,q)}}(x) &= \{ D^\alpha_i : x \in POS^\alpha_{A_{(p,q)}}(D^\alpha_i) \}, \forall x \in U, \\
B^\alpha_{A_{(p,q)}}(x) &= \{ D^\alpha_i : x \in BND^\alpha_{A_{(p,q)}}(D^\alpha_i) \}, \forall x \in U, \\
N^\alpha_{A_{(p,q)}}(x) &= \{ D^\alpha_i : x \in NEG^\alpha_{A_{(p,q)}}(D^\alpha_i) \}, \forall x \in U.
\end{align*}
\]
Then we have
1. \( A \) is a \( p \)-positive distribution consistent set \( \iff \ p^\alpha_{A_{(p,q)}}(x) \subseteq p^\alpha_{A_{(p,q)}}(x), \forall x \in U, \)
2. \( A \) is a \( (p, q) \)-boundary distribution consistent set \( \iff \ B^\alpha_{A_{(p,q)}}(x) \subseteq B^\alpha_{A_{(p,q)}}(x), \forall x \in U, \)
3. \( A \) is a \( q \)-negative distribution consistent set \( \iff \ N^\alpha_{A_{(p,q)}}(x) \subseteq N^\alpha_{A_{(p,q)}}(x), \forall x \in U. \)

**Proof.** The theorem can be proved directly from the definitions of \( p^\alpha_{A_{(p,q)}}(x) \), \( B^\alpha_{A_{(p,q)}}(x) \) and \( N^\alpha_{A_{(p,q)}}(x) \).

**Theorem 5.6.** Let \( S = (U, A, d, V, F) \) be a SDT, \( \alpha \in (0.5, 1], p, q \in (0, 1], A \subseteq A_t \) and \( \text{Inc} \) be a maximal, HM and \( \alpha \)-transitive inclusion measure. Let
\[
S_\alpha(x, y) = \min \{ \text{Inc}(f_\alpha(x), f_\alpha(y)), \text{Inc}(f_\alpha(y), f_\alpha(x)) \}, \forall x, y \in U.
\]
1. \( A \) is a \( p \)-positive distribution consistent set \( \iff \) for all \( x, y \in U, \) if \( p^\alpha_{A_{(p,q)}}(x) \neq p^\alpha_{A_{(p,q)}}(y), \) there exists \( a \in A \) such that \( S_\alpha(x, y) < \alpha, \)
2. \( A \) is a \( (p, q) \)-boundary distribution consistent set \( \iff \) for all \( x, y \in U, \) if \( B^\alpha_{A_{(p,q)}}(x) \neq B^\alpha_{A_{(p,q)}}(y), \) there exists \( a \in A \) such that \( S_\alpha(x, y) < \alpha, \)
3. \( A \) is a \( q \)-negative distribution consistent set \( \iff \) for all \( x, y \in U, \) if \( N^\alpha_{A_{(p,q)}}(x) \neq N^\alpha_{A_{(p,q)}}(y), \) there exists \( a \in A \) such that \( S_\alpha(x, y) < \alpha, \)

**Proof.**
1. can be proved similarly to the proof of Theorem 5.7 in [70].
2. \( (\Rightarrow) \) Assume that \( A \) is a \( (p, q) \)-boundary distribution consistent set.
\[
\begin{align*}
\text{Inc}(f_\alpha(x), f_\alpha(y)) &\geq \alpha \quad \text{and} \quad \text{Inc}(f_\alpha(y), f_\alpha(x)) \geq \alpha.
\end{align*}
\]
With the hybrid monotonicity of inclusion measure, we have \( [y]^A_{\alpha} \subseteq [x]^A_{\alpha}, [x]^A_{\alpha} \subseteq [y]^A_{\alpha}, \) then \( [x]^A_{\alpha} = [y]^A_{\alpha}. \)
As a result, we have \( B^\alpha_{A_{(p,q)}}(x) = B^\alpha_{A_{(p,q)}}(y). \) Since \( A \) is a \( (p, q) \)-boundary distribution consistent set, we obtain
\[
\begin{align*}
B^\alpha_{A_{(p,q)}}(x) &= B^\alpha_{A_{(p,q)}}(y) \quad \text{and} \quad B^\alpha_{A_{(p,q)}}(y) = B^\alpha_{A_{(p,q)}}(y).
\end{align*}
\]
Therefore, \( B^\alpha_{A_{(p,q)}}(x) = B^\alpha_{A_{(p,q)}}(y). \)

\( (\Leftarrow) \) For any \( j \leq r, \) we need to prove \( BND^\alpha_{A_{(p,q)}}(D^\alpha_j) = BND^\alpha_{A_{(p,q)}}(D^\alpha_j). \)

We have \( BND^\alpha_{A_{(p,q)}}(D^\alpha_j) \subseteq BND^\alpha_{A_{(p,q)}}(D^\alpha_j), \) then \( D^\alpha_j \subseteq B^\alpha_{A_{(p,q)}}(x). \) For all \( x, y \in U, \) if \( S_\alpha(x, y) \geq \alpha, \forall a \in A, \) then, \( [x]^A_{\alpha} = [y]^A_{\alpha}, \) we have \( B^\alpha_{A_{(p,q)}}(x) = B^\alpha_{A_{(p,q)}}(y) = B^\alpha_{A_{(p,q)}}(x). \) From Theorem 5.5, we have \( D^\alpha_j \subseteq B^\alpha_{A_{(p,q)}}(x), \) which implies \( x \in BND^\alpha_{A_{(p,q)}}(D^\alpha_j). \) So we have \( \forall x \in BND^\alpha_{A_{(p,q)}}(D^\alpha_j) \Rightarrow x \in BND^\alpha_{A_{(p,q)}}(D^\alpha_j). \)

Similarly, (3) can be proved.
Theorem 5.6 provides approaches to judge whether a subset of attributes is p-positive, (p, q)-boundary and q-negative distribution consisten set respectively. To get the approaches to knowledge reduction based on α-dominance-based quantitative rough set model, we further define the discernibility set and discernibility matrix in Definition 5.5.

Definition 5.5. Let \( S = (U, At \cup d, V, F) \) be a SDT, \( \alpha \in (0.5, 1) \) and \( p, q \in (0, 1) \). We denote

\[
D^{(p)}_{At} = \{ (x)_{At}^{[\alpha]} : \ P^{[\alpha]}_{At}(x) \neq P^{[\alpha]}_{At}(y), \ \forall x, y \in U \},
\]

\[
D^{(-q)}_{At} = \{ (x)_{At}^{[\alpha]} : \ N^{[\alpha]}_{At}(x) \neq N^{[\alpha]}_{At}(y), \ \forall x, y \in U \},
\]

\[
D^{(q)}_{At} = \{ (x)_{At}^{[\alpha]} : \ B^{[\alpha]}_{At}(x) \neq B^{[\alpha]}_{At}(y), \ \forall x, y \in U \}.
\]

Let \( l = 1, 2, 3 \), the discernibility set between \( x \) and \( y \) is defined as

\[
D^{(p)}_{ly}(x, y) = \begin{cases} \{ & \text{if } \exists \alpha \in [a_{l}, b_{l}]: S_{b_{l}}(x, y) < \alpha \}, \ (x)_{At}^{[\alpha]} \in D^{(p)}_{ly}, \\ \emptyset, & \text{if } (x)_{At}^{[\alpha]} \notin D^{(p)}_{ly}. \end{cases}
\]

Obviously, the discernibility matrix \( D^{(p)}_{ly} = (D^{(p)}_{ly}(x, y)) \) is symmetrical.

Theorem 5.7. Let \( S = (U, At \cup d, V, F) \) be a SDT, \( \alpha \in (0.5, 1) \), \( p, q \in (0, 1) \), \( A \subseteq At \) and \( Inc \) be a maximal, HM and \( \alpha \)-transitive inclusion measure. Then

1. \( A \) is a p-positive distribution consistent set iff \( A \cap D^{(p)}_{At}(x, y) \neq \emptyset \) for all \((x)_{At}^{[\alpha]} \in D^{(p)}_{At}, \ \forall x, y \in U \).
2. \( A \) is a q-negative distribution consistent set iff \( A \cap D^{(-q)}_{At}(x, y) \neq \emptyset \) for all \((x)_{At}^{[\alpha]} \in D^{(-q)}_{At}, \ \forall x, y \in U \).
3. \( A \) is a (p, q)-boundary distribution consistent set iff \( A \cap D^{(q)}_{At}(x, y) \neq \emptyset \) for all \((x)_{At}^{[\alpha]} \in D^{(q)}_{At}, \ \forall x, y \in U \).

Proof. (1) \( \Rightarrow \) Let \( A \) be a p-positive distribution consistent set. By the definition of \( D^{(p)}_{At} \), we conclude that \( \forall (x)_{At}^{[\alpha]} \in D^{(p)}_{At} \), we have \( P^{[\alpha]}_{At}(x) = P^{[\alpha]}_{At}(y) \). Combined with the property (1) in Theorem 5.6, we obtain that there exists \( a \in A \) such that \( S_{a}(x, y) < \alpha \). Hence \( a \in D^{(p)}_{At}(x, y) \) and \( A \cap D^{(p)}_{At}(x, y) \neq \emptyset \).

(\( \Rightarrow \)) If \( A \) is not a p-positive distribution consistent set, then by the property (1) in Theorem 5.6, we obtain that for any \( a \in A \), there exists \( x, y \in U \), if \( P^{[\alpha]}_{At}(x) = P^{[\alpha]}_{At}(y) \), then \( S_{a}(x, y) \geq \alpha \). Thus we conclude \( A \cap D^{(p)}_{At}(x, y) = \emptyset \), which contradicts the assumption.

(2) and (3) can be similarly proved. \( \square \)

Definition 5.6. Let \( S = (U, At \cup d, V, F) \) be a SDT, \( \alpha \in (0.5, 1) \), \( p, q \in (0, 1) \), \( A \subseteq At \). \( \{ l = 1, 2, 3 \} \) are the p-positive, q-negative and (p, q)-boundary distribution discernibility matrices respectively. p-positive, q-negative and (p, q)-boundary distribution discernibility functions are defined as

\[
M^{(p)}_{ly}(x, y) = \bigwedge_{a_{l} \in D^{(p)}_{ly}} \left( \left[ a_{l} \wedge a_{l} \in D^{(p)}_{ly} \right] \right) \ (l = 1, 2, 3), \ \forall x, y \in U.
\]

Combined with the above theories about attribute reduction based on \( \alpha \)-dominance-based quantitative rough sets in SDT, we in what follows present the approaches to obtain the attribute reduction by the three regions of a quantitative rough sets.

Theorem 5.8. Let \( S = (U, At \cup d, V, F) \) be a SDT, \( \alpha \in (0.5, 1) \), and \( A \subseteq At \). The minimal disjunctive normal form of each \( M^{(p)}_{ly} \) is

\[
M^{(p)}_{ly} = \bigwedge_{l=1}^{t} \bigwedge_{k=1}^{q_{l}} \bigwedge_{i=1}^{s} \left( a_{l} \right).
\]

Let \( B_{lk} = \{ a_{lk} : l = 1, 2, \ldots, q_{l} \} \), then \( \{ B_{lk} : k = 1, 2, \ldots, t \} \) are the set of all p-positive, q-negative and (p, q)-boundary distribution reductions, respectively.

6. Approximate reasoning based on \( \alpha \)-dominance quantitative rough sets

Rule acquisition is one of the most important objectives in SDT [23,25,72]. However, a real-world decision system is generally inconsistent, which may lead to some possible rules. To obtain the granular rules with some degree of confidence, Kryszkiewicz [23] proposed a confidence measure by conditional probability. Zhang et al. [75] extended the confidence rule to interval-valued decision tables. In what follows, we will extend the \( \alpha \)-dominance decision rules to the set-valued granular rules with the framework of \( \alpha \)-dominance-based quantitative rough sets.

Let \( (U, At \cup d, V, F) \) be a SDT and \( A \subseteq At \), an \( \alpha \)-dominance decision rule has the form of \( \text{des}(x)_{\alpha}^{[\alpha]} \rightarrow \text{des}(y)_{\alpha}^{[\alpha]} \), where \( \alpha \in [0, 1] \). In fact, \( \text{des}(x)_{\alpha}^{[\alpha]} \) is the description or semantic explanation of the granule \( x_{\alpha}^{[\alpha]} \) and possesses all the properties implied by \( x_{\alpha}^{[\alpha]} \). From the conceptual point of view [52], \( x_{\alpha}^{[\alpha]} \) and \( \text{des}(x)_{\alpha}^{[\alpha]} \) will be regarded as the extension and the intensification of \( (x)_{\alpha}^{[\alpha]} \) respectively. i.e., \( x_{\alpha}^{[\alpha]} \) and \( \text{des}(x)_{\alpha}^{[\alpha]} \) are mutually characterized. So we denote \( \text{des}(x)_{\alpha}^{[\alpha]} \rightarrow \text{des}(y)_{\alpha}^{[\alpha]} \) by \( \text{des}(x)_{\alpha}^{[\alpha]} \rightarrow x_{\alpha}^{[\alpha]} \) and call it an \( \alpha \)-dominance decision rule. \( x_{\alpha}^{[\alpha]} \) and \( y_{\alpha}^{[\alpha]} \) are the premise and conclusion of \( x_{\alpha}^{[\alpha]} \rightarrow y_{\alpha}^{[\alpha]} \) respectively.
From the definition of partial orders defined in Definitions 3.2 and 3.3, we can obtain two types of semantic explanation of the implication relation between $\alpha$-dominance decision rules under disjunctive and conjunctive semantics respectively.

**Type 1** (disjunctive semantic analysis). If $y \in \{x\}_A^{\Delta, \alpha}$, a new $\alpha$-dominance decision rule $\{y\}_A^{\Delta, \alpha} \rightarrow \{y\}_D^{\alpha}$ can be obtained from $\{x\}_A^{\Delta, \alpha} \rightarrow \{y\}_D^{\alpha}$. Otherwise for any $y \in \{x\}_A^{\Delta, \alpha}$, $\{y\}_A^{\Delta, \alpha} \rightarrow \{y\}_D^{\alpha}$ will be obtained from $\{x\}_A^{\Delta, \alpha} \rightarrow \{x\}_D^{\alpha}$.

**Type 2** (conjunctive semantic analysis). For any $y \in \{x\}_A^{\alpha}$, we can obtain $\{y\}_A^{\Delta, \alpha} \rightarrow \{y\}_D^{\alpha}$ from $\{x\}_A^{\alpha} \rightarrow \{x\}_D^{\alpha}$. However, if $y \in \{x\}_A^{\alpha}$, des($\{y\}_A^{\alpha}$) can only reflect some properties that is dominant than des($\{x\}_A^{\alpha}$). So a new $\alpha$-dominance decision rule $\{y\}_A^{\Delta, \alpha} \rightarrow \{y\}_D^{\alpha}$ from $\{x\}_A^{\alpha} \rightarrow \{x\}_D^{\alpha}$ cannot be guaranteed.

The conclusion measure defined in this paper takes all the elements of a set into account, while the inclusion measure in [75] just focuses on the end points of two intervals. We conclude that $R^\alpha_A(R^\Delta_A, R^\alpha_A, R^\alpha_A)$ has better capacities to depict the implication relationship between dominant decision rules. So in what follows, we will present the $\alpha$-dominance decision rules mainly from the point of $R^\alpha_A$, where $\Delta = \forall \circ \alpha$, $\forall \circ \alpha$ or $\leq \alpha$.

**Definition 6.1.** Let $(U, At \cup d, V, F)$ be a SDT and $U/d = \{D_k : k = 1, \ldots, r\}$ be the decision partition of $U$. For $A \subseteq At$ and $x_i \in U$, we define

$$\text{conf}(\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}) = \text{Inc}(\{x_i\}_A^{\alpha} D_k^{\alpha}).$$

and called it the confidence of the $\alpha$-dominance decision rule $\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}$. If $\text{conf}(\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}) = 1$, then the $\alpha$-dominance decision rule is a certain rule; otherwise, it is possible one, where $\Delta = \forall \circ \alpha$, $\forall \circ \alpha$ or $\leq \alpha$.

According to Definition 6.1 and quantitative rough sets, for $p, q \in [0, 1]$ and $k = 1, \ldots, r$, three regions of $D_k^{\alpha}$ based on the $\alpha$-dominance relation $R^\alpha_A$ can be defined as follows:

$$\text{POS}^\alpha_A(D_k^{\alpha}) = \{x_i \in U | \text{conf}(\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}) \geq p \};$$

$$\text{NEG}^\alpha_A(D_k^{\alpha}) = \{x_i \in U | \text{conf}(\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}) \leq q \};$$

$$\text{BND}^\alpha_A(D_k^{\alpha}) = \{x_i \in U | \text{conf}(\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}) < p \} \land \{x_i \in U | \text{conf}(\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}) > q \}.$$

$\text{POS}^\alpha_A(D_k^{\alpha})$ is the set of the objects in which each object $x_i$ induces a possible $\alpha$-dominance decision rule $\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}$ in $D_k^{\alpha}$ with the confidence at least $p$. $\text{NEG}^\alpha_A(D_k^{\alpha})$ is the set of objects in which each object $x_i$ induces a possible $\alpha$-dominance decision rule $\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}$ in $D_k^{\alpha}$ with the confidence at least $q$. The remaining objects are classified in the $\text{BND}^\alpha_A(D_k^{\alpha})$.

Let $(U, At \cup d, V, F)$ be a SDT and $U/d = \{D_k : k = 1, \ldots, r\}$ be the decision partition of $U$. For $A \subseteq At$ and $x_i \in U$, we define

$$R^\alpha(A, d, \alpha) = \{\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha} | x_i \in \text{POS}^\alpha_A(D_k^{\alpha})\}$$

as the class of decision rules by the positive region of quantitative rough sets.

**Definition 6.2.** Let $(U, At \cup d, V, F)$ be a SDT and $U/d = \{D_k : k = 1, \ldots, r\}$ be the decision partition of $U$. For $A \subseteq At$ and $x_i \in U$, given a $\alpha$-dominance decision rule $\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha} \in R^\alpha(A, d, \alpha)$, if $\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}$ satisfies $\text{conf}(\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}) = \text{conf}(\{x_i\}_A^{\Delta} \rightarrow D_k^{\alpha})$, we say the $\alpha$-dominance decision rule $\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}$ can be implied by the rule $\{x_i\}_A^{\Delta} \rightarrow D_k^{\alpha}$ and denote the implication relationship by $\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha} \Rightarrow \{x_i\}_A^{\Delta} \rightarrow D_k^{\alpha}$. If $\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}$ cannot be implied by the rule $\{x_i\}_A^{\Delta} \rightarrow D_k^{\alpha}$ and we denotes it by $\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha} \Rightarrow \{x_i\}_A^{\Delta} \rightarrow D_k^{\alpha}$.

**Definition 6.3.** Let $(U, At \cup d, V, F)$ be a SDT and $U/d = \{D_k : k = 1, \ldots, r\}$ be the decision partition of $U$. For $A \subseteq At$ and $x_i \in U$, if for each $\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha} \in R^\alpha(A, d, \alpha)$, there exists a $\alpha$-dominance decision rule $\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha} \in R^\alpha(A, d, \alpha)$ such that $\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha} \Rightarrow \{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}$, we say that $R^\alpha(A, d, \alpha)$ can be implied by $R^\alpha(A, d, \alpha)$ and denote the implication relationship by $R^\alpha(A, d, \alpha) \Rightarrow R^\alpha(A, d, \alpha)$. Otherwise, $R^\alpha(A, d, \alpha)$ can not be implied by $R^\alpha(A, d, \alpha)$, we denote it as $R^\alpha(A, d, \alpha) \not\Rightarrow R^\alpha(A, d, \alpha)$.

In the following definition, we propose an approach to obtain some compact dominant decision rules.

**Definition 6.4.** Let $(U, At \cup d, V, F)$ be a SDT and $U/d = \{D_k : k = 1, \ldots, r\}$ be the decision partition of $U$. For $A \subseteq At$ and $x_i \in U$, $\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha} \in R^\alpha(A, d, \alpha)$ is said to be redundant if there exists $\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha} \subseteq R^\alpha(A, d, \alpha) \setminus \{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}$ such that $\{x_i\}_A^{\alpha} \subseteq \{x_i\}_A^{\alpha}$ and $\text{conf}(\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha}) = \text{conf}(\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha})$. Otherwise, $\{x_i\}_A^{\alpha} \rightarrow D_k^{\alpha} \in R^\alpha(A, d, \alpha)$ is said to be optimal.

To derive compact rules with positive region of quantitative rough sets, we investigate the attribute reduction of SDT under the framework of the implication relationship in Definition 6.5.

**Definition 6.5.** Let $(U, At \cup d, V, F)$ be a SDT and $U/d = \{D_k : k = 1, \ldots, r\}$ be the decision partition of $U$. If $R^\alpha(A, d, \alpha) \Rightarrow R^\alpha(A, d, \alpha), A \subseteq At$ is called a p-positive consistent set of $(U, At \cup d, V, F)$. Otherwise, $A \subseteq At$ is called a p-positive inconsistent set of $(U, At \cup d, V, F)$, Furthermore, if for any $a \in A$, $R^\alpha(A, d, \alpha) \Rightarrow R^\alpha(A \setminus \{a\}, d, \alpha) \Rightarrow R^\alpha(A, d, \alpha), A$ is said to be a p-positive reduction of $(U, At \cup d, V, F)$.

The intersection of all the p-positive reductions is called the positive core of $(U, At \cup d, V, F)$ and is denoted by $\text{Core}_p^\alpha(A, p)$.

The following theorem can be obtained by the similar way to Theorem 1 of [75].
Theorem 6.1. Let \((U, A \cup d, V, F)\) be a SDT and \(U/d = \{D_k : k = 1, \ldots, r\}\) be the decision partition of \(U\). Then the \(p\)-positive reduction must exist, i.e., \(\text{Red}_D^p(A, p) \neq \emptyset\).

In this subsection, we presented the \(\alpha\)-dominance decision rule of SDT both under disjunctive and conjunctive semantics and analyzed the approaches to the attribute reduction based on the rule acquisition.

7. Experiments of attribute reduction for a target concept

Many algorithms of attribute reduction for SDT have been developed. Hereafter we introduce the classic heuristic reduction methods of rough set theory and ranking mutual information (RMI).

Much attention to heuristic reduction in rough set theory has been paid [1,15,16,19–21,31], in which a forward greedy search strategy is frequently employed [15,16,19–21]. In this kind of attribute reduction approaches, significance measures of conditional attributes are employed for heuristic functions, which is defined by dependency function. The dependency function is defined to characterize the dependency degree of an attribute subset with respect to a given decision [45]. Nevertheless, in the framework of ordered information systems, the value of the dependency degree is equal to 1 with respect to the single decision attribute. Xu [57] proposed the \(p\)-dominance dependency degree and \(p\)-dominance significance measure in ordered information system as follows:

**Definition 7.1.** Let \(S = (U, A \cup d, V, F)\) be a SDT. For \(\forall A \subseteq At\), the \(p\)-dependency degree of \(A\) with respect to \(d\) is defined as follows:

\[
\text{DEP}_{A(p)}^{d \geq \alpha}(d) = \frac{1}{|U|} \sum_{D \in U/d} \frac{|\text{POS}_{A(p)}^{d \geq \alpha}(D)|}{|D|}
\]

**Definition 7.2.** Let \(S = (U, A \cup d, V, F)\) be a SDT. For \(\forall A \subseteq At\) and \(\forall A \subseteq A - A\), the \(p\)-dominance significance measure of \(A\) in \(At\) is defined as

\[
\text{Sig}_{A(p)}^{d \geq \alpha}(A, d, U) = \text{DEP}_{A|A(p)}^{d \geq \alpha}(d) - \text{DEP}_{A(p)}^{d \geq \alpha}(d)
\]

RMI was first proposed by Hu et al. [14], which is based on mutual information to calculate the relevance between two sets of variables. Mutual information derived from Shannon entropy outperforms the measures of Gini and dependency in decision tree construction [32,33]. The measure is stable and robust for sample perturbation and noise. In addition, the measure of mutual information also do well in feature selection. The definition of RMI is defined as follows:

**Definition 7.3.** [14] Let \(S = (U, A \cup d, V, F)\) be a SDT. For \(\forall A \subseteq At\), the RMI of \(A\) with respect to \(d\) is defined as follows:

\[
\text{RMI}_{A,d} = -\sum_{i=1}^{n} \frac{1}{|X|} \log_{2} \left( \frac{|X_i|^{d \geq \alpha} \times |X_i|_A^{d \geq \alpha} \cap |X_i|_B^{d \geq \alpha}}{|X_i|^2} \right).
\]

And the significance of attribute \(A \subseteq At\) with respect to \(B\) is defined as \(\text{Sig}(A, B, D) = \text{RMI}_{B|A,d} - \text{RMI}_{B,d}\).

In what follows, we formulate an algorithm of heuristic attribute reduction by quantitative rough set model (QRS) and further compare the results with the time obtained by the variable precision rough sets (VPRS) and RMI.

7.1. Computing the positive region and attribute reduction of a target concept

The information granules and the positive region of a target concept \(X\) have been defined by the QRS. Now we give algorithms of calculating the positive region of QRS and attribute reduction, respectively, which are depicted as follows.

We obtain the whole positive region of a target concept by Algorithm 1. In what follows, we employ the forward greedy attribute reduction algorithm to get the selected conditional attributes.

**Algorithm 1** Computing positive region of a target concept with QRS.

**Input:** An SIT \(S = (U, A, V, F)\), a target concept \(X\) \((X \subseteq U)\) and the parameter values \(\alpha, p, q\).

**Output:** Positive region (PR) of \(X\) with respect to the attribute sets \(At\).

**Step 1:** From \(i = 1\) to \(|X|\) Do

- \{Generating \(\alpha\)-dominance classes to approximate the target concept\}

**Step 2:** \(PR \leftarrow \emptyset, i \leftarrow 1\).

**Step 3:** While \(i < |X|\) Do

- \{If \(\text{Inc}([x_i]_A^{\alpha}, X) \geq p\) and \(\text{Inc}([x_i]_A^{\alpha}, X^c) < q\), then \(PR \leftarrow PR \cup \{x_i\}\).

**Step 4:** Return \(PR\) and end.
Algorithm 2 A forward greedy attribute reduction algorithm with QRS.

**Input:** An STS \( S = (U, At, V, F) \), a target concept \( X \) (\( X \subseteq U \)) and the parameter values \( \alpha \), \( p \), \( q \).

**Output:** one reduction \( \text{red} \).

**Step 1:** \( \text{red} \leftarrow \emptyset; \) ///reduction is the pool vector to record the selected attributes.

**Step 2:** While \( \text{DEP}^{\alpha, \alpha}_{\text{red}(p)}(d) \neq \text{DEP}^{\alpha, \alpha}_{\text{At}(p)}(d) \) Do// This controls the stopping criterion.

\[
A \leftarrow \text{At} - \text{red}.
\]

Select \( a_0 \in A \) which satisfies \( \text{Sig}^{\alpha, \alpha}_{\text{At}(p)}(a_0, \text{red}, d, U) = \max \{ \text{Sig}^{\alpha, \alpha}_{\text{At}(p)}(a_k, \text{red}, d, U), a_k \in A \} \).

if \( \text{Sig}^{\alpha, \alpha}_{\text{At}(p)}(a_0, \text{red}, d, U) > 0 \), then \( \text{red} \leftarrow \text{red} \cup \{ a_0 \} \).

**Step 3:** Return \( \text{red} \) and end.

### Table 3

Data sets description.

<table>
<thead>
<tr>
<th>data sets</th>
<th>Cases</th>
<th>Features</th>
<th>Classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Congressvoting</td>
<td>435</td>
<td>16</td>
<td>2</td>
</tr>
<tr>
<td>Dermatology</td>
<td>366</td>
<td>34</td>
<td>6</td>
</tr>
<tr>
<td>Audiology</td>
<td>225</td>
<td>69</td>
<td>24</td>
</tr>
<tr>
<td>Soybean</td>
<td>307</td>
<td>35</td>
<td>17</td>
</tr>
</tbody>
</table>

### Table 4

The computation time of the algorithms QRS, VPRS and RMI for \( q = 0.15 \).

<table>
<thead>
<tr>
<th>Data sets</th>
<th>( p = 0.55 )</th>
<th>( p = 0.85 )</th>
<th>( p = 0.95 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>QRS</td>
<td>VPRS</td>
<td>RMI</td>
</tr>
</tbody>
</table>

### 7.2. Experimental analysis.

In this subsection, we take four UCI data sets in Table 3 to compare the computational performance of QRS, VPRS and RMI. The missing information of a given attribute is replaced by the whole values of this attribute.

Firstly, we normalize the attribute values \( f_{a_k}(x_i) \) (\( i = 1, \ldots, n \), \( k = 1, \ldots, m \)) into subsets of the interval [0,1] by the following way

\[
f_{a_k}(x_i) = \frac{f_{a_k}(x_i) - \min_j f_{a_k}(x_j)}{\max_j f_{a_k}(x_j) - \min_j f_{a_k}(x_j)}
\]

The value of the HM inclusion measure of \( x_i \) and \( x_j \) with respect to numerical attribute \( a \) is computed as

\[
\text{Inc}(f_a(x_i), f_a(x_j)) = \text{Inc}_{-1}(f_a(x_i), f_a(x_j))
\]

\[
= \left\{ \begin{array}{ll}
\bigwedge_k \mathcal{S}(f_a(x_i)(k), f_a(x_j)(k)) & , \quad k = 1, 2, \ldots, |f_a(x_i)|, \quad \text{if} \quad |f_a(x_i)| \leq |f_a(x_j)|, \\
\bigwedge_k \mathcal{S}(\beta_{f_a(x_i)}(k), f_a(x_j)(k)) & , \quad k = 1, 2, \ldots, |f_a(x_j)|, \quad \text{else},
\end{array} \right.
\]

where \( \mathcal{S} \) is the L-implicator, i.e., \( \mathcal{S}(x, y) = \min(1, 1 - x + y), x, y \in [0, 1] \).

In what follows, we compare the results of the algorithms QRS, VPRS and RMI in terms of the computation time and the number of selected attributes. Without loss of generality, we assume \( \alpha = 0.95 \). In the experiment, we obtain the experimental results with different values of the parameters \( p \) and \( q \) under the models of QRS, VPRS and RMI. Tables 4–6 present the computation time of the attribute reduction and Tables 7–9 are the numbers of selected attributes with respect to three models.

On the whole, we conclude from Tables 4–6 that the computation time of algorithm with QRS is less than that with VPRS except for the following data sets Congressvoting (when \( p = 0.55, q = 0.15 \)), Audiology (when \( p = 0.85, q = 0.15 \) and \( p = 0.95, q = 0.25 \)) and Soybean (when \( p = 0.85, q = 0.15 \) and \( p = 0.95, q = 0.15 \)). While the computation time of algorithm with RMI is the most compared with QRS and VPRS.

Please cite this article as: H.-Y. Zhang, S.-Y. Yang, Feature selection and approximate reasoning of large-scale set-valued decision tables based on \( \alpha \)-dominance-based quantitative rough sets, Information Sciences (2016), http://dx.doi.org/10.1016/j.ins.2016.06.028
Table 5
The computation time of the algorithms QRS, VPRS and RMI for q = 0.25.

<table>
<thead>
<tr>
<th>Data sets</th>
<th>p=0.55</th>
<th>p=0.85</th>
<th>p=0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>QRS</td>
<td>VPRS</td>
<td>RMI</td>
</tr>
</tbody>
</table>

Table 6
The computation time of the algorithms QRS, VPRS and RMI for q = 0.55.

<table>
<thead>
<tr>
<th>Data sets</th>
<th>p = 0.55</th>
<th>p = 0.85</th>
<th>p = 0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>QRS</td>
<td>VPRS</td>
<td>RMI</td>
</tr>
</tbody>
</table>

Table 7
Comparison of numbers of the attributes based on the algorithms QRS, VPRS and RMI for q=0.15.

<table>
<thead>
<tr>
<th>Data sets</th>
<th>Original Features</th>
<th>p=0.55</th>
<th>p=0.85</th>
<th>p=0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>QRS</td>
<td>VPRS</td>
<td>RMI</td>
<td>QRS</td>
</tr>
<tr>
<td>Congressvoting</td>
<td>16</td>
<td>6</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>Dermatology</td>
<td>34</td>
<td>9</td>
<td>9</td>
<td>34</td>
</tr>
<tr>
<td>Audiology</td>
<td>69</td>
<td>23</td>
<td>11</td>
<td>46</td>
</tr>
<tr>
<td>Soybean</td>
<td>35</td>
<td>11</td>
<td>6</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 8
Comparison of numbers of the attributes based on the algorithms QRS, VPRS and RMI for q = 0.25.

<table>
<thead>
<tr>
<th>Data sets</th>
<th>Original features</th>
<th>p=0.55</th>
<th>p=0.85</th>
<th>p=0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>QRS</td>
<td>VPRS</td>
<td>RMI</td>
<td>QRS</td>
</tr>
<tr>
<td>Congressvoting</td>
<td>16</td>
<td>2</td>
<td>6</td>
<td>16</td>
</tr>
<tr>
<td>Dermatology</td>
<td>34</td>
<td>13</td>
<td>9</td>
<td>34</td>
</tr>
<tr>
<td>Audiology</td>
<td>69</td>
<td>18</td>
<td>11</td>
<td>46</td>
</tr>
<tr>
<td>Soybean</td>
<td>35</td>
<td>7</td>
<td>6</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 9
Comparison of numbers of the attributes based on the algorithms QRS, VPRS and RMI for q = 0.55.

<table>
<thead>
<tr>
<th>Data sets</th>
<th>Original features</th>
<th>p=0.55</th>
<th>p=0.85</th>
<th>p=0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>QRS</td>
<td>VPRS</td>
<td>RMI</td>
<td>QRS</td>
</tr>
<tr>
<td>Congressvoting</td>
<td>16</td>
<td>6</td>
<td>6</td>
<td>16</td>
</tr>
<tr>
<td>Dermatology</td>
<td>34</td>
<td>6</td>
<td>9</td>
<td>34</td>
</tr>
<tr>
<td>Audiology</td>
<td>69</td>
<td>11</td>
<td>11</td>
<td>46</td>
</tr>
<tr>
<td>Soybean</td>
<td>35</td>
<td>6</td>
<td>6</td>
<td>10</td>
</tr>
</tbody>
</table>

We can conclude from Tables 7–9 that the numbers of attributes based on the algorithms with QRS are much smaller than that with RMI and the numbers of the attributes based on the algorithms with QRS and VPRS in Tables 7–9 are almost same.

8. Conclusion

In this paper, we have proposed a general framework of the α-dominance-based quantitative rough set approach for SITs under conjunctive and disjunctive semantics and have presented a method to rank objects by using the concept of α-dominance degree of each object. The α-dominance-based quantitative rough approximations have been proposed via α-dominance relations under disjunctive and conjunctive semantics respectively. Furthermore, we have analyzed attribute reduction in SITs by α-dominance-based quantitative rough set approach. In addition, approximate reasoning based on the

Please cite this article as: H.-Y. Zhang, S.-Y. Yang, Feature selection and approximate reasoning of large-scale set-valued decision tables based on α-dominance-based quantitative rough sets, Information Sciences (2016), http://dx.doi.org/10.1016/j.ins.2016.06.028
\(\alpha\)-dominance-based quantitative rough set has been studied. Finally, experiments validate the performance of our proposed method.

Acknowledgements

The authors wish to thank the anonymous reviewers and the editors for their constructive comments on this study. This work was supported by grants from the National Natural Science Foundation of China (Nos. 61005042), the Natural Science Foundation of Shaanxi Province (Nos. 2014JQ8348) and the Fundamental Research Funds for the Central Universities.

References


Please cite this article as: H.-Y. Zhang, S.-Y. Yang, Feature selection and approximate reasoning of large-scale set-valued decision tables based on α-dominance-based quantitative rough sets, Information Sciences (2016), http://dx.doi.org/10.1016/j.ins.2016.06.028